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EUCLIDEAN FIELDS: VECTOR MESONS AND PHOTONS

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Евклидовые поля: Векторные мезоны и фотоны

Рассматривается свободное поперечное векторное поле частиц масс $m \geq 0$. Модель относится к обычной теории квантовых полей свободного векторного мезонного поля и свободного электромагнитного поля посредством расширения от поперечных к произвольным пробным функциям. Мы описываем одночастичные состояния в поперечной калибровке и их локализацию. Докажем физическую положительность. Мы даем евклидов подход к фотонному полю в сферическом мире, используя дилатационную инвариантность и инверсии.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

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Euclidean Fields: Vector Mesons and Photons

We study free transverse vector fields of mass $m \geq 0$. The model is related to the usual free vector meson and electromagnetic quantum field theories by extension of the field operators from transverse to arbitrary test functions. We describe the one particle states in transverse gauge and their localization. We prove reflexion positivity and derive free Feynman-Kac-Nelson fotmulas. We give an Euclidean approach to a photon field in a spherical world using dilatation covariance and inversions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

0. Introduction and content

It was Schwinger in 1958 who first argued that Euclidean Q.F.T. may be a "possible avenue for future development of field theory". Now, 1979, it has already a long history. Stimulated by the progress in scalar $P(\phi)_2$-theory from 1972, the techniques exploited by Nelson and others have been used to describe in the Euclidean region also Fermi and other higher-spin fields even if gauge fields are included in the interactions.

The basic idea is to pass from Minkowski to Euclidean space-time which brings the structural simplification of positive-definite metric and, in the case of Boson models we will discuss, the main technical advantage of commuting random variables. However, once having the Euclidean formulation of a quantum field theoretical problem it yields an interesting mathematical model itself, and we can forget about the "physical background". In fact, the reflection positivity - together with Euclidean covariance and a regularity condition - always guaranties the backward step to Minkowski space-time.

In particular, the free vector meson and electromagnetic quantum field theories have unique Euclidean formulations via the reconstruction of Osterwalder and Schrader. They were given by Yao, Gross, Guerra and others. Independently, in 1974 we have found the quantum and Euclidean models for the photon field in terms of transverse potentials and derived them from corresponding massive theories as limits $m \downarrow 0$. We avoided the well known problem of quantizing the free Maxwells equations, restricting the vector potentials to transversal test functions in the Lorentz or Coulomb gauge.

Applying the above idea to the massive case we have defined also the quantum and Euclidean models for the free vector meson field in the transverse gauge. The main result of our paper is that reflection positivity and reflexivity hold for $m \geq 0$, even for the Euclidean models on a lattice.

However, the Markoff property can be recovered exactly for the Euclidean photon field theory. We have studied a nice analogy to classical magnetostatics and end with the following interpretation of reflection positivity:

Let $\{m_i\}_{i \in I}$ a finite set of magnetic dipoles localized in the open right half-space $\Lambda_+ = \{r=(r^0,\underline{r}): r^0>0\}$ and $\overset{e}{A}(h)$, $h=\sum_I \nabla x(m_i \delta_{a_i})$, the related Euclidean random field. Then

$$\langle \overset{e}{A}(h)\overset{e}{A}(\theta_0 h)\rangle = -\sum_{i,j \in I} m_i . K(a_i-\theta_0 a_j)\ \theta_0 m_j \geq 0 ,$$

because it is just the expression for the Coulomb interaction energy of the set of electric dipoles $\{m_i\}_I$ with their mirror images with respect to the hyperplane $S_0=\{r:\ r^0=0\}$. Above θ_0 denotes reflection at S_0 and K the coupling matrix of the dipoles.

The paper consists of three parts which are organized as follows: (cF. JINR preprints E5-I2779, I278I for the parts II and III)

I. Vector Mesons and Photons: 1. One-particle states in the transverse gauge, 2. Localization, Euclidean and dilatation covariance, 3. Reflection positivity and Hamiltonian semi-groups,

II. Quantum and Random Fields: 1. Vector meson field in the Stückelberg gauge and photon field in terms of $F^{\mu\nu}$, 2. Second quantization, 3. Transversal self-interaction,

III. Lattice models: 1. Transversal lattice fields, 2. Steady currents and magnetic dipoles, 3. Conclusions.

1. One-particle states in the transverse gauge

Let be $\{n_\mu:\ \mu=0,1,2,3\}$ the Carthesian basis in R^4 and $a,b \to ab= g^{\mu\nu}a^\mu b^\nu=-a^0b^0+\underline{a}.\underline{b}$ the indefinite metric. In the momentum space we have the upper mass hyperboloids $V_+^m=\{p=(w,\underline{p}):\ w=+\sqrt{\underline{p}^2+m^2}\}$, $m>0$, and the forward light cone $V_+^0=\{p=(k,\underline{p}):\ k=+/\underline{p}/\}$. For $m \geq 0$ we introduce the tangent planes with respect to points $p \in (V_+^m \setminus \mathring{p})$, $\mathring{p}=(m,\underline{0})$,

(1) $$T_p=\{h\in C^4:\ ph=-wh^0+\underline{p}.\underline{h}=0\}.$$

Note that $T_p \cong C^3$. Moreover, $T_{\mathring{p}}$ is well defined if $m>0$. We construct a helicity basis $\{v_\lambda:\ \Lambda v_\lambda=\lambda v_\lambda,\ \lambda=0,\pm 1\}$ on T_p as follows. For given $\underline{n}\in R^3$ non-colinear in $\underline{p}$

(2) $$v_0=(1,\underline{p}w/k^2),\quad v_{\pm 1}=(\underline{n}-\underline{p}(\underline{p}.\underline{n})/k^2)\pm i\underline{p}x\underline{n}/k .$$

We now define the one-particle Hilbert space in the transverse gauge for the free vector meson quantum field of mass $m>0$ as the direct integral

(3) $$M_m=\int_{V_+^m}^{\oplus} T_p \frac{d^3p}{2p_0} .$$

As an operator in M_m Λ is called helicity and the square $C=\Lambda^2$ Coulomb projection, where

(4) $$C^{\mu\nu}(p)=\begin{cases} \delta^{\mu\nu}-\dfrac{p^\mu p^\nu}{\underline{p}^2} & \mu,\nu=1,2,3 \\ 0 & \text{otherwise.}\end{cases}$$

They allow a unique decomposition $\tilde{h}=\tilde{t}\oplus\tilde{l}$, $\tilde{t}\in M_m^C$, where

(5) $$M_m^C=\operatorname{ran} C=\{\tilde{h}\in M_m:\ \tilde{h}^0=0\}$$

and $l\in \ker C$. The restriction to V_+^m defines a unitary equivalence

(6) $$M_m \cong \sum_\mu^{\oplus} L^2(R^3,\frac{d^3p}{2w})\cap\{\tilde{h}:\ -w\tilde{h}^0+\underline{p}.\tilde{\underline{h}}=0\ \text{a.e.}\}.$$

In particular $//1// = m\,//\frac{\tilde{1}^{o}}{k}//_{L^2(\frac{d^3p}{2w})}$, hence for $m=0$ - - i.e. for the one- photon space - in the Lorentz gauge we put the factor space

(7) $$M_{ph} = {}^{M_m}/_{M_m^O}, \quad m=0,$$

and in the Coulomb gauge $M^C_{ph} = M^C_m$, $m=0$. We write shortly M for M_m and M^C_{ph}, respectively. By the conjugation

(8) $$h \rightarrow (\tilde{h}^*)^{\mu}(\underline{p}) = g^{\mu\nu}\bar{h}^{\nu}(-\underline{p})$$

we select a $*$-real part M_r. For $m>0$ we have dense subspaces $S^T(V^m_+)$ in M_m of smooth transverse test functions $\tilde{h}$ on V^m_+ which satisfy $\tilde{h}(p)=0$, and similar $\tilde{h}(0)=0$ for $\tilde{h}\in S^T(V^o_+)$. Gross has identified M_m with a Hilbert space of weak vector-valued solutions of the Procca field equations and M_{ph} with a Hilbert space of Maxwells equations in vacuum [Gr]. Using the $L^{\uparrow}_+$-invariant tempered measure $dw^m_+ = \theta(p^o)\,\delta(p^2+m^2)d^4p$ in R^4 and the projections $\tilde{P}(p): C^4 \rightarrow T_p$

(9) $$\tilde{P}^{\mu\nu}(p) = g^{\mu\nu} + \frac{p_\mu p_\nu}{m^2},$$

we can redefine the one-vector-meson space as $M_m = L^2(\tilde{P}dw^m_+)$ not specifying any gauge.

Therefore in the Euclidean formulation it seems natural to introduce for $m\geq 0$ the Hilbert spaces

(10) $$N_m = L^2\left(\frac{\tilde{E}\,d^4p}{p^2+m^2}\right), \qquad \tilde{E}^{\mu\nu}(p) = \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2},$$

where $p^2 = \underline{p}^2 + (p^o)^2 \geq 0$ and for $p \neq_e 0$ $E(p)$ are the projections onto the Euclidean transverse planes $T_p = \{h\in C^4:\ p.h=0\}$. Passing to spherical coordinates $p=(\rho,\underline{\omega})\in R^4$, $\rho=/p/$, the Euclidean counterpart of definition (3) is $N_m(\rho) = \int^{\oplus}_{S_\rho} \overset{e}{T}_p\, d\underline{\omega}$, where $S_\rho = \{p:\ /p/=\rho\}$. However

(11) $$N_m = \pi^2/_2 \int_0^{\infty}{}^{\oplus}\, N_m(\rho)\,\frac{\rho^3 d\rho}{\rho^2+m^2},$$

i.e. N_m is reducible with respect to the Euclidean group.

We write shortly N for N_m, $m\geq 0$. The natural conjugation in N is given by the map $h \rightarrow \tilde{h}^*(p) = \bar{\tilde{h}}(-p)$. We have a dense subspace S^T in N_m - independend of the mass m - of smooth transverse test functions h in R^4 which satisfy point-wise $p.\tilde{h}(p)=0$ and $\tilde{h}(0)=0$. The set

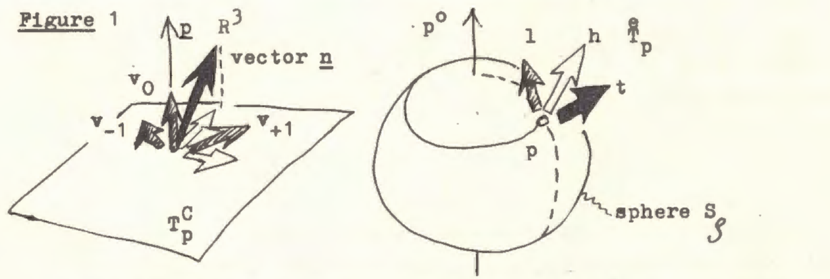
(12) $$N^C = \{h\in N:\ h^o=0\}$$

is a closed subspace in N and it defines a projection $\overset{e}{C}=C$. Let be $N(s) = L^2(R^4, \frac{d^4p}{p^2+m^2})$ Nelsons one-particle Hilbert space for the free Eucl. scalar field of mass $m\geq 0$. Then

(13) $$J^O_+:\ \tilde{u} \rightarrow (1, -pp^o/_{k^2})\frac{k\tilde{u}}{ip^o+k}, \quad k = +/\underline{p}/,$$

is a unitary map from $N(s)$ onto $M^O = \ker \overset{e}{C}$.

Figure 1



In the next section we pass to the description of the one-particle states in the coordinate representation, by the formal Fourier transform in R^n

(14) $$h^{\mu}(x) = (2\pi)^{-n/2}\int_{R^n} e^{ip.x}\,\tilde{h}^{\mu}(p)\,d^np.$$

Using the correspondence $w \rightarrow (-\Delta + m^2)^{1/2}$ in R^3, we can identify $M(s) = L^2(R^3, \frac{d^3p}{2w})$ with the dual of the Sobolev space $H_{+1/2}(R^3)$ and we denote the norm by $//.//_{-1/2}$. Similar $p^2 \rightarrow -\Delta$ in space-time R^4. We write $//.//_{-1}$ for the norm in $N(s)$, even for $m=0$.

2. Localization, Euclidean and dilatation covariance

By Fourier transform we identify the Euclidean one-particle spaces N_m, $m \geq 0$, for the free Euclidean vector meson fields of masses m with the Hilbert spaces of tempered vector-valued distributions

$$(15) \qquad \sum_{\mu}^{\oplus} S(R^4)' \cap \{h=(h^0,\underline{h}):\ \nabla . h=0,\ \ \|h^\mu\|_{-1}^2 = \int_{R^4} |\tilde{h}^\mu(p)|^2 \frac{d^4p}{p^2+m^2} < \infty \}$$

We introduce a localization in N - different from the usual one - as follows: For given closed subsets Λ in R^4 we let

$$(16) \qquad N_\Lambda = N \cap \sum_{\mu}^{\oplus} S(\Lambda)' = \{h \in N:\ \mathrm{supp}\ h^\mu \subset \Lambda\}.$$

Those subspaces N_Λ are closed in N and we associate with the above localization the family of projections $\{E_\Lambda:\ \Lambda \text{ closed}\}$. If N_Λ then also $N_{S=\partial\Lambda}$ is non-trivial, where $\partial\Lambda$ denotes the boundary of Λ. We have $N = \bigcup_{\text{all } S} N_S$, however, in contrast to the scalar case

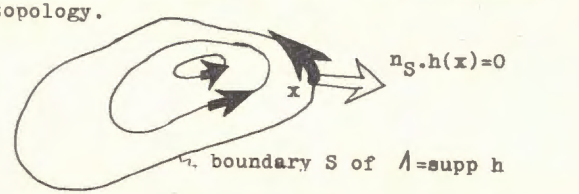
$$(17) \qquad \bigcup_{t \in R} N_{S_t} = N^C, \quad S_t = \{x=(x^0,\underline{x}):\ x^0=t\}$$

is a propper subspace. More general, by Gauß' law the elements h in N_Λ should have vanishing normal component $h_{n_S}=0$ with respect to the boundary surface $S=\partial\Lambda$. N is localizable in the sense that the set of smooth transverse test functions

$$(18) \qquad \{h = \Delta E n_\mu u:\ u \in S(\mathcal{O}),\ \mathcal{O} \text{ open}\}$$

is just S^T which is dense in N. Fixing $\mathcal{O}$ with closure Λ, the Hilbert space completion $N(\mathcal{O})$ in N coincides with N_Λ. Note that the operator ΔE in (18) is local and continuous in the Schwartz topology.

Figure 2



The localization of particle states in quantum field theory is still problematic. The Wightman approach uses a suitable family of projection-valued measures on the σ-algebra of Borel sets in R^3 and their eigenstates in the corresponding one-particle Hilbert space for a free field. In this sense vector mesons are localizable.

However, the photon position operator $\underline{Q}^C$ in M^C_{ph} has non-commuting components and Fleming argued maximal localizability of photons to be on hyperplanes. But contrary to that conjecture, Jadczyk and Jancewicz found generalized helicity states of photons on a curve $\mathcal{C}$ in R^3 $\underline{h}^{\mathcal{C}}_\lambda$, $\lambda = \pm 1$, for $\mathcal{C}$ a stright line and a circle. Due to (2) they are

$$(19) \qquad \underline{h}^{\mathcal{C}}_{\pm 1} = (\Lambda \pm 1)\Lambda U \underline{t}^{\mathcal{C}},$$

$$\underline{Q}^C = U C \underline{x} C U^{-1} = i\nabla + i\underline{p} \times \underline{\Sigma}/k^2, \quad (\Sigma_\alpha)^{\mu\nu} = \varepsilon^{\alpha\mu\nu}$$

where $\underline{t}^{\mathcal{C}} \in \sum^{3\,\oplus} S(\mathcal{C})'$ is a normalized tangential vector field on $\mathcal{C}$ and $U:\ M^C_{ph} \to L^2(\mathcal{C}d^3x)$ a unitary mapping.

In the next section we prove that for $m \geq 0$

$$(20) \qquad J_0 \tilde{\underline{h}} = (0, \delta_0 \otimes \underline{h})$$

defines an isometric embedding of M^C in N with ran $J_0 = N_{S_0}$, hence we can identify the photon states described above with the elements $J_0 \underline{h}^{\mathcal{C}}_\lambda \in N$. It would be nice to derive them from the localization in the Euclidean model.

Next we describe some covariance properties of the one-particle spaces. For $m > 0$ the Hilbert space M_m carries an irreducible unitary representation $g \to U_g$, $g=(a,L) \in iL^\uparrow_+$, of the inhomogeneous restricted Lorentz group defined formally by

$$(21) \qquad (U_g \tilde{h})^\mu(p) = e^{iap} L^{\mu\nu} \tilde{h}^\nu(L^{-1}p), \quad p \in V^m_+ .$$

In particular $U_t = e^{-itH_0}$ denotes the one-parameter group of time evolution with the flat Hamiltonian $H_0 = +\sqrt{\underline{p}^2+m^2}$. Time and space reversal are excluded from $L^\uparrow_+$. Let be θ_μ, $\mu=0,1,2,3$ reflections at the hyperplanes $S_\mu = \{x \in R^4:\ x^\mu=0\}$.

The helicity operator Λ is a Casimir for $m=0$. Since Λ anti-commutes with space reflection $U_{\underline{\theta}}$ the one-photon space M^C_{ph} is irreducible for the representation

(22) $$g \to CU_gC^{-1}, \quad g \in iL_+U\underline{\theta}$$

$$U_{\underline{\theta}}: M^{+1} \to M^{-1},$$

where C^{-1} denotes the inverse of the unitary map from the Hilbert space M_{ph} of one-photon states in the Lorentz gauge to M^C_{ph}. The full covariance group in M^C_{ph} is the conformal group.

For $m \geq 0$ the Euclidean one-particle space N_m carries a reducible unitary representation $g \to T_g$, $g=(a,R) \in iO(4)$, of the inhomogeneous Euclidean group defined formally by

(23) $$(T_g h)^{\mu}(x) = R^{\mu\nu} h^{\nu}(R^{-1}(x-a)),$$

and T_{θ_μ} are also well defined unitary self-adjoint operators in N. In particular for reflections at S_o

(24) $$(T_{\theta_o} h)^{\mu}(x) = g^{\mu\nu} h^{\nu}(-x^o, \underline{x})$$

we write shortly θ_o. The localization in N is related with covariance by

(25) $$T_g E_{\Lambda} T_g^{-1} = E_{\Lambda_g}, \quad g \in iO(4) \cup \{\theta_\mu\}.$$

Let $E_{\pm 0}$ denote the projections in N onto the subspaces spanned by elements with support in the half-spaces

(26) $$\Lambda_+ = \{x \in R^4: x^o \geq 0\}$$

Λ_- and on $S_o = \Lambda_+ \cap \Lambda_-$, respectively. Then $\theta_o E_+ \theta_o = E_-$, however, the reflexivity $\theta_o E_o = E_o$ is a non-trivial property. Note that the intersection of the relativistic and Euclidean groups $,O(3) = L_+^{\uparrow} \cap O(4)$, consists exactly of all transformations which leave the hyperplane $S_o \cong R^3$ invariant.

For $m=0$ the full covariance group in N is the Euclidean conformal group. We discuss dilatations $x \to \lambda x$, $0 < \lambda < \infty$, inversions $\theta_I: x \to \frac{-x}{/x/^2}$ and reflections at the spheres $S_R = \{x \in R^4: /x/=R\}$

(27) $$\theta_R: x \to \left[\frac{R}{/x/}\right]^2 x.$$

For space-time R^n, $n \geq 3$, on the domain $\sum^{n}\oplus C_o^{\infty}(R^n \setminus 0)$ we introduce the diagonal operators

(28) $$T_\lambda h(x) = \lambda^{\frac{n+2}{2}} h(\lambda x), \quad \theta_I h(x) = /x/^{-(n+2)} h(\theta_I x)$$

and $\theta_R = \theta T_\lambda \theta_I T_\lambda^{-1} \theta$, $\lambda = R$, where $\theta: x \to -x$. Note that

(29) $$\theta_I T_\lambda \theta_I = T_\lambda -1.$$

The Fourier transform gives $T_\lambda \tilde{h}(p) = \lambda^{1-n/2} \tilde{h}(\lambda^{-1}p)$, hence the operators of dilatations conserve the transversality $p.\tilde{h}(p)=0$ and build a multiplicative unitary group in N, $m=0$, while for $m>0$ they rescale the mass parameter, too.

Since θ_I does not commute with the projection E, we shall study these operators first in $N' = \sum^{n}\oplus N(s)$ with the scalar product given by the kernel

(30) $$\frac{\delta^{\mu\nu}}{c_n /x-y/^{n-2}},$$

where c_n is the area of the unit sphere in R^n.

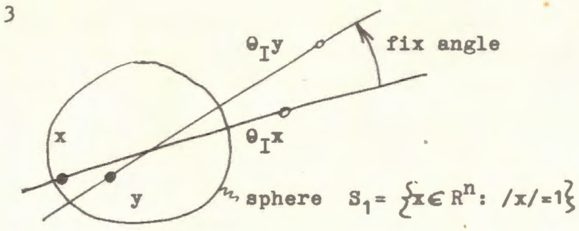
Lemma 1 θ_I is a unitary self-adjoint operator in N'.

Proof: On the dense set $\{h = n^{\mu} e^{/x/^2} D(x) e^{-2/x/^2}\}$, where D is any differential operator with constant coefficients in R^n, θ_I is well defined and symmetric. The fact that θ_I is isometric depends on $d^n(\theta_I x) = (-1)^{n+1} /x/^{-2n} d^n x$ and

(31) $$/\theta_I x - \theta_I y/ = \frac{/x-y/}{/x/./y/} \quad \text{for } x \neq y.$$

Q.E.D.

__Figure__ 3



Of course for $a \in R^n$, $\Theta_I T_a \Theta_I$ are unitary operators in N', m=0, of special Euclidean __conformal transformations__. Since dilatations leave the hyperplane S_o invariant

(32) $\quad U^C_\lambda = J_o^* T_\lambda J_o$

are unitary operators in M^C_{ph}. In the scalar case, let us denote by $j_o: \tilde{u} \to \delta_o \otimes u$ the isometric embedding of $L^2(d^3\underline{p}/k)$ in N(s) with ran $j_o = N_{S_o}(s)$. Then in coordinate representation and for space-time of dimension $n=d+1 \geq 3$, we find

(33) $\quad j_o^* \Theta_I j_o: u \to /\underline{x}/^{-(d+1)} u(\frac{-\underline{x}}{/\underline{x}/^2})$.

This result coincides with the unitary representation of inversion on the one-particle Hilbert space of the __free scalar massless field__ given by Swieca and Völkel.

Note that Θ_I and Θ_R are well defined in the subspace $\bigcup_{R>0} N_{S_R}$ of N spanned by elements with vanishing radial component. In particular we have $\Theta_R E_{S_{R_1}} \Theta_R = E_{S_{R_2}}$, $R_1 . R_2 = R^2$, and we have the __inversion property__

__Lemma 2__

(34) $\quad \Theta_R E_{S_R} = E_{S_R}$, for $m \geq 0$.

Proof: For $h \in N_{S_R}$ the transversality $\nabla . h=0$ implies just $x.h=0$, hence passing to spherical coordinates $x=(r,\underline{\omega}) \in R^n$, $h = J_R \underline{t} = (0, \delta_R \otimes \underline{t})$.

Q.E.D.

3. __Reflection positivity and Hamiltonian semi-groups__

First we remark that the localizations of the transverse Euclidean one-particle states in Λ_+, S_o and in some other closed subsets of R^4 we can describe in momentum representation, and we need not refer to coordinate representation at all.

Using a version of the __Paley-Wiener-Schwartz__ theorem, we find that the components $\tilde{h}^\mu(.,\underline{p})$ for $h \in N_+$, $m>0$, have analytic continuations to the complex half-plane $C_+ = \{p^o: \text{Im } p^o > 0\}$ a.e. with respect to the Lebesque measure d^3p. More precisely

(35) $\quad \tilde{h}^\mu(.,\underline{p}) \in HL^2(\frac{dp^o}{(p^o)^2+w^3})$, $w = +\sqrt{\underline{p}^2+m^2}$,

in particular $\check{h}^\mu \in L^2(d^3p/_{2w})$, where $\check{h}^\mu(\underline{p}) = \tilde{h}^\mu(iw(\underline{p}),\underline{p})$ denotes the restriction of the Laplace transform of $h^\mu(x)$ to V^m_+. A similar result we get for m=0.

Then by continuity arguments, for $h \in N_{S_o} = N_+ \cap N_-$

(36) $\quad \tilde{h}^\mu(p) = \sum_{c^\mu=0,1,2...}^{\text{finite}} (ip^o)^{c^\mu} \tilde{f}^\mu(\underline{p})$,

which in coordinate representation expresses the fact that $h^\mu(.,\underline{x})$ is a finite derivative of $\delta_0(x^o)$.

From $//\tilde{h}^\mu// < \infty$ follows $c^\mu=0$, $\mu=0,1,2,3$. From the transversality[1] $p.\tilde{h}(p)=0$ a.e. we derive $h^o=0$. Then we get $//\tilde{h}^\beta// = //\tilde{f}^\beta//_{-1/2}$, $\beta=1,2,3$ and $\underline{p}.\underline{\tilde{f}}=0$. Therefore, in momentum[1] representation, any element $\tilde{h} \in N_{S_o}$ has the form $\tilde{h} = J_o \underline{\tilde{f}} = (0, 1 \otimes \underline{\tilde{f}})$, $f \in M^C$.

__Lemma 3__ $\quad E_+ \Theta_o E_+ \geq 0$.

Proof: For real $h \in N_+$ by the __Cauchy integral formula__

(37) $$\langle h, \Theta_o h \rangle_N = \int d^3p \int \frac{dp^o}{p^o+iw} \left[\frac{g^{\mu\nu} \tilde{h}^\mu(p) \tilde{h}^\nu(\underline{\theta}p)}{p^o+iw}\right]$$
$$= \pi \int d^3p/_w \, (-/\check{h}^o/^2 + /\underline{\check{h}}/^2).$$

The property $\nabla . h=0$ implies

$$(38) \qquad w|\check{h}^{o}| = |\underline{p}.\check{\underline{h}}| \leq w|\check{\underline{h}}| \quad \text{a.e.,}$$

and since Θ_o is symmetric in the real part N_r, (37) generalizes to complex h as well. With the decomposition $h = \underline{t} \oplus 1$ we can rewrite the nominator as

$$(39) \qquad \underline{\tilde{t}}.\underline{\tilde{t}}^{\Theta} - |\hat{v}|^{2} \tilde{1}^{o}(\tilde{1}^{o})^{\Theta}, \quad v = (1, -\underline{p}p^{o}/_{k}2),$$

hence

$$(40) \quad \langle h, \Theta_o h\rangle_N = \sum_{\beta=1,2,3} \| \check{t}^{\beta} \|^{2}_{-1/2} + m^{2} \| k^{-1} \check{1}^{o} \|^{2}_{-1/2} . \qquad \text{Q.E.D.}$$

From the formula (40) we expect $E_+\Theta_o E_+ \longrightarrow E_{S_o}$ as $m \downarrow 0$. By an argument of Hegerfeldt this may happen if and only if $E_+E_- = E_{S_o}$, m=0. Conversely, for $m>0$ the Markoff property with respect to the hyperplane S_o is valid in the subspace N^C only. Note that the discussion on page 12 explains the reflexivity $\Theta_o E_{S_o} = E_{S_o}$, $m \geq 0$.

Moreover, $E_{S_o} = Ce_{S_o}$ since C and e_{S_o} commute in N' and

$$(41) \qquad N_{S_o} = \{ h \in N' : h^{o}=0, \ \underline{\nabla}.\underline{h}=0 \text{ and supp } h \subset S_o \} .$$

<u>Lemma 4</u> The localization $\Lambda \rightarrow N_{\Lambda}$ defined by formula (16) satisfies the <u>Markoff property</u>, <u>exactly for m=0</u>.

Proof: We shall prove the Markoff property with respect to S_o, however, our proof generalizes to any smooth hypersurface. Let be φ a vector test function with components $\varphi^{\mu} \in S(\Lambda_+^{int})$. Then $E\varphi \in N$, and for given $h \in N_-$

$$(42) \qquad (E_+h, \varphi) = \langle E_+h, (-\Delta+m^2)E\varphi\rangle_N$$

vanishes exactly for m=0, since then $(-\Delta+m^2)E\varphi$ has support in Λ_+^{int}, too. Therefore, $E_+E_- = E_-(E_+E_-) = E_{S_o}$, m=0.

Q.E.D

In the momentum representation the Markoff property for m=0 results from the fact that $p^2 \tilde{E}^{\mu\nu}(p)$ is a <u>polynomial</u>.

Naturally there arises the question of a possible orthogonal local decomposition in N with respect to a given sufficiently smooth hypersurface. For the hyperplane S_o - and similar for any other by Euclidean covariance - let us regard the subspaces $N_> = N_+ \ominus N_{S_o}$ and $N_<$ with the corresponding projection operators in N, $E_{\gtrless}$, where $E_< = \Theta_o E_> \Theta_o$.

For m=0 the problem is solved by the identity

$$(43) \qquad E_> \oplus E_< \oplus E_{S_o} = 1_N,$$

which results from the Markoff property. However, for $m>0$ (40) implies that $\langle 1, \Theta_o 1\rangle_N \neq 0$ for some $1 \in N_+^{0}$. Hence in this case $N_>$ and $N_<$ are not orthogonal to each other and there is no such kind of <u>partition of unity</u>.

Explicitely, using that the Fourier transform of $\dfrac{|\tilde{v}(p)|^2}{p^2+m^2}$ for m=0 is just $\dfrac{\delta(x^0)}{4\pi|\underline{x}|}$

$$(44) \quad \langle 1, \Theta_o 1\rangle_N = - \iint_{R^3 \times R^3} \frac{d^3x \, d^3y}{|\underline{x}-\underline{y}|} \int_{-\infty}^{+\infty} 1^{o}(s,\underline{x}) 1^{o}(-s,\underline{y}) ds = 0.$$

Having reflection positivity in the Euclidean one-particle space N we can start the Osterwalder Schrader <u>reconstruction</u>. First we regard the "<u>physical subspaces</u>" in N which we shall denote by N_{M_m} and $N_{M_{ph}}C$, respectively.

<u>Lemma 5</u> Let us define for $m \geq 0$ $N_M = N_+ \ominus \{h : E_-h=0\}$. Then $N_{M_{ph}}C = N_{S_o}$ and using $J_+^{0}\restriction N_{S_o}(s) =: J_{S_o}^{0}$

$$(45) \qquad N_{N_m} = N_{S_o} \oplus \text{ran } J_{S_o}^{0}, \text{ for } m>0.$$

Proof: The Markoff property in N^C implies that $\{ t \in N_+^C : E_-t=0 \} = N_>^C$, hence $N_M C = N_{S_o}$ for $m \geq 0$.

Then ran $J^0_+\restriction_{N_+(s)} = N_+ \cap \ker C$ gives for $1 = J^0_+ u$

(46) $$E_+\theta_o E_+ \tilde{1} = m^2 J^0_+ \frac{\check{u}}{(w+/\underline{p}/)^2}, \quad \text{supp } u \subset \Lambda_+ .$$

For $m>0$ (46) vanishes exactly if $\tilde{u}(iw,\underline{p})=0$ a.e., and the restriction of the operator $(E_+\theta_o E_+)^{1/2}$ to $N^0_{M_m}$ corresponds to the map $\tilde{u} \longrightarrow \frac{m.\tilde{u}}{w+/\underline{p}/}$ in $L^2(d^3p/2w)$. Performing the Fourier transformation in the <u>Euclidean time variable</u> we find for $1 \in N^0_{M_m}$

(47) $$1(x^o,\underline{p}) = (1,-i\underline{p}\nabla^o/_k2)\,\theta(x^o)e^{-/\underline{p}/x^o}/\underline{p}/\tilde{u}(\underline{p}). \quad \text{Q.E.D.}$$

The lemma 5 allows an <u>isometric embedding</u> of the relativistic one-particle space M into the Euclidean. However, it is more convenient to regard the <u>canonical map</u> $W: N_+ \to M$ defined by the relation

(48) $$\langle h,\theta_o h\rangle_N = //Wh//^2_M .$$

<u>Lemma 6</u>

(49) $$W\tilde{h} = (-i\check{h}^o,\check{\underline{h}}), \text{ for } m>0.$$

Proof: Formally, W is analytic continuation from T^e_p to T_p, $p \in V^m_+$ and $\underline{p}$ fix. Using

(50) $$W: (1,-\underline{p}p^o/_k2)\frac{ik}{p^o+ik} \longrightarrow (1,\underline{p}w/_k2)\frac{-ik}{w+k}$$

we see that $\langle 1,\theta_o 1\rangle_N = m^2 //k^{-1}\check{1}^o//^2$. Moreover, W correctly changes the metric, $^{-1/2}$ transversality and conjugation from relativistic to Euclidean.

Alternatively, W will define a map to M_{ph} in the Lorentz gauge and for $m>0$ to the factor space $L^2(P^{\mu\nu}dv^m_+)$, starting from the <u>factor space</u>

(51) $$N_+/\{h: E_-h=0\} . \qquad \text{Q.E.D.}$$

In the section 2 we remarked that those Euclidean rotations, translations and reflections which leave the hyperplane S_o invariant - here shortly denoted by $\underline{g}=(\underline{a},\underline{R},\underline{\theta})$ - - <u>induce unitary representations</u> $\underline{g} \to U_{\underline{g}}$ in M_m. Using the canonical map W we get

(52) $$U_{\underline{g}}W = WT_{\underline{g}} \text{ in } N_+ .$$

<u>Lemma 7</u> For Euclidean <u>time translations</u> $g_t: S_o \to S_t$, the relation (52) generalizes to $U_s W = WT_t$, for $t=is$ with Im $s < 0$. Hence for $h,h' \in N_+$

(53) $$\langle h',\theta_o T_t h\rangle_{N_m} = (Wh', e^{-tH_o} Wh)_{M_m}, \quad t \geq 0.$$

For m=0

(54) $$U^C_{\underline{g},\lambda} = J^*_o T_{\underline{g},\lambda} J_o ,$$

$$e^{-/t/H^C_o} = J^*_o T_t J_o \qquad \text{(free <u>Feynman-Kac-Nelson</u> formula),}$$

in M^C_{ph}. Moreover, the <u>dilatations</u> T_λ generate a continuous multiplicative semi-group of self-adjoint contractions in the subspace $N_{S_R} = \{h \in N: \nabla.h=0 \text{ and supp } h^\mu \subset S_R\}$

(55) $$D_\lambda = E_{S_R}T_\lambda, \quad \lambda \geq 1.$$

Proof: The first part of the lemma simply generalizes Nelsons reconstruction of the free scalar Hamiltonian to the vector model. Since T_t and e_{S_o} commute with the projection C and $E^C_+\theta_+E^C = E_{S_o} = Ce_{S_o}$, (53) follows from

(56) $$E_+\theta_o E_+ T_t\tilde{1} = \frac{m^2}{(w+/\underline{p}/)^2} J^0_{S_o}(T_t u)^{\vee},$$

for $1 = J^0_S u$, $\tilde{u} \in L^2(d^3p/2w)$.

The fact that $\{D_\lambda: \lambda \geq 1\}$ - and similar for $\lambda \leq 1$ - defines a self-adjoint semi-group in N_{S_R}, m=0, follows from the Markoff and inversion properties. Q.E.D.

In a forthcoming publication we will prove that in the parametrization $\lambda = e^{ct/R}$, $t \geq 0$, the semi-group can be expressed in the form

$$(57) \qquad D_t = e^{-\frac{tH_R}{\hbar}}, \quad H_R \geq \frac{2\hbar c}{R}.$$

Summary: In part I of our paper we have shown that the notion of transverse gauge is a good concept for the description of one-particle states of the free vector meson quantum field of mass $m > 0$ and the photon field, respectively. We derived the Hamiltonian semi-groups from a corresponding Euclidean model.

In the parts II and III we shall discuss the main properties of Euclidean transverse random fields.(JINR-preprints E5-I2779,I278I)

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