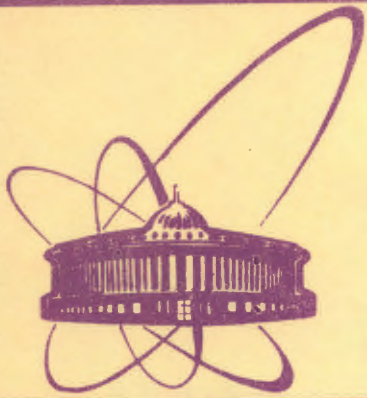

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ASYMPTOTIC FUNCTIONS AND MULTIPLICATION OF DISTRIBUTIONS

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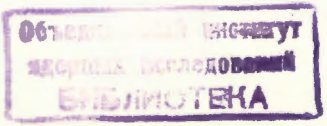

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E5 - 12859

Асимптотические функции и умножение распределений

Асимптотические функции представляют собой новый тип обобщенных функций. Но они не являются, как распределения Шварца, функционалами на некотором пространстве пробных функций. Асимптотические функции - это отображения в себя множества A – асимптотических чисел, введенных Христовым [1,2,3]. A, в свою очередь, есть вполне упорядоченное множество обобщенных чисел, включающее в себя как множество вещественных чисел, так и бесконечно малые и бесконечно большие числа. Любые две асимптотические функции можно умножать друг на друга. С другой стороны, распределения могут быть реализованы как асимптотические функции некоторого частного вида. Данная серия работ была инициирована некоторыми физическими проблемами, возникающими в квантовой теории. В будущем предполагается применить аппарат асимптотических функций вместо распределений Шварца в некоторых разделах квантовой теории, используя существование умножения между асимптотическими функциями. Данная работа является продолжением [5] и [6]. Для ее понимания необходимо знакомство с первыми десятью страницами работы [5] и §4 работы [6].

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Asymptotic Functions and Multiplication of Distributions

The asymptotic functions are a new type of generalized functions. But they are not functionals on some space of test-functions as the Schwartz distributions. They are mappings of the set A into itself, where A is the set of the asymptotic numbers introduced by Christov [1,2,3]. On its part, A is a totally-ordered set of generalized numbers including the system of real numbers R as well as infinitesimals and infinitely large numbers. Every two asymptotic functions can be multiplied. On the other hand, the distributions have realizations, in a certain sense, as asymptotic functions. The motivations of this work are connected with some physical problems of quantum theory. In further investigations we intend to probe the asymptotic functions instead of the distributions in some topics of quantum theory and to make use of the existence of multiplication between the asymptotic functions. The paper is a continuation of refs. [5,6]. To understand it, the knowledge of the first ten pages of ref. [5] and Sec. 4 of ref. [6] is presupposed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

As soon as the book "Théorie des distributions" by L.Schwartz has appeared, it has been realized that the multiplication of distributions is not always possible. For example, the products

$$\delta^2(x), \delta(x)P(\frac{1}{x}), \Theta(x)P(\frac{1}{x}), \delta(x)\Theta(x), [P(\frac{1}{x^n})]^m, \text{ etc.},$$

having significance in the quantum theory [18,22-27], cannot be satisfactorily defined. What is more, in the rare cases, when the multuplication is possible, it is not associative. The following well-known example is given by Schwartz:

$$P(\frac{1}{x})[x\delta(x)] \neq [P(\frac{1}{x}) \cdot x]\delta(x).$$

Several papers have been written by mathematicians as well as by physicists in order to introduce in some way a natural operation of multiplication of distributions: a) The most popular approach is based on an approximation of distributions by regular sequences [7-12]. The exchange formula relating (formally) the product and the convolution [13,15] as well as the point of view treating the distributions as boundary values of analytic functions [14,15] deliver us another base for introducing a multiplication. A typical feature of the approaches from this group is that the product (in the cases it exists) belongs to the space of distributions. Unfortunately, only a rather small number of products can be defined in these ways; b) In another group of works [16-22] an enlargement of the space of distributions $\mathcal{D}'$ is performed and the product, in general, does belong to this larger space. The methods used here involve either an axiomatic-algebraic technique [16,17], or non-standard analysis [19-22]. However, the operation of multiplication introduced in these works is not defined on the whole enlargement of $\mathcal{D}'$ which is the reason why the problem of associativity cannot be even formulated (see, for example, [22]); c) In the third group of approaches

the space of distributions is entirely abandoned. It is replaced by another space of generalized functions similar, in a certain sense, to the distributions and a commutative and associative multiplication is introduced for any two generalized functions. These approaches are based usually on a non-archimedean enlargement of the system of real as well as of complex numbers and are connected first of all with the non-standard analysis /19-22/. At first sight these methods are more successful, because very strong results about the existence of multiplication are established in their framework. They possess, in a certain sense, a disadvantage, lying in the fact that comparatively few /19-24/ numbers of applications to the other branches of mathematics as well as of physics have been performed up to now. It is not quite sure whether this kind of generalized functions will turn out to be so interesting and so useful for both mathematics and physics, as the distributions are, although the latter cause well-known troubles.

The approach we use belongs to the third group of works. The asymptotic functions are generalized functions similar to the distributions. However, they do not coincide with the distributions, i.e., they are not functionals on some space of test-functions. The asymptotic functions are mappings of the set of asymptotic numbers A into itself. On its part, A /1,3,5,6/ is a totally-ordered set of generalized numbers which includes isomorphically the field of the real numbers R as well as infinitely small (infinitesimals) and infinitely large numbers. Every two asymptotic functions can be multiplied (because every two asymptotic numbers can be multiplied) and the distributions have realizations, in a certain sense, as asymptotic functions. The motivations of this work are, in fact, connected with some problems in quantum theory /1-4,18,23-25/. In a further work we intend to probe the asymptotic functions instead of the distributions in some topics of quantum theory and to make use of the existence of multiplication between the asymptotic functions.

The present paper is organized as follows:

(i) In Sec. 1 the most general notion of asymptotic function is introduced (Definition (1.1)). This type of asymptotic functions is different from the asymptotic functions introduced in refs. /2,4/, although the notion of asymptotic numbers (1,3,5,6) is the common base idea for both types of asymptotic functions (see Remark (1.13)).

(ii). In Sec. 2 a particular type of sets of asymptotic numbers (subsets of A), called extended sets, are separated and their properties are studied. These sets (of asymptotic numbers) will play the role of domains of asymptotic functions of a particular type defined in a next paper.

(iii). In Sec. 3 a particular type of asymptotic functions called extended asymptotic functions, are considered.

Our plane for the future is the following: In a series of papers we are going to define two particular classes of asymptotic functions: the class F of quasi-extended asymptotic functions and the class Φ_{as} of the so-called quasi-distributions. The asymptotic functions of these two classes are very similar to the Schwartz distributions and at the same time every two such functions can be multiplied as opposed in the theory of distributions. In further investigations, as we already said, we intend to apply asymptotic functions instead of distributions to some topics of quantum mechanics and quantum theory. This paper is a continuation of refs. /5,6/. To understand it, the knowledge of the first ten papers of /5/ and Sec. 4 of /6/ is presupposed. For the sake of convenience, we will reproduce here some of the most important notations and terms (key works) of refs. /5,6/.

KEY WORDS: A = the set of the real asymptotic numbers (ref. /5/, Sec. 1); A^* = the set of the complex asymptotic numbers (ref. /5/, Sec. 2); $a = \sum_{k=\mu}^{\nu} \alpha_k s^k + o^\nu$ is the normal additive form of the asymptotic number a (ref. /5/, Sec. 6). Here $\mu, \nu \in Z \cup \{\infty\}$, $\mu \leq \nu$, $\alpha_k \in R$ (or $\alpha_k \in C$); μ is the power; ν is the order and $\lambda = \nu - \mu$ ($\infty - n = \infty$, $n \in Z$, and $\infty - \infty = 0$) is the relative order of a.

CHANGE OF TERMINOLOGY: In refs. /5,6/ we use the terms "accuracy of a" and "relative accuracy of a" for ν and λ, respectively. In the present paper ν will be called the order of a (instead of accuracy) and λ will be called relative order of a (instead of relative accuracy). We apologize for changing the terminology;

$$R^\circ = \{x + o^\circ : x \in R\}; \qquad C^\circ = \{c + o^\circ : c \in C\};$$

$$R^\infty = \{x + o^\infty : x \in R\}; \qquad C^\infty = \{c + o^\infty : c \in C\}$$

(ref.[5], Definition 10). Recall that R° and R^∞ are isomorphic to R and C°, and C^∞ are isomorphic to C (ref.[5], Theorem 20);

$\Theta=\{o^\nu : \nu\in Z\cup\{\infty\}\}$ is the set of the asymptotic zeros (ref.[5], Definition 5,v);

$I=\{1^\lambda : \lambda\in N_0\cup\{\infty\}\}$ is the set of the asymptotic units (ref.[5], Definition5,vi), where $N_0=\{0,1,2,\dots\}$; A^∞= the set of all asymptotic numbers with $\nu=\infty$ (ref.[5], Definition 11, Theorem 23, ii). Recall that

$$A^\infty \cong \{\sum_{k=\mu}^{\infty} a_k s^k ;\ \mu\in Z,\ a_k\in R\},$$

(ref.[5], Definition 12, Sec. 6); Ω_0 is the set of all infinitesimals (infinitely small asymptotic numbers) (Ref.[6], Sec. 4); Ω is the set of all finite asymptotic numbers (ref.[6] Sec. 4); Ω_∞ is the set of all infinitely large asymptotic numbers (ref.[6], Sec. 4). Recall that

$$\Omega_0\subset\Omega,\quad \Omega\cap\Omega_\infty=\emptyset,\quad A=\Omega\cup\Omega_\infty;$$

"f is an ordinary function" = "f is a complex-valued function of real variables", i.e., $f:X\to C$, where $X\subseteq R$, or $X\subseteq R\times R\times..\times R$.

1. ASYMPTOTIC FUNCTIONS

(1.1) DEFINITION (Asymptotic Function). (i) Every mapping of the type

$$f:\ D\to A^*, \tag{1.2}$$

where $D\subseteq A$ will be called an asymptotic function of one variable and if $D\subseteq A\times A\times\dots\times A$ (n -times), f will be called an asymptotic function of n variables. As usual, we shall often write $f(a)$, $a\in D$ or $f(a_1,a_2,\dots,a_n)$, $(a_1,a_2,\dots,a_n)\in D$, respectively, instead of (1.2).

(1.3) REMARK: Corresponding to the above definition, the values of the asymptotic functions are asymptotic numbers. Let $\mu(a)$, $\nu(a)$ and $\lambda(a)$ be the power, the order and the relative order of $f(a)$, respectively, for some $a\in D$. It is clear that μ, ν and λ are mappings of the following type:

$$\mu:\ D\to Z\cup\{\infty\}, \tag{1.4}$$

$$\nu:\ D\to Z\cup\{\infty\}; \tag{1.5}$$

$$\lambda:\ D\to N_0\cup\{\infty\}=\{0,1,2,\dots,\infty\}. \tag{1.6}$$

(1.7) DEFINITION (Algebraic Operations). The algebraic operations: addition, subtraction, multiplication and division between two asymptotic functions f and g defined on D and E, respectively, (where $D,E\subseteq A$, or $E,D\subseteq A\times A\times\dots\times A$) will be introduced by means of their values (i.e., just like the ordinary functions are added, subtracted, multiplied and divided). In other words

$$(f\pm g)(a)=f(a)\pm g(a),\qquad a\in D\cap E, \tag{1.8}$$

$$(f\cdot g)(a)=f(a)\cdot g(a),\qquad a\in D\cap E, \tag{1.9}$$

$$(f/g)(a)=f(a)/g(a),\qquad a\in D\cap E\setminus E_0, \tag{1.10}$$

where

$$E_0=\{a:\ a\in E,\ g(a)\in\Theta\}. \tag{1.11}$$

(1.12) REMARK: It is clear that every two asymptotic functions (for which $D\cap E\neq\emptyset$) can be added (subtracted) and multiplied (since every two asymptotic numbers can be added and multiplied). As we know, the set of the real numbers R is isomorphically embedded in the set of real asymptotic numbers A (ref.[5], Theorem 20) and the set of the complex numbers C is isomorphically embedded in the set of the complex asymptotic numbers A^*. In other words, $R\subset A$ and $C\subset A^*$ (and $R\subset C$ as well as $A\subset A^*$, of course). Consequently, the set of all ordinary functions (we mean the complex-valued functions of real variables) is isomorphically embedded in the set of all asymptotic functions (of the above-mentioned type).

The notion of asymptotic functions was introduced for the first time by Christov as equivalence classes of sequences of ordinary functions [2,4]. The notion of asymptotic function just introduced (1.1) is obviously different from that one given in refs. [2,4]. There exists a connection between these two types of asymptotic functions, but this will become clear only when analytic operations are introduced (which will be done in a next paper).

(1.13) LEMMA: Let M be a set (a subset) of asymptotic functions and let M be closed with respect to the algebraic operations: addition; multiplication; or addition and multiplication. Then M has the same algebraic properties as A and A* have (we mean the identities (ref./5/, Theorem 6)) which are valid in A and A* are valid in M, too). In particular, the set of all asymptotic functions has the same algebraic structure as A and A* have /5/.

PROOF: The lemma follows directly from the fact that the asymptotic functions are A* -valued functions.

2. ASYMPTOTIC EXTENSION OF SUBSETS OF R

As is known, the ordinary functions (as well as the distributions) are interesting first of all because of their analytic operations: differentiation, integration, Fourier-transformation, convolution and so on. That is why we must introduce somehow the analytic operations for just defined asymptotic functions. But there are some difficulties connected with this problem since A (as well as A*) is a non-archimedean set (containing infinitesimals) and consequently A (and A*) is not Dedekind completed. What is more, A (and A*) is also disconnected (ref./6/,Theorem 44).These features are typical not only for the asymptotic numbers but also for any non-archimedean extension of the real or complex numbers /19/. So, we cannot introduce the analytic operations by the standard way (by a given measure, the set of the measurable functions and so on). We shall introduce the analytic operations in another way: We are going to separate some very special classes of asymptotic functions (closely connected with the ordinary functions) in which classes the analytic operations can be naturally defined. The domains of the asymptotic functions of these particular classes cannot be arbitrary subsets of A. This section is devoted to these rather special subsets of A, called extended sets, because they are obtained as asymptotic extension of ordinary sets of real numbers.

(2.1) DEFINITION (Asymptotic Extension of Open Subsets). (i) Let X be an open subset of R. The set of all real asymptotic numbers $a \in A$ for which

$$\{s:\ a(s) \in X\} \in \mathfrak{E} \tag{2.2}$$

for all $a \in a$, where $\mathfrak{E}$ is the filter defined in ref./6/ will be called asymptotic extension of X and will be denoted by X_{as} ; (ii). Let X_i , $i=1,2,\dots,n$ $(n \in N)$ be open subsets of R and let

$$X = X_1 \times X_2 \times \dots \times X_n \,. \tag{2.3}$$

The set of all n -tuples $(a_1, a_2, \dots, a_n)$, where $a_k \in A$, $k=1,2,\dots,n$ for which

$$\{s:\ (a_1(s), a_2(s), \dots, a_n(s)) \in X\} \in \mathfrak{E} \tag{2.4}$$

for all $a_k \in a_k$, $k=1,2,\dots,n$ will be called an asymptotic extension of X and will be denoted also by X_{as} ; (iii). A subset D of A (or of $A \times A \times \dots \times A$) for which there exists an open subset X or R (or X is of the type of (2.3)) such that $D = X_{as}$ will be called an extended set.

(2.5) REMARK: For the sake of convenience we shall remind the definition of the filter $\mathfrak{E}$ (ref./6/,Definition 2): $\mathfrak{E}$ is the set of all subsets E of (0,1) which contains an interval of the type $(0,\epsilon)$, where $\epsilon \in R$, $\epsilon > 0$. As any filter, $\mathfrak{E}$ possesses the following (filter) properties:

$$\emptyset \notin \mathfrak{E}, \tag{2.6}$$

$$E, F \in \mathfrak{E} \quad \text{inplies} \quad E \cap F \in \mathfrak{E}, \tag{2.7}$$

$$E \subseteq F \subseteq (0,1) \quad \text{and} \quad E \in \mathfrak{E} \ \text{implies} \ F \in \mathfrak{E}. \tag{2.8}$$

(2.9) REMARK: It is easy to see that

$$(X_1 \times X_2 \times \dots \times X_n)_{as} = X_{1\,as} \times X_{2\,as} \times \dots \times X_{n\,as} \,. \tag{2.10}$$

That is why we shall concentrate our attention only on the case n=1. The generalization to the case $n > 1$ is done immediately by means of (2.10).

The following Lemma replaces the expression "for all $a \in a$ " in the above definition by the more simple one "there exists $a \in a$ ". It will help us to construct the asymptotic extensions of concrete open subsets of R .

(2.11) LEMMA: Let X be an open subset of R. Then: (i) $\emptyset_{as} = \emptyset$ and $X_{as} = \emptyset$ if and only if $X = \emptyset$; (ii) $R_{as} = A$ and $X_{as} = A$ if and only if $X = R$; (iii) $o^{-n} \in X_{as}$ for some $n \in N$

if and only if $X = R$ (and consequently, $X_{as} = A$); (iv) Let $X \neq R$ and $a \in A$. Then $a \in X_{as}$ if and only if the following two conditions (denoted by a) and b)) are valid:

a) $a - x \notin \Theta$ for all $x \in \bar{X} \setminus X$,

where $\bar{X}$ is the closure of X;

b) There exists $\alpha \in a$ for which (2.2) holds.

(2.12) REMARK: The conditions a) are equivalent to the following two conditions (denoted by a)* and a)**, respectively):

a)* $a \notin \{o^{-n} : n \in N\}$;

a)** $a \notin \{x + o^{\nu} : x \in \bar{X} \setminus X\}$.

PROOF: (i) and (ii) are obvious. Let us consider (iii): If $X = R$, then $o^{-n} \in X_{as}$, of course, since $X_{as} = A$, corresponding to (ii). Let $o^{-n} \in X_{as}$ for some $n \in N$, i.e., (2.2) holds for all $\alpha \in o^{-n}$. In particular, for $\alpha(s) \equiv x$, where x is any real number (we mean $\lim_{s \to 0} s^n x = 0$ for any $x \in R$), (2.2) implies $x \in X$ for any $x \in R$, i.e., $X = R$; (iv) Let $a \in X_{as}$, i.e., (2.2) holds for all $\alpha \in a$. We must show only the condition a) (which, on its part, is equivalent to a)* and a)**) since b) follows obviously from $a \in X_{as}$. But $X \neq R$ and $a \in X_{as}$ implies a)*, corresponding to (iii). If a is infinitely large, i.e., $a \in \Omega_{\infty}$, then a)** holds obviously. Let a be finite, i.e., $a \in \Omega$. Corresponding to (ref. /6/ Theorem 36, (iv)), a can be represented in the form $a = x_0 + h$, where $x_0 \in R$ and $h \in \Omega_0$ are uniquely determined by a ($x_0 = r(a)$ and $h = a - x_0$ (ref. /6/, Definition 10). So (2.2) reduces to

$$\{s : x_0 + \chi(s) \in X\} \in \mathfrak{F} \tag{2.13}$$

for all $\chi \in h$. If $h \notin \Theta$, then a)** holds obviously. Let $h \in \Theta$. Then we must show that $x_0 \notin \bar{X} \setminus X$. Indeed, we can put $\chi = \chi_0$ in (2.13), where $\chi_0(s) \equiv 0$ (since $h \in \Theta$), and we shall obtain $x_0 \in X$. Let now a) and b) be valid (or, which is the same, let a)* , a)** and b) be valid); It is necessary to consider the cases $a \in \Omega$, $a \in \Omega_{\infty}^{-}$ and $a \in \Omega_{\infty}^{+}$ separately, where Ω_{∞}^{-} and Ω_{∞}^{+} are the sets of infinitely large negative and positive asymptotic numbers, resp. Let $a \in \Omega$ and let $a = x_0 + h$ be the representation we have talked above; In

this case b) reduces to

$$\{s : x_0 + \chi_0(s) \in X\} \in \mathfrak{F} \tag{2.13'}$$

for some $\chi_0 \in h$ and consequently, $x_0 \in \bar{X}$ (since $\lim_{s \to 0} \chi_0(s) = 0$). If $x_0 \in X$, then (2.13') will be valid for all $\chi_0 \in h$ (since X is open), i.e., $a = x_0 + h \in X_{as}$. If $x_0 \notin X$ (i.e., $x_0 \in \bar{X} \setminus X$), then the condition a) (for $x = x_0$) will imply $h \notin \Theta$, i.e., h can be represented in the form $h = r s^n + \Delta s^n$, where $n \in N$, $r \in R$, $r \neq 0$, and Δ is an infinitesimal. In the case $r < 0$, (2.13) will imply $X \supset (-\epsilon, x_0)$ for some $\epsilon \in R$, $\epsilon > 0$ and in the case $r > 0$, , (2.13) implies $X \supset (x_0, \epsilon)$ for some $\epsilon \in R$, $\epsilon > 0$. In both cases we obtain

$$\{s : x_0 + \chi(s) \in X\} \in \mathfrak{F}$$

for all $\chi \in h$, i.e., $a = x_0 + h \in X_{as}$. Let $a \in \Omega_{\infty}^{-}$, i.e., a can be represented in the form $a = r s^{-n} + \Delta s^{-n}$ for some $r \in R$, $r < 0$. and some infinitesimal Δ. In the same way we conclude that X contains an interval of the type $(-\infty, t)$ for some $t \in R$ and consequently, b) implies (2.2) for all $\alpha \in a$, i.e., $a \subset X_{as}$. The case $a \in \Omega_{\infty}^{+}$ is treated in the same way. The proof is finished.

(2.14) SOME EXAMPLES: If $x_1, x_2 \in R$ and $x_1 < x_2$, then:

$$\{r \in R : x_1 < r < x_2\}_{as} = \{a \in A : x_1 < a < x_2,\ a - x_1 \notin \Theta\}, \tag{2.15}$$

$$\{r \subset R : r < x_1\}_{as} = \{a \subset A : a < x_1\}, \tag{2.16}$$

$$\{r \in R : x_1 < r\}_{as} = \{a \in A : x_1 < a,\ a - x_1 \notin \Theta\}. \tag{2.17}$$

In the special case $x_1 = 0$ and $x_2 = 1$ we obtain

$$(0,1)_{as} = \{a \in A : 0 < a < 1,\ a \notin \Theta\}; \tag{2.18}$$

$$\{(x,y) : x, y \in R,\ x \neq 0 \text{ or } y \neq 0\}_{as} = \tag{2.19}$$

$$= \{(a,b) : a, b \in A,\ a \notin \Theta \text{ or } b \notin \Theta\}.$$

(2.20) THEOREM (X_{as} contains X). Let X be a non-empty open subset of R. Then X_{as} contains X properly in the sense of the isomorphism $R^{\circ} \cong R$ and $R^{\infty} \cong R$ (ref./5/, Theorem 20). Namely, $X^{\circ} \subset X_{as}$ and $X^{\infty} \subset X_{as}$ where

$$X^{\circ} = \{ x + o^{\circ} : \quad x \in X \}, \tag{2.21}$$

$$X^{\infty} = \{ x + o^{\infty} : \quad x \in X \}. \tag{2.22}$$

Moreover, if $a \in A$, then $a \in R^{\circ} \cap X_{as}$ implies $a \in X^{\circ}$ and $a \in R^{\infty} \cap X_{a}$ implies $a \in X^{\infty}$.

PROOF: Elementary (by means of Lemma (2.11)).

(2.23) REMARK: Let us restrict only one of the above-mentioned isomorphisms: wither $R \cong R^{\circ}$ or $R \cong R^{\infty}$. Then Theorem (2.20) can be formulated as follows: If X is a non-empty open subset of R, then $X \subsetneq X_{as}$. Moreover, $a \in R \cap X_{as}$ implies $a \in X$.

(2.24) THEOREM (Some Properties of X_{as}). Let X and Y be two open subsets of R. Then: (i) $X \subseteq Y$ implies $X_{as} \subseteq Y_{as}$; (ii) $(X \cap Y)_{as} = X_{as} \cap Y_{as}$.

PROOF: (i) Let $X \subseteq Y$ and $a \in X_{as}$, i.e., (2.2) holds for all $\alpha \in a$. Let us set

$$E(\alpha) = \{ s : \alpha(s) \in X \}, \qquad \alpha \in a,$$

$$F(\alpha) = \{ s : \alpha(s) \in Y \}, \qquad \alpha \in a,$$

Obviously, $X \subseteq Y$ implies $E(\alpha) \subseteq F(\alpha)$ and $a \in X_{as}$ implies $E(\alpha) \in \mathfrak{E}$ for all $\alpha \in a$. Corresponding to (2.8), we obtain $F(\alpha) \in \mathfrak{E}$ for all $\alpha \in a$, i.e., $a \subseteq Y_{as}$; (ii) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$ imply $(X \cap Y)_{as} \subseteq X_{as}$ and $(X \cap Y)_{as} \subseteq Y_{as}$ respectively, corresponding to (i), i.e., $(X \cap Y)_{as} \subseteq X_{as} \cap Y_{as}$. Let $a \in X_{as} \cap Y_{as}$. We have $E(\alpha)$, $F(\alpha) \in \mathfrak{E}$ for all $\alpha \in a$ which implies $E(\alpha) \cap F(\alpha) \in \mathfrak{E}$ for all $\alpha \in a$, corresponding to (2.7). On the other hand,

$$E(\alpha) \cap F(\alpha) = \{ s : \alpha(s) \in X \cap Y \}, \qquad \alpha \in a$$

which implies $a \in (X \cap Y)_{as}$. The theorem is proved.

(2.25) REMARK: The facts worth keeping in mind from this section are the following: $R_{as} = A$; $(X \cap Y)_{as} = X_{as} \cap Y_{as}$; all examples (2.15)-(2.19). More especially, we are going

to use in future the fact that $(0,1)_{as}$ contains all positive infinitesimals h which are not asymptotic zeros. In particular,

$$r s^{n} + o^{\nu} \in (0,1)_{as} , \quad n \in N , \quad \nu \in N \cup \{\infty\}, \quad r \in R , \quad r > 0. \tag{2.26}$$

(2.27) REMARK: The reader who knows the non-standard analysis (Robinson's theory of infinitesimals)/19-24/ will probably observe that the notion of "asymptotic extension X_{as} of a set X " just introduced (2.1), in analogous, to a certain degree, to the so-called "non-standard extension $^{*}X$ of a set X" in the framework of the non-standard analysis. The lemma (2.11) as well as the theorems (2.20) and (2.24) have also counterparts in the nonstandard analysis. This analogy will continue in the future developments, too. For example, we are going to introduce the notions of "asymptotic extension of a given ordinary function" and "quasi-extended asymptotic functions" which will have analogs in the non-standard analysis *.

3. EXTENDED ASYMPTOTIC FUNCTIONS

In this section we are goung to consider a very special type of asymptotic functions which are obtained as an extension (continuation) of continuous ordinary functions (we mean complex-valued functions of real variables). These functions are called extended asymptotic functions.

(3.1) DEFINITION (Extended Asymptotic Functions). (i) Let X be an open subset of R and $\phi : X \to C$ be a continuous ordinary function defined on X. We shall say that an asymptotic function of the type $\phi_{as} : D \to A^{*}$ is an asymptotic extension of ϕ if the following two conditions are valid: a) $X \subseteq D \subsetneq X_{as}$ (2.1) and for every $a \in D$ the set (of functions):

$$a^{*} = \{ \phi(\alpha) : \quad \alpha \in a \} \tag{3.2}$$

possesses an asymptotic cover as a^{*} (ref./5/ Definition 7); b) If $a \in D$, then $\phi_{as}(a)$ is (by definition) the asymptotic cover of a^{*}, i.e.,

* Some of the results of this paper were reported by the authos at the Conference "Operatoren-Distributionen und verwandte Non-Standard Methoden", Oberwolfach, 2-8 July, 1978, BRD.

$$\phi_{as}(a) = as\,a^*, \quad a \in D; \tag{3.3}$$

(ii) Let X_k, $k=1,2,\ldots,n$ be open subsets of R and let $\phi: X \to C$ be a continuous ordinary function defined on X, where

$$X = X_1 \times X_2 \times \ldots \times X_n .$$

We shall say that an asymptotic function of the type ϕ_{as}: $D \to A^*$ is an asymptotic extension of ϕ if the following two conditions are valid: a) $X \subseteq D \subseteq X_{as}$ (2.10) and for every point $(a_1, a_2, \ldots, a_n) \in D$ the set (of functions):

$$a^* = \{\phi(\alpha_1, \alpha_2, \ldots, \alpha_n): \ \alpha_k \in a_k, \ k = 1,2,\ldots,n\} \tag{3.4}$$

possesses an asymptotic cover $as\,a^*$ (ref. /5/, Definition 7); b) If $(a_1, a_2, \ldots, a_n) \in D$, then (by definition):

$$\phi_{as}(a_1, a_2, \ldots, a_n) = as\,a^*; \tag{3.5}$$

(iii) The asymptotic functions obtained as an asymptotic extension of some continuous ordinary functions will be called extended asymptotic functions.

(3.6) REMARK: It is clear that (i)-part of the above definition is a particular case of its (ii)-part for $n=1$. On this point onwards, we shall consider the case $n = 1$ only, but all results established further can be easily generalized for $n > 1$.

(3.7) REMARK: Let us remind (ref. /5/, Definition 7), that the asymptotic cover $as\,a^*$ of a subset a^* of A_s^* (ref. /5/, Sec. 2) is an asymptotic number, i.e., $as\,a^* \in A^*$, which contains a^*, i.e., $a^* \subseteq as\,a^*$ and such that there is no other (different from $as\,a^*$) asymptotic number a' for which $a^* \subset a' \subsetneq as\,a^*$. Let us recall further (ref. /5/ Theorem 5), that a^* has a unique asymptotic cover if and only if there exists $a' \subseteq A^*$ such that $a \subseteq a'$. The following three lemmas will help us to construct the asymptotic extensions of some ordinary functions.

(3.8) LEMMA: Let $\phi(x)$, $x \in X$ be an ordinary function differentiable any number of times defined on the open subset X of R. Then $\phi_{as}(a)$ exists for every finite asymptotic number of the type $a = x+h$, where $x \in X$ and h is an infinitesimal. Moreover,

$$\phi_{as}(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} \phi^{(k)}(x) h^k, \quad x \in X, \ h \in \Omega_0 . \tag{3.9}$$

(3.10) REMARK: The series on the right-hand side of (3.9) is convergent with respect to the interval topology of A (ref. /6/, Sec. 5).

PROOF: (3.9) is, in fact, a translation of the well-known formula for the asymptotic expansion of $\phi(x+\epsilon)$:

$$\phi(x+\epsilon) \underset{\epsilon \to 0}{\sim} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(x)}{k!} \epsilon^k \tag{3.11}$$

in terms of the asymptotic number terminology. Notice that the series on the right-hand side of (3.11) is divergent (in general) with respect to the ordinary topology of R, in contrast to the series in (3.9) which is convergent with respect to the topology of A.

(3.12) LEMMA: Let $\phi(x)$, $x \in X$ be a continuous ordinary function defined on the open subset X of R and let $\bar{x} \in \bar{X}$, where $\bar{X}$ be the closure of X.
(i) If there exists $m \in Z$ for which

$$\left| \begin{array}{l} \lim t^{-m} \phi(\bar{x}+t) = 0, \\ {\scriptstyle t \to 0} \\ \bar{x}+t \in X \end{array} \right. \tag{3.13}$$

then the value $\phi_{as}(\bar{x}+h)$ exists for all infinitesimals h for which $\bar{x}+h \in X_{as}$ and the inequality (an estimate for ϕ_{as}):

$$\phi_{as}(\bar{x}+h) \subseteq o^{m\mu_h}, \quad \bar{x}+h \in X_{as} \tag{3.14}$$

holds, where μ_h is the power of h; (ii) Let, moreover, ϕ have an asymptotic expansion of the type:

$$\phi(\bar{x}+t) = \sum_{k=\mu}^{n} c_k t^k + \Delta_n(t), \quad t \in R, \ \bar{x}+t \in X \tag{3.15}$$

for all $n \in Z$, $n \le \nu$, where $\mu, \nu \subseteq Z \cup \{\infty, +\infty\}$, $\mu \le \nu$, $c_k \in C$ and

$$\left| \begin{array}{l} \lim \Delta_n(t)/t^n = 0. \\ {\scriptstyle t \to 0} \\ \bar{x}+t \in X \end{array} \right. \tag{3.16}$$

And let, finally, ϕ has no asymptotic expansion of the type (3.15) by a higher order than ν. (The last is not a restriction in the case $\nu = \infty$). Then:

$$\phi_{as}(\bar{x}+h) = \sum_{k=\mu}^{\nu} c_k h^k + o^{\nu}, \quad \bar{x} \subset \bar{X}, \ h \in \Omega_0, \ \bar{x}+h \in X_{as} . \tag{3.17}$$

PROOF: The above lemma follows immediately from (ref. /6/, Theorem 41). It is, in fact, a periphrasis of the notion of asymptotic expansion of a given function in our "asymptotic number language".

(3.18) REMARK: The above Lemma (3.12) reduces to Lemma (3.8) in the following special case: $\bar{x} \in X$ and $\phi^{(k)}(\bar{x}), k=0,1,...$ exist. In this case we have $\phi^{(k)}(\bar{x}) = c_k$. We wrote out Lemma (3.8) only for the sake of convenience. We are going to use Lemma (3.12) (but not Lemma (3.8)) only in the case $\bar{x} \in \bar{X} \setminus X$.

(3.19) LEMMA: Let $\phi(x), x \in X$, be a continuous ordinary function defined on X, where X is an open subset of R. Let X contain intervals of the type (t, ∞) or $(-\infty, t)$ for some $t \in R$ (the case $X = R$ is included here). If there exists $m_{\pm} \in Z$ (respectively) such that

$$\lim_{x \to \pm\infty} x^{-m_{\pm}} \phi(x) = 0 \tag{3.20}$$

then $\phi_{as}(a)$ exists for every positive (or negative, resp.) infinitely large asymptotic numbers a and

$$\phi_{as}(a) \subseteq o^{\mu \cdot m_{\pm}} \tag{3.21}$$

holds, where μ is the power of a.

PROOF: Elementarily, it is sufficient to put $a = \pm r s^{\mu} + s^{\mu} h$ for some $\mu \in Z$, $\mu < 0$, $r \in R$, $r > 0$ and some infinitesimal h, to replace any $\alpha \in a$ (which can be represented in the form $\alpha(s) = \pm r s^{\mu} + s^{\mu} \chi(s)$ for $\lim_{s \to 0} \chi(s) = 0$) in (3.20) and use the definition (3.1).

Recall the definition of the asymptotic numbers (ref. /5/, Definition 5). The following lemma establishes a connection between this definition and the notion of asymptotic extension of a function.

(3.22) LEMMA: Let $\phi(x), x \in X$ be a continuous ordinary function defined on the open subset X of R and let $\phi_{as}(a), a \in D$ be an asymptotic extension of ϕ. Then

$$\phi_{as}(a) = as\{\phi(\alpha)\} + o^{\nu(a)}, \quad a \in D \tag{3.23}$$

for any $\alpha \in a$, where $\nu(a)$ is the order of $\phi_{as}(a)$.

(3.24) COROLLARY: Let $a \in A^*$. Then

$$a = \alpha_{as}(s + o^{\infty}) + o^{\nu} \tag{3.25}$$

for any $\alpha \in a$, where ν is the order of a.

PROOF: (3.23) follows immediately from (ref. /5/, Theorem 3).

(3.26) REMARK: Recall (ref. /5/, Sec. 6) that $s + o^{\infty}$ is the asymptotic number defined as follows:

$$s + o^{\infty} = \{\alpha : \alpha \in A_s, \ \alpha(s) = s + \Delta(s), \ \lim_{s \to 0} \Delta(s)/s^n = 0 \text{ for all } n \in Z\} \tag{3.27}$$

Recall, as well, that $s + o^{\infty} = s.1^{\infty}$. Moreover, corresponding to (ref. /5/, Definition 12), the number $s + o^{\infty} \equiv s1^{\infty}$ can be denoted simply as s, i.e.,

$$s + o^{\infty} = s1^{\infty} = s. \tag{3.28}$$

(3.29) REMARK: In the n-dimensional case (3.23) could be replaced by:

$$\phi_{as}(a_1, ..., a_n) = as\{\phi(\alpha_1, ..., \alpha_n)\} + o^{\nu(a)}, \quad (a_1, ..., a_n) \in D \tag{3.30}$$

for any $\alpha_k \in a_k$, $k = 1, 2, ..., n$.

(3.31) THEOREM (ϕ is the Restriction of ϕ_{as}). Let $\phi(x)$, $x \in X$ be a continuous ordinary function defined on the open subset X of R and let $\phi_{as}(a)$, $a \in D$ be an asymptotic extension of ϕ. Then:

$$\phi_{as}(x) = \phi(x), \quad x \in X, \tag{3.32}$$

PROOF: (3.32) follows directly from the fact that ϕ is continuous and X is an open set, bearing in mind the isomorphism $R \cong R^{\circ}$ (ref. /5/, Theorem 20) as well as Theorem (2.20)).

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