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EUCLIDEAN FIELDS:
QUANTUM AND RANDOM FIELDS

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Евклидовые поля: квантованные и случайные поля

Рассматривается свободное поперечное векторное поле частиц массы $m \geq 0$. Модель относится к обычной теории квантованных полей свободного векторного мезонного поля и свободного электромагнитного поля посредством расширения от поперечных к произвольным пробным функциям. Мы вводим квантованные поля в калибровке Штюккельберга и поля $F_{\mu\nu}$ соответственно и доказываем возможность перехода к поперечным потенциалам. Мы относим евклидовые однофотонные состояния к статическим потокам в классической магнитостатике и даем альтернативное доказательство для свойства физической положительности, используя аппроксимацию потоков магнитными диполями. Конструируется поперечное самодействие.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Euclidean Fields: Quantum and Random Fields

We study free transverse vector fields of mass $m \geq 0$. The model is related to the usual free vector meson and electromagnetic quantum field theories by extension of the field operators from transverse to arbitrary test functions. We introduce quantum fields in Stückelberg gauge and in terms of $F_{\mu\nu}$, respectively, and show how to pass to the transverse potentials. We relate the one photon to steady currents of classical magnetostatics and give an alternative proof of reflexion positivity. We discuss transverse self-interactions with spatial cut-off.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

Introduction

Nelson proposed a reconstruction of Bose quantum fields from Euclidean Markoff fields via the following idea: Given a random field ϕ with underlying probability space (Q,B,β) one introduces a localization by the σ-subalgebras $B_{\mathcal{O}} \subseteq B$, $\mathcal{O}$ open in R^{d+1}, which are generated by all $\phi(\varphi)$, $\varphi \in S(\mathcal{O})$.

In particular, let $\Lambda_+ = \{x=(x^0,\underline{x}): x^0 \geq 0\}$ and S_o the hyperplane $x^0=0$. If ϕ is a Markoff field and it satisfies reflexivity with respect to the above localization, then $L^2(Q,B_{S_o},\beta)$ is identified with the physical Hilbert space H for the related Q.F.T. in Minkowski space-time.

Our paper gives two generalizations: The first is due to Hegerfeldt and will be applied to the free Euclidean transversal vector field of mass $m>0$. Using reflection positivity we find a canonical map

$$\Gamma(W): \quad L^2(Q,B_{\Lambda_+},\beta^m) \longrightarrow H^m$$

to the Fock space of non-interacting vector mesons. For the Euclidean photon field in terms of transverse potentials $\overset{e}{A}$ which is the formal limit $m \downarrow 0$, we recover Nelsons procedure. However, by the dilatation invariance of the measure $d\beta^m$, $m=0$, we can reconstruct a model of Q.F.T. in curved space-time in $L^2(Q,B_{S_R},\beta^0)$, where $S_R = \{x: |x|=R\}$ is a sphere in R^{d+1}. We shall exploit this idea in a forthcoming paper.

1. Vector meson field in the Stückelberg gauge and the photon field in terms of $P^{\mu\nu}$

The starting point for the construction of the one-particle Hilbert space M_m for the free vector meson quantum field of mass $m>0$ is the two-point Wightman function

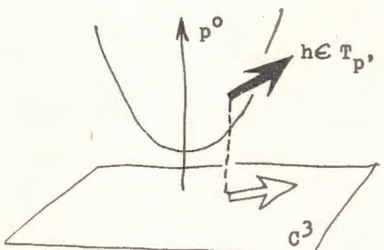
(1) $\langle A^\mu(x)A^\nu(y)\rangle_o = (g^{\mu\nu}-\nabla_\mu\nabla_\nu/m^2)1/i\ D_+^m(x-y)$,

which is the Fourier transform of the measure $\tilde{P}^{\mu\nu}dw_+^m$ with support on T_p, $p\in V_+^m$. We observe that in the sense of a positive-definite matrix

(2) $J^{*\nu}PJ\restriction_{V_+^m} > 0$,

where $J\underline{h} = (0,\underline{h})$ denotes the embedding $J: C^3 \to C^4$. The projection $JJ^*: T_p \to C^3$ - which is different from C - is non-zero for all $p\in V_+^m$.

Figure 1



Hence elements $h\in M_m \cong L^2(\tilde{P}^{\mu\nu}dw_+^m)$ are completely given by their spatial components $\underline{h}$. Conversely, there are canonical mappings to the Stückelberg and to the transverse gauges

(3) $J^T\underline{h} = (h^o,\underline{h})$ and $J^S\underline{h} = PJ\underline{h}$.

Then

(4) $\underline{Y} = J^{*\nu}PJ = \underline{C}\oplus\left(\frac{w}{m}\right)^2\underline{C}^\perp$, $\underline{Y}^{\alpha\beta} = \delta^{\alpha\beta}+p^\alpha p^\beta/m^2$ for $\alpha,\beta=1,2,3$

is an unbounded self-adjoint operator in $\sum^{3\oplus} L^2(d^3p/2w)$.

(5) $\|J^{S/T}\underline{h}\|_{M_m^{S/T}} = \|\underline{Y}^{\pm1/2}\underline{h}\|_{\sum^{3\oplus} L^2(d^3p/2w)}$,

hence the Hilbert spaces $M_m^{S/T}$ are dual to each other as remarked by Gross. The reconstruction of the Euclidean Green's functions via analytic continuation of (1) by the procedure of Osterwalder and Schrader gives

(6) $S^{2,\mu\nu}(x,y) = (\delta^{\mu\nu}-\nabla^\mu\nabla^\nu/m^2)\ G^m(x-y)$,

where G^m is the Green's function for $-\Delta+m^2$ with free boundary conditions. This result was found independently by Gross, Yao, Velo and Ginibre.

Passing to the momentum representation (6) defines a Hilbert space $N^S = L^2(\frac{Y\,dp}{p^2+m^2})$, where

(7) $Y^{\mu\nu}(p) = \delta^{\mu\nu}+p^\mu p^\nu/m^2$

is uniquely defined by $J^*YJ = \underline{Y}$ and Euclidean covariance.

Lemma 1 For $m>0$, there is an isometric embedding of N_m into N^S. Then

(8) $\langle h,\Theta_o h\rangle_{N_m} = \|E_o^S f\|^2_{N^S}$, $f = h\oplus g \in N_+^S$.

Proof: It was argued by Karwowski that for any model showing reflection positivity, the operator $E_+\Theta_oE_+ > 0$ extends to a projection with the same structure in some larger Hilbert space. For the model of the free Euclidean vector meson field this is true, and we have to pass from the transverse to the Stückelberg gauge, to recover the Markoff property.

By the ultra-locality in $N^S\ominus N$, for real $g=\nabla u$ with supp $u\subset\Lambda_+$

(9) $\langle g,\Theta_o g\rangle_{N^S} = g^{\mu\nu}/m^2(\nabla^\mu u,\nabla^\nu\Theta_o u)_{L^2(d^4x)} = 0$.

Yao proved that the free Euclidean vector meson field in the Stückelberg gauge $B^{\mu}(x)$ with two-point function (6) satisfies reflexivity and the Markoff property. With the remark that in N^S the reflection operator Θ_o commutes with the projection $E: N^S \to N_m$, the proof is complete. Q.E.D.

The projection E_o^S can be expressed in the form $J_o^S J_o^{S*} = C^S e_{S_o}$, where $J_o^S: M_m^S \to N^S$ and C^S denotes the symmetrization of C in the scalar product of N^S, explicitely

$$(10) \qquad C^S \tilde{f} = (0, \underline{\tilde{f}} + \underline{p}\,\frac{p^o \tilde{f}^o}{w^2}), \quad w = \sqrt{\underline{p}^2 + m^2}\ .$$

Note that for $m \geq 0$ the two-point function $E^{\mu\nu} G^m(x-y)$ - which is different from $S^{2,\mu\nu}(x,y)$ - does not satisfy the <u>Osterwalder-Schrader-positivity</u>, however, for transverse test functions the non-local projection cancels and by the reflection positivity in N_m in the sense of a vector-valued positive-definite function

$$(11) \qquad g^{\mu\nu} G^m(x - \Theta_o y) \geq 0 \quad \text{in} \quad S^T(\Lambda_+).$$

The free electromagnetic field - we say shortly "<u>photon field</u>" - in terms of $F^{\mu\nu}$ is given by the two-point Wightman function

$$(12) \qquad \langle F^{\alpha\beta}(x) F^{\mu\nu}(y) \rangle_o = (g^{\alpha\mu} \nabla^{\beta} \nabla^{\nu} + \ldots)\, 1/i\, D_+^o(x-y),$$

where $+\ldots$ contains the other permutations of the indices $\alpha, \beta, \mu, \nu = 0,1,2,3$ so that the resulting expression is anti-symmetric for exchanges $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$. The corresponding one-photon Hilbert space in the momentum representation is

$$(13) \qquad M_{ph}^F = L^2(P_{ph} dw_+^o)\ ,$$

where $P_{ph}^{\alpha\beta/\mu\nu}(p)$, $p \in V_+^o$, denotes the Fourier transform of that expression. By the map

$$(14) \qquad (If)^{\mu} = i p_{\nu} (f^{\mu\nu} - f^{\nu\mu})$$

we can pass to tne Lorentz gauge, and by C to Coulomb gauge.

The relations $\bar{f}.P_{ph} f = /If/^2$ and $\operatorname{ran} I(p) = T_p$, $p \in V_+^o$, show that I is a unitary map, and we arrive at the following commutative <u>diagram</u>:

$$(15) \qquad \begin{array}{ccc} & & M_{ph} \\ & \overset{I}{\nearrow} & \\ M_{ph}^F & & \downarrow C \\ & \underset{CI}{\searrow} & \\ & & M_{ph}^C \end{array} \quad \text{(Coulomb gauge).}$$

Via analytic continuation of (12) to the <u>Euclidean region</u> we find the two-point function

$$(16) \qquad \langle \overset{e}{F}{}^{\alpha\beta}(x) \overset{e}{F}{}^{\mu\nu}(y) \rangle_o = (\delta^{\alpha\mu} \nabla^{\beta} \nabla^{\nu} + \ldots)\ \frac{1}{c_n /x-y/^{n-2}}\ ,$$

where $\overset{e}{F}{}^{\mu\nu}(x)$ is the free Euclidean electromagnetic field in space-time of dimension $n \geq 3$, studied first by Yao. c_n is the area of the unit sphere in R^n, so that the expression outside the brackets is just the normalized Green's function for the <u>Laplacean</u> with free boundary conditions.

We denote by $N^F = L^2(\overset{e}{P}_{ph} /p/^{-2} d^n p)$ the corresponding one-particle Hilbert space, where $\overset{e}{P}{}_{ph}^{\alpha\beta/\mu\nu}(p)$, $p \in R^n$, is the Fourier transform of the antisymmetric expression inside the brackets of the r.h.s. of (16).

<u>Lemma 2</u> The Euclideanization $\overset{e}{I}$ of the map I defined in formula (14) is a <u>local unitary</u> map from N^F onto N, m=0.

Proof: The relation $\bar{f}.\overset{e}{P}_{ph} f = /\overset{e}{I} f/^2$ implies that $\overset{e}{I}$ is isometric from N^F to N'. Using $\overset{e}{I}{}^* \overset{e}{I} = E$ we get $\operatorname{ran} \overset{e}{I} = N$. The formal adjoint of $\overset{e}{I}$ is

$$(17) \qquad \overset{e}{I}{}^*: f \to -i/p/^{-2}(p^{\nu} f^{\mu} - p^{\mu} f^{\nu}), \quad f \in N'$$

and has no relativistic counterpart since $p^2 \restriction_{V^o} = 0$.
A proof for the locality of $\overset{e}{I}$ has been anounced[+] by Guerra, in particular for localization in $\Lambda_{\pm}$ and S_o it has been discussed by Yao.
We give here only an argument for the case n=3.

Of course, for any open region $\mathcal{O} \subset R^3$, $\{\overset{e}{I} f : f^{\mu\nu} \in S(\mathcal{O})\}$ is a dense set in $N(\mathcal{O})$. Conversely, any transverse $h \in N(\mathcal{O})$ is a limit of smooth curls localized in $\mathcal{O}$. Using the Hodge duality $* : f \to (*f)^{\mu\nu} = \varepsilon^{\mu\nu\beta} f^{\beta}$

$$(18) \qquad h = \nabla \times f = \overset{e}{I}(*f), \quad f^{\beta} \in S(\mathcal{O}). \qquad \text{Q.E.D.}$$

2. Second quantization

Let us introduce for $m \geq 0$ the Fock representation of the CCR (Γ, M_r, Ω), where Γ denotes the functor of second quantization, M_r stands for the real part of M_m and M_{ph} and Ω is the Fock vacuum. Let be $A^T(0,.)$ the associated free Hermitean field.

Lemma 3 For $m > 0$, the restriction of the free vector meson quantum field to transverse test functions coincides with the transverse potentials, i.e. for $h_t = \delta_t \otimes h$, $\tilde{h} \in S_r^T(V_+^m)$

$$(19) \qquad A(h_t) = e^{itH_o} A^T(0,\tilde{h}) e^{-itH_o}.$$

Proof: $\langle A(h_t) A(g_s) \rangle_o = g^{\mu\nu} \int_{R^3} \frac{d^3p}{2w} e^{iw(s-t)} \tilde{h}^{*\mu} \tilde{g}^{\nu}$. Q.E.D.

For $m=0$ we can pass from the transverse potentials in Coulomb gauge to the $F^{\mu\nu}$'s as follows: As operators in Fock space $\Gamma(M^C_{ph,r})$

$$(20) \qquad \Gamma(CI) F(f_t) \Gamma(CI)^{-1} = e^{itH_o^C} A^C(0, CI\tilde{f}) e^{-itH_o^C}, \quad f^{\alpha\beta} \in S_r(R^3),$$

where we have used $S^C(V_+^o) \cong S^T(R^3)$. The well known formula $F^{\mu\nu} = \nabla^{\mu} A^{T\nu} - \nabla^{\nu} A^{T\mu}$ relates the $F^{\mu\nu}$'s with the Lorentz gauge.

For the photon field , gauge invariance is equivalent to the following algebraic result. Let us define

$$(21) \qquad D_{ph} = \sum_{n \geq 0}^{fin \, \oplus} S^T(\overset{n}{X}\, V_+^o).$$

Lemma 4 D_{ph} can be equipped with the structure of a topological $*$-tensor algebra. The Coulomb projection C the extends to a continuous $*$-homomorphism $\Gamma(C)$ with the image ran $\Gamma(C) = D^C_{ph}$. The kernel L_{ph} is a $L_+^{\uparrow}$-invariant closed two-sided ideal, i.e.

$$(22) \qquad D^C_{ph} \cong D_{ph} / \ker \Gamma(C).$$

Proof: $\Gamma^{(0)} = 1$ and for $n \geq 1$, $\Gamma^{(n)}(C) = C \otimes \ldots \otimes C$, hence L_{ph} is algebraicaly spanned by $1 = (1^{(n)})_{n \geq o}$ with

$$(23) \qquad 1^{(0)} = 0,$$
$$1^{(n)} = ip_s^{\beta(s)} u^{\beta(1)} \ldots \hat{\beta}^{(s)} \ldots \beta^{(n)} \quad \text{for some } 1 \leq s \leq n. \quad \text{Q.E.D.}$$

Expanding $A^C(0,.)$ in the Fourier coefficients $a^{\lambda}(\underline{p})$, where $\lambda = \pm 1$ denotes the helicity index and $\underline{p} \in R^3$ the momentum variable, we find

$$(24) \qquad \langle a^{\lambda}(\underline{p})^* a^{\lambda'}(\underline{q}) \rangle_o = \delta^{\lambda\lambda'} \delta^3(\underline{p}-\underline{q}) / 2w, \quad w = +/\underline{p}/.$$

Hence the $a^{\lambda}(\underline{p})$'s commute and act irreducibely in the Fock space $\Gamma(M^C_{ph,r})$. In the Schrödinger representation we redefine them as coordinate functions on the probability space (Q_o, B_o, β_o), where $Q_o = \underset{\lambda,\underline{p}}{X} R$ and B_o is the σ-algebra of measurable sets w.r.t. $\lambda, \underline{p}$

$$(25) \qquad d\beta_o = \prod_{\lambda,\underline{p}} \frac{e^{-\frac{w}{2} q_o^{\lambda}(\underline{p})^2}}{\sqrt{2\pi/w}} \, dq_o^{\lambda}(\underline{p}).$$

In this picture the $a^{\lambda}(\underline{p})$ are Gaussian random variables with the covariances (24).

Now we are prepared to relate the time-zero potentials of the photon field in Coulomb gauge $A^C(0,.)$ with the free Euclidean electromagnetic field. Let be $\overset{e}{A}$ the real Gaussian random field indexed by S_r^T with mean zero and covariance

$$\int_Q \overset{e}{A}(h)\overset{e}{A}(g)\, dB = \langle h,g\rangle_N, \quad m=0, \tag{26}$$

and we denote by (Q,B,dB) the underlying probability space. The lemma 2 tells us that for any $\overset{e}{A}(h)$, $h\in S_r^T$, there is a unique $\overset{e}{F}(f)\in\Gamma(N_r^F)$, $\overset{e}{f}=h$. We call $\overset{e}{A}$ <u>transverse potentials</u> of the free Euclidean electromagnetic field.

As a consequence of lemma 4 of part I we have the following result:

<u>Lemma 5</u> The transverse potential field $\overset{e}{A}$ is a <u>Markoff field</u> and satisfies reflexivity, with respect to the local structure introduced in B via sub-σ-algebras $B_{\mathcal{O}}\subset B$, $\mathcal{O}\subset R^4$ open - generated by all $\overset{e}{A}(h)$ with $h\in S_r^T(\mathcal{O})$, that is h with $\nabla.h=0$ and supp $h\subset\mathcal{O}$.

To have a <u>full set of random variables</u> with respect to B, we regard the basis vectors on the Euclidean transverse planes $\overset{e}{T}_p$

$$\overset{e}{v}_{\pm1}\in T_{\underline{p}}^C \quad\text{and}\quad \overset{e}{v}_0=(1,-\underline{p}p^0/k^2)\frac{ik}{p^0+ik}. \tag{27}$$

Then

$$\langle \overset{e}{A}_\lambda(t,\underline{p})\overset{e}{A}_\varrho(s,\underline{q})\rangle_B=\delta_{\lambda\varrho}\,\delta^3(\underline{p}-\underline{q})\frac{e^{-w|s-t|}}{2w}, \quad \lambda,\varrho=0,\pm1 \tag{28}$$

where $\overset{e}{A}_\lambda(t,\underline{p})=\overset{e}{A}(T_t\overset{e}{v}_\lambda\tilde{\delta}_{\underline{p}})$. Comparing the covariances (24) with (28) we can identify the Fourier coefficients $a_{\pm1}(\underline{p})$ of the potentials $A^C(0,.)$ with $\overset{e}{A}_{\pm1}(0,\underline{p})$. Moreover,

$$Q^C=\mathop{\times}_{\substack{t,\underline{p}\\ \lambda=\pm1}} R \cong \mathop{\times}_{t} Q_0 \tag{29}$$

is just the <u>path space</u> over the photon quantum field theory in Coulomb gauge. For any $t\in R$, there is an isometric embedding $\Gamma(J_t)$ of the Fock space $\Gamma(M^C_{ph,r})$ in $L^2(dB)$, in particular for the one-photon states we get

$$\Gamma^{(1)}(J_0):\ A^C(0,\underline{h})\Omega \to \overset{e}{A}(J_0\underline{h}),\quad \underline{h}\in S_r^T(R^3). \tag{30}$$

We generalize the definition of a random field $\overset{e}{A}$ by (26) to the <u>massive case</u> $m>0$.

As above we can realize $\overset{e}{A}(h)$, $h\in S_r^T$, as <u>coordinate functions</u> on a probability space (Q,B^m,B^m) - where B^m is the formal <u>Gaussian measure</u>

$$dB^m=\frac{1}{Z^m}\,e^{-\frac{1}{2}(q,(-\Delta+m^2)q)}\,dq,\quad q\in Q=\mathop{\times}_{\{t,\underline{p};\lambda=0,\pm1\}} R . \tag{31}$$

Then

$$\overset{e}{A}_{\pm1}(0,\underline{p})(q)=q_{\pm1}(0,\underline{p}), \tag{32}$$

$$\overset{e}{A}_0(0,\underline{p})(q)=i\underline{p}.\underline{q}(0,\underline{p})/k+\int_0^{+\infty}dt\, e^{-kt}\left\{q^0(t,p)+i\underline{p}.\underline{q}(t,\underline{p})/k\right\},$$

where $k=+|\underline{p}|$. Hence $\overset{e}{A}_\lambda(0,\underline{p})$ - when smeared in $\underline{p}\in R^3$ - are B_+^m-measurable functions. In particular $\overset{e}{A}_{\pm1}(0,\underline{p})$ can be identified with the Fourier coefficients of the free vector <u>meson quantum field</u> , of momentum $\underline{p}$ and helicity $\lambda=\pm1$. Smearing with h, $\tilde{h}(p)\in T_p^C$, we get

$$A^T(0,\underline{h})=\overset{e}{A}(J_0\underline{h}),\quad h\in S_r^C(V_+^m). \tag{33}$$

However, $\Theta_0\overset{e}{A}_0(0,.)\Theta_0=\ldots+\int_{-\infty}^{0}dt\ \ldots$, and for $m>0$ the <u>Markoff property breakes down</u> to reflection positivity. Using (56) of part I we find the conditional expectation

$$E(\Theta_0\overset{e}{A}_0(t,\underline{p})\Theta_0/B_+^m)=\left\{\frac{m}{w+k}\right\}^2 e^{-wt}\,\overset{e}{A}_0(0,\underline{p}),\quad t\geq 0. \tag{34}$$

<u>Lemma 6</u> Let $l_{(s)}=v.u_{(s)}$, $u_{(s)}=l^0_{(s)}\in S(\Lambda_+^{int})$; $s=1,2,..,n$. Then for $t_1<t_2\ \ldots<t_n$ holds

$$\int_Q dB^m\,\overset{e}{A}(T_{t_1}l_{(1)})\ldots\overset{e}{A}(T_{t_n}l_{(n)})=(\Omega,A^T(0,Wl_{(1)})e^{-(t_2-t_1)d\Gamma(H_0)}\,A^T(0,Wl_{(2)})\ldots e^{-(t_n-t_{n-1})d\Gamma(H_0)}A^T(0,Wl_{(n)})\Omega). \tag{35}$$

Proof: Identifying $\overset{e}{A}(h_{(1)})\overset{e}{A}(h_{(2)})\ldots\overset{e}{A}(h_{(n)})\in L^2(d\beta^m)$ with $\Psi[h_{(1)}\otimes\ldots\otimes h_{(n)}]$ in the <u>Fock space</u> $\Gamma(N_{m,r})$, we can rewrite the l.h.s. of (35) in the form

(36) $\langle 1,\Gamma(\theta_o)\Psi[T_{t_1}1_{(1)}\otimes\ldots\otimes T_{t_n}1_{(n)}]\,1\rangle_{\Gamma(N_{m,r})}$.

Since $Wl_{(s)}\in S^T(V^m_+)\cap\ker C$ are smooth elements in M_m, (35) is nothing than the generalization of the free Feynman-Kac-Nelson formula (53) of part I (JINR E5-12780) to Fock space.

Q.E.D.

Due to Hegerfeldt there is another possible reconstruction of the quantum field operators at time zero $A^T(0,l)$, which for $l\in S^T(V^m_+)\cap\ker C$ describe the degrees of freedom of the free vector meson field with <u>helicity</u> $\lambda=0$.

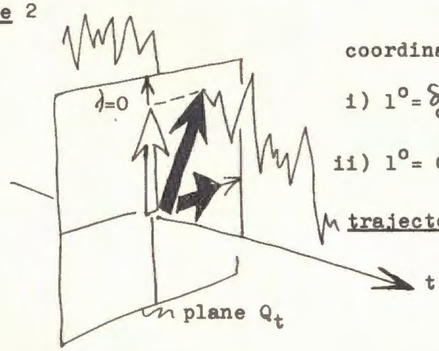
In the lemma 6 we have used $l=v.u\in N_m$ with supp $l^\mu\subset\Lambda^{int}_+$. We remark that for $m>0$ and

(37) $l=v(\delta_o\otimes\varphi)$, $\varphi\in S(R^3)$,

$$\int|\tilde{l}(p)|^2\frac{d^4p}{(p^2+m^2)^2}\le 2\pi\,\|k^{-1}\tilde{\varphi}\|^2_{L^2(\frac{d^3p}{2w})}<\infty,$$

supp $l^\mu\subset S_o$ and $\|\tilde{l}^\mu\|<\infty$. Hence we can define fields $\overset{e}{A}(l)$ at <u>sharp time</u> as ${}^{-2}$ forms. But we stop here.

<u>Figure</u> 2



coordinate functions $\overset{e}{A}(l)$:

i) $l^o=\delta_o\otimes\tilde{\varphi}$, $l^\beta=-\delta_o'\otimes ip^\beta\tilde{\varphi}$; $\beta=1,2,3$

ii) $l^o=\theta(x^o)e^{-kx^o}\tilde{u}(\underline{p})$, $l^\beta=\ldots$

<u>trajectory</u> $q\in Q$

3. <u>Transversal self-interaction</u>

We ask the question whether for $m\geq 0$ reflexion positivity as a property of the restriction $d\beta^m\restriction_{B^m_e}$ of the free measure for the Euclidean vector meson field $\overset{e}{A}$ is stable with respect to transversal pertubations.

First a negative result:

<u>Lemma 7</u> It is not possible to give the formal expression

(38) $F=e^{-\frac{1}{2}q.Eq}$, $q.Eq=\iint_{R^4\times R^4}dxdy\;q^\mu(x)E^{\mu\nu}(x-y)q^\nu(y)$

the precise meaning of a <u>multiplicative functional</u> on (Q,B^m,β^m). This means that for a given disjoint covering $\{\mathcal{O}_i\}$ of Euclidean space-time R^4, we cannot express F as a product of $B^m_{\mathcal{O}_i}$-measurable random variables F_i.

Proof: With suitable normalization holds $F\,d\beta^0=d\beta^m$, m=1. Then by a theorem of Nelson F cannot be multiplicative - even with respect to the decomposition $R^4=\Lambda_+\cup\Lambda_-$ - because $\overset{e}{A}$ is not a Markoff field, except in the case m=0.

Q.E.D.

Intuitively, the non-local projection E causes that q.Eq is <u>not additive</u>. Note that $\{q(t,\underline{h}):\ t\in R,\ \underline{h}\in S^C_r(R^3)\}$ is not a full set of random variables.

Below we construct functionals on (Q,B^m,β^m) multiplicative with respect to the crucial hyperplane $S_o=\{x=(t,\underline{x}):\ t=0\}$ yielding a <u>space cut-off</u>. Hence there will be no Euclidean covariance.

Given a real measurable function V on the real line bounded from below and $h\geq 0$, $\tilde{h}\in S^T_r(V^m_+)$, we define the Q-space function

(39) $F^+_h=e^{-\int_0^{+\infty}V[q(J^m_t h)]\,dt}$,

where

(40) $J^m_o(t\oplus l)=(0,l\otimes\underline{t})+(1,-\underline{p}p^o/k^2)\,\frac{ik(w+k)}{m(p^o+ik)}\,\tilde{l}^o$.

By J_0^m we denote the inverse of the restriction of the canonical map W to the physical subspace N_{M_m}, which for m=0 coincides with the embedding $J_0: M^C_{ph} \to N$.

Then $F = F_+^h . \Theta_0 F_+^h$ is multiplicative with respect to S_0, and with a suitable normalization $F d\beta^m$ becomes a probability measure. We have reflection positivity, i.e. for all B_+^m-measurable functions ψ

$$(41) \qquad \int_Q \psi^* . \Theta_0 \psi \, F \, d\beta^m \geq 0.$$

Lemma 8 For m=0 $\overset{e}{A}$ is a Markoff field on $(Q, B^0, F\, d\beta^0)$ with respect to any disjoint covering $R^4 = \bigcup \mathcal{O}_i$ by the Euclidean time strips $\mathcal{O}_i = \{x: t_i < t \leq t_{i+1}\}$.

Proof: For m=0, $q[V(J_t \underline{h})]$, $\tilde{\underline{h}} \in S_r^C(R^3)$, are $B^0_{S_t}$-measurable. Moreover, $\Theta_0 J_t \Theta_0 = J_{-t}$. Hence $F = \prod_i F_i$, where

$$(42) \qquad F_i = e^{-\int_{t_i}^{t_{i+1}} V[(J_t \underline{h})] dt} . \qquad \text{Q.E.D.}$$

We remark that the cut-off function $\underline{h}$, $\nabla . \underline{h} = 0$, is needed to make F transversal, hence B^0-measurable. Our definition (39) of the Euclidean action from the very beginning avoids the troubles with products of fields in coinciding points.

If $V = V(q^2)$ we can study $q \otimes q\, (\Gamma^{(2)} J_t h)$, for test functions $h \in \Gamma^{(2)}(S_r^T(V_+^m))$ with $h \geq 0$, or more singular.

$$(43) \qquad \tilde{h}^{\mu\nu}(\underline{p}, \underline{q}) = \tilde{\chi}_{\Lambda}(\underline{p} - \underline{q}) . (\delta^{\mu\nu} - \frac{p^\mu q^\nu}{\underline{p} . \underline{q}}); \quad \mu, \nu = 1,2,3$$

is a pathological example, since for finite Λ (43) is not transversal and as $\Lambda \nearrow R^3$, we obtain the very singular expression $q . Cq(t, .)$.

We can generalize our method in the case m=0 to construct for given V and $h \in N_{S_r}$, where $S_r = \{x: /x/ = r\}$, r=1 and $h \geq 0$, the functional

$$(44) \qquad G = e^{-\int_0^{+\infty} V[q(T_r h)] \, d\varsigma(r)}, \quad d\varsigma(r) = /S_r/ dr,$$

where $r \to T_r$ is the unitary group of dilatations in N described in the section 2 of part I. (JINR E5-12780) Then $\overset{e}{A}$ is a Markoff field on $(Q, B^0, G d\beta^0)$ with respect to any covering $\bigcup \mathcal{O}_i$ by the shells $\mathcal{O}_i = \{x: r_i < /x/ \leq r_{i+1}\}$. Since $/S_r/ < \infty$, the function h is not needed for a cut-off but to make G transversal. The derivation of corresponding Feynman-Kac-Nelson formulas is straightforward.

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