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Submitted to "Reports on Math.Phys."

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Оснащенные гильбертовы пространства и топологии в операторных алгебрах

Любой Op*-алгебре $\mathcal{A}$ неограниченных операторов соответствует оснащенное гильбертово пространство $\mathfrak{D}[t_{\mathcal{A}}] \to \mathcal{H} \to \mathfrak{D}'[t^{\mathcal{A}}]$ такое, что все операторы $A \in \mathcal{A}$ являются непрерывными относительно топологий $t_{\mathcal{A}}$ и $t^{\mathcal{A}}$. Это приводит к различным топологиям равномерной сходимости на каждом ограниченном множестве в операторной алгебре $\mathcal{A}$, которые исследованы в этой статье. В частности, полные результаты получены, если $\mathfrak{D}[t_{\mathcal{A}}]$ является рефлексивным пространством.

Работа выполнена в Лаборатории теоретической физики, ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

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Rigged Hilbert Spaces and Topologies on Operator Algebras

To any Op*-algebra $\mathcal{A}$ of unbounded operators there corresponds a rigged Hilbert space $\mathfrak{D}[t_{\mathcal{A}}] \to \mathcal{H} \to \mathfrak{D}'[t^{\mathcal{A}}]$ so that all operators $A \in \mathcal{A}$ are continuous with respect to the topologies $t_{\mathcal{A}}$, $t^{\mathcal{A}}$. This leads to different topologies of uniformly bounded convergence on the operator algebra $\mathcal{A}$, which are investigated in this paper. Particularly consistent results are obtained if $\mathfrak{D}[t_{\mathcal{A}}]$ is a reflexive space.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

1. INTRODUCTION

In a previous paper [2] we have investigated rigged Hilbert spaces $\mathfrak{D}[t_{\mathcal{A}}] \to \mathcal{H} \to \mathfrak{D}'[t^{\mathcal{A}}]$ associated with algebras $\mathcal{A}$ of operators on a dense domain $\mathfrak{D}$ of a Hilbert space $\mathcal{H}$. In this paper we show that the so-called quasi-uniform topologies on algebras of unbounded operators [5,6] are related to this rigged Hilbert space structure. These quasi-uniform topologies on operator agrebras have found different applications in the algebraic approach to quantum field theory and statistics (see, e.g., [3,7,8,12]). Particularly consistent results are ontained if $\mathfrak{D}[t_{\mathcal{A}}]$ is a reflexive space. The problem of reflexity of $\mathfrak{D}[t_{\mathcal{A}}]$ is regarded in the last section, where especially a counterexample is given and some open questions are formulated.

Let us recall some basic definitions of ref. [2]. For a dense linear subspace $\mathfrak{D}$ of a Hilbert space $\mathcal{H}$ over the field $\mathbf{C}$ of complex numbers with the scalar product $\langle \cdot , \cdot \rangle$ we denote by $\mathcal{L}^+(\mathfrak{D})$ the set of all linear operators $A \in \mathrm{End}\,\mathfrak{D}$, for which an $A^+ \in \mathrm{End}\,\mathfrak{D}$ exists with $\langle \phi, A\psi \rangle = \langle A^+\phi, \psi \rangle$ for all $\phi, \psi \in \mathfrak{D}$. $\mathcal{L}^+(\mathfrak{D})$ is a $*$-algebra with respect to the usual algebraic operations with operators and the involution $A \longrightarrow A^+$. A $*$-subalgebra $\mathcal{A}$ containing the identity operator is called Op $*$-algebra. An Op$*$-algebra $\mathcal{A}$ over $\mathfrak{D}$ generates in a natural way a topology $t_{\mathcal{A}}$ on $\mathfrak{D}$ by all seminorms $\|\phi\|_{\mathfrak{A}} = \|A\phi\|$, $\phi \in \mathfrak{D}$, where A runs over all operators of $\mathfrak{A}$. $\mathfrak{D}[t_{\mathfrak{A}}]$ is continuously imbedded into $\mathcal{H}$. The strong dual space of $\mathfrak{D}[t_{\mathfrak{A}}]$ is denoted by $\mathfrak{D}'[t^{\mathfrak{A}}]$. We write the linear functionals $F \in \mathfrak{D}'$ on $\phi \in \mathfrak{D}$ in the form $<F,\phi>$ and equip $\mathfrak{D}'$ with the (anti-) linear structure defined by $<\lambda F + \mu G, \phi> = \bar{\lambda}<F,\phi> + \bar{\mu}<G,\phi>$. Then $\psi \to F_\psi$, $<F_\psi, \phi> = <\psi, \phi>$ for $\psi \in \mathcal{H}$, defines a linear imbedding of $\mathcal{H}$ into $\mathfrak{D}'$. So we get in dependence on $\mathfrak{A}$ the rigged Hilbert space

$$\mathfrak{D}[t_{\mathfrak{A}}] \subset \mathcal{H} \subset \mathfrak{D}'[t^{\mathfrak{A}}].$$

An Op $*$-algebra $\mathfrak{A}$ is called closed if $\mathfrak{D}[t_{\mathfrak{A}}]$ is a complete space. $\mathfrak{D}$ is called a closed domain if $\mathfrak{L}^+(\mathfrak{D})$ is closed on $\mathfrak{D}$. Then we have the following Lemma (/2/ Lemma 7,8 /11/).

Lemma 1.1

Let $\mathfrak{D}$ be a closed domain and $\mathfrak{A}$ a closed Op$*$-algebra on $\mathfrak{D}$.

i) $\mathfrak{D}[t_{\mathfrak{A}}]$ is semi-reflexive.

ii) If $\mathfrak{A}_1$ is yet another closed Op$*$-algebra on $\mathfrak{D}$, then $t_{\mathfrak{A}}$ and $t_{\mathfrak{A}_1}$ have the same bounded sets. Therefore, the dual topologies $t^{\mathfrak{A}}, t^{\mathfrak{A}_1}$ coincide on $\mathfrak{D}[t_{\mathfrak{A}}]' \quad \mathfrak{D}[t_{\mathfrak{A}_1}]'$, and we denote it by $t' = t^{\mathfrak{A}} = t^{\mathfrak{A}_1}$.

2. TOPOLOGIES OF UNIFORMLY BOUNDED CONVERGENCE ON $\mathfrak{A}$

For two locally convex spaces E, F the topology of uniformly bounded convergence (/10/, ch. III,3) on the space $\mathfrak{L}(E,F)$ of continuous linear mappings of E into F is defined by all semi-norms

$$q_{\alpha,M}(A) = \sup_{\phi \in M} p_\alpha(A\phi),$$

where p_α runs over all semi-norms defining the topology of F and M runs over all bounded sets in E.

If $\mathfrak{A}$ is an Op$*$-algebra over the domain $\mathfrak{D}$, then we have on $\mathfrak{D}$ three topologies $t_{\mathfrak{A}} \geq \|\cdot\| \geq t^{\mathfrak{A}}$ in correspondence with the rigged Hilbert space defined by $\mathfrak{A}$. The system of $t_{\mathfrak{A}}$-bounded sets will be denoted by $\mathfrak{M}_{\mathfrak{A}}$ and the system of $t^{\mathfrak{A}}$-bounded sets by $\mathfrak{M}^{\mathfrak{A}}$.

Lemma 2.1

An Op $*$-algebra $\mathfrak{A}$ is a subspace of the following five linear spaces of continuous linear mappings with the corresponding topologies of uniformly bounded convergence, defined by the given seminorms:

$$\mathfrak{L}(\mathfrak{D}[t_{\mathfrak{A}}], \mathfrak{D}[t_{\mathfrak{A}}])[\tau^{\mathfrak{D}}]: \quad \|A\|^{M,B} = \sup_{\phi \in M} \|BA\phi\|, \; B \in \mathfrak{A}, \; M \in \mathfrak{M}_{\mathfrak{A}}$$

$$\mathfrak{L}(\mathfrak{D}[t_{\mathfrak{A}}], \mathfrak{D}[\|\cdot\|])[\tau^{(\mathfrak{D})}]: \quad \|A\|^{M} = \sup_{\phi \in M} \|A\phi\|, \; M \in \mathfrak{M}_{\mathfrak{A}}$$

$$\mathfrak{L}(\mathfrak{D}[t_{\mathfrak{A}}], \mathfrak{D}[t^{\mathfrak{A}}])[\tau_{\mathfrak{D}}]: \quad \|A\|_M = \sup_{\phi,\psi \in M} |<\phi, A\psi>|, \; M \in \mathfrak{M}_{\mathfrak{A}}$$

$$\mathfrak{L}(\mathfrak{D}[\|\cdot\|], \mathfrak{D}[t^{\mathfrak{A}}])[\tau_+^{(\mathfrak{D})}]: \quad \|A\|_+^M = \sup_{\phi \in M} \|A^+\phi\|, \; M \in \mathfrak{M}_{\mathfrak{A}}$$

$$\mathfrak{L}(\mathfrak{D}[t^{\mathfrak{A}}], \mathfrak{D}[t^{\mathfrak{A}}])[\tau_0^{\mathfrak{D}}]: \quad \|A\|_{\mathfrak{N}}^M = \sup_{\phi \in M, \psi \in \mathfrak{N}} |<\phi, A\psi>|, \; M \in \mathfrak{M}_{\mathfrak{A}}, \mathfrak{N} \in \mathfrak{M}^{\mathfrak{A}}$$

It can be proven by simple calculations that $\mathfrak{A}$ is a subset of $\mathfrak{L}(\mathfrak{D}[t_{\mathfrak{A}}], \mathfrak{D}[t_{\mathfrak{A}}])$ and of $\mathfrak{L}(\mathfrak{D}[t^{\mathfrak{A}}], \mathfrak{D}[t^{\mathfrak{A}}])$. For the other three spaces it is then a consequence of the relation between the topologies $t_{\mathfrak{A}} \geq \|\cdot\| \geq t^{\mathfrak{A}}$. If τ

is a locally convex topology given by the semi-norms $\|A\|^{a}$, $\{a\}$ a set of indices, on an Op $*$-algebra $\mathcal{A}$, then by τ^{+} we denote the topology on $\mathcal{A}$ defined by the semi-norms $\|A^{+}\|^{a} = \|A\|^{a}_{+}$. The involution $A \to A^{+}$ is then a continuous mapping of $\mathcal{A}[\tau]$ onto $\mathcal{A}[\tau^{+}]$. With the help of this notation we get as a consequence of Lemma 2.1 the following relations between the topologies on $\mathcal{A}$:

Lemma 2.2

i) $\tau_{\mathfrak{D}} \leq \begin{array}{l} \tau^{(\mathfrak{D})} \leq \tau^{\mathfrak{D}} \\ \tau^{(\mathfrak{D})}_{+} \leq \tau^{\mathfrak{D}}_{0} \end{array}$

ii) $\tau^{+}_{\mathfrak{D}} = \tau_{\mathfrak{D}}$

iii) $(\tau^{(\mathfrak{D})})^{+} = \tau^{(\mathfrak{D})}_{+}$

The second and the fourth topology $\tau^{(\mathfrak{D})}$ and $\tau^{(\mathfrak{D})}_{+}$ of Lemma 2.1 are connected by the continuity of the involution. For the case that $\mathfrak{D}[t_{\mathcal{A}}]$ is a reflexive space we have also such a relation between the first and the fifth topology $\tau^{\mathfrak{D}}$ and $\tau^{\mathfrak{D}}_{0}$, as we shall see below. In general this symmetry is broken. In what follows we shall study this situation. First we state the following Lemma:

Lemma 2.3

Let $\mathcal{A}(\mathfrak{D})$ be an arbitrary Op $*$-algebra and $K = \{\phi \in \mathfrak{D}, \|\phi\| \leq 1\}$ the unit ball in $\mathfrak{D}$. Then all subsets of the form $A(K), A \in \mathcal{A}$, are $t^{\mathcal{A}}$-bounded in $\mathfrak{D}$.

Proof: Every $t^{\mathcal{A}}$-semi-norm on $A(K)$ is bounded. In fact:

$$\sup_{\phi \in K} p_{M}(A\phi) = \sup_{\phi \in K, \psi \in M} |\langle A\phi, \psi\rangle| = \sup_{\phi \in K, \psi \in M} |\langle \phi, A^{+}\psi\rangle| =$$

$$= \sup_{\chi \in A^{+}M} \|\chi\| \leq C_{A^{+}M} < \infty,$$

since for $M \in \mathfrak{M}_{\mathcal{A}}$ it is $A^{+}M \in \mathfrak{M}_{\mathcal{A}}$ for all $A \in \mathcal{A}$. The inversion of this Lemma is not true even in the case of closed algebras. To it we refer to example 3 of ref. [2].

Example:

Let $\mathcal{H}_0, \mathcal{H}_1, \ldots$ be a sequence of Hilbert spaces, and let in every $\mathcal{H}_n$ be given an unbounded selfadjoint operator $T_n \geq 1$. We define $\mathfrak{D}_n = \underset{k \geq 0}{\ } \mathfrak{D}(T^{k}_{n})$ and form

$$\mathcal{H} = \sum_{n \geq 0}{}^{\oplus} \mathcal{H}_n \qquad \text{(Hilbert direct sum) and}$$

$$\mathfrak{D} = \sum_{n \geq 0} \mathfrak{D}_n \qquad \text{(algebraic direct sum).}$$

Then $\mathfrak{D}$ is dense in $\mathcal{H}$. Every vector $\phi \in \mathfrak{D}$, $\phi = \sum_{n \geq 0} \phi_n, \phi_n \in \mathfrak{D}_n$, has only a finite number of components which are different from zero. For an arbitrary sequence $\{a_n\}_{n=0,1,..}$ of complex numbers and an arbitrary sequence $\{k_n\}_{n=0,1,..}$ of non-negative integer numbers we define by

$$A\phi = \sum_{n \geq 0} a_n T^{k_n}_{n} \phi_n$$

the operator

$$A = \sum_{n \geq 0} a_n T^{k_n}_{n} \quad \text{on } \mathfrak{D}.$$

The sum of this definition breaks off for every $\phi \in \mathfrak{D}$.

Now we consider

$$\mathcal{A}_1 = \{ A = \sum_{n \geq 0} a_n T^{k_n}_{n}, \{a_n\}, \{k_n\} \text{ arbitrary}$$

$$\mathcal{A}_2 = \{ A = \sum_{n \geq 0} a_n T^{k_n}_{n}, \{a_n\} \text{ arbitrary}, \{k_n\} \text{ with } k_n \leq N(A)\},$$

where N(A) is a bound for the grads k_r, which depends on A.
$\mathfrak{A}_1$ $\mathfrak{A}_2$ are closed Op $*$-algebras over $\mathfrak{D}$. They generate on $\mathfrak{D}$ $\mathfrak{D}'$ the same strong topology t'. But for an operator $A_1 \in \mathfrak{A}_1 \setminus \mathfrak{A}_2$ we can forms the set $A_1(K)$ which cannot be absorbed by any set $A_2(K)$, $A_2 \in \mathfrak{A}_2$.

Lemma 2.4

For an arbitrary Op$*$-algebra $\mathfrak{A}$ the following is true:
(i) $(r^{\mathfrak{D}})^+ \leq r_0^{\mathfrak{D}}$
(ii) $(r_0^{\mathfrak{D}})^+ \geq r^{\mathfrak{D}}$
The equalities hold, if the sets A(K), $A \in \mathfrak{A}$, form a basis of the $t^{\mathfrak{A}}$-bounded subsets in $\mathfrak{D}$.

Proof: If every $t^{\mathfrak{A}}$-bounded set $\mathfrak{N}$ is contained in a set B(k) then we have

$$\|A\|_{\mathfrak{N}}^{M} = \sup_{\phi \in M, \psi \in \mathfrak{N}} |<\phi, A\psi>| \leq \sup_{\phi \in M, \psi \in B(K)} |<\phi, A\psi>| = \sup_{\phi \in M, \chi \in K} |<\phi, AB\chi>| =$$

$$= \sup_{\phi \in M, \chi \in K} |<B^+A^+\phi, \chi>| = \sup_{\phi \in M} \|B^+A^+\phi\| = \|A^+\|^{M, B^+} = \|A\|_{+}^{M, B^+}$$

for a suitable $B \in \mathfrak{A}$ and every $M \in \mathfrak{M}_{\mathfrak{A}}$, $\mathfrak{N} \in \mathfrak{M}^{\mathfrak{A}}$. So we can estimate every $r_0^{\mathfrak{D}}$ seminorn by a suitable $(r^{\mathfrak{D}})^+$-seminorm. On the other hand, $r_0^{\mathfrak{D}}$ is not coarser than $(r^{\mathfrak{D}})^+$ in consequence of the foregoing Lemma. (ii) can be proved analogously. Thus we can complete the assertion (i) in Lemma 2.2 as follows:

$$(*) \quad r_{\mathfrak{D}} \leq \begin{matrix} r^{(\mathfrak{D})} \leq r^{\mathfrak{D}} \leq (r_0^{\mathfrak{D}})^+ \\ r_+^{(\mathfrak{D})} \leq (r^{\mathfrak{D}})^+ \leq r_0^{\mathfrak{D}} \end{matrix}$$

Now we are going to show that for reflexive $\mathfrak{D}[t_{\mathfrak{A}}]$ the topologies $(r^{\mathfrak{D}})^+$ and $r_0^{\mathfrak{D}}$ coincide. First we prove the following Lemma:

Lemma 2.5

Let $\mathfrak{A}$ be an Op$*$-algebra and K the unit ball in $\mathfrak{D}$. For any $A \in \mathfrak{A}$ we regard the set A(K) as a subset of $\mathfrak{D}'$ and form the bipolar $A(K)^{00}$ in $\mathfrak{D}'$ with respect to the dual pair $(\mathfrak{D}', \mathfrak{D})$. Then the sets $A(K)^{00}$, $A \in \mathfrak{A}$, form a basis of the $t_{\mathfrak{A}}$-equicontinuous subsets in $\mathfrak{D}'$.

Proof: In the dual pair $(\mathfrak{D}', \mathfrak{D})$ we consider the sets A(K), $A \in \mathfrak{A}$, as subsets of $\mathfrak{D}'$ and their polars in $\mathfrak{D}$:

$$\mathfrak{D} \supset A(K)^0 = \{\phi \in \mathfrak{D} : \sup_{\psi \in A(K)} |<\psi, \phi>| \leq 1\} = \{\phi \in \mathfrak{D} : \sup_{\chi \in K} |<A\chi, \phi>| \leq 1\} =$$

$$= \{\phi \in \mathfrak{D} : \sup_{\chi \in K} |<\chi, A^+\phi>| \leq 1\} = \{\phi \in \mathfrak{D} : \|A^+\phi\| \leq 1\} = \{\phi \in \mathfrak{D} : \|\phi\|_{A^+} \leq 1\}.$$

Consequently, the sets $A(K)^0$, $A \in \mathfrak{A}$, form a $t_{\mathfrak{A}}$-neighbourhood base of zero. The $t_{\mathfrak{A}}$-equicontinuous subsets in $\mathfrak{D}'$ are those subsets A', for which exists a $t_{\mathfrak{A}}$-neighbourhood of 0 $U \subset \mathfrak{D}$ with $A' \subset U^0$ in $\mathfrak{D}'$ (/9/, ch. II,4). Therefore, the sets $A(K)^{00}$, $A \in \mathfrak{A}$, form a basis of the $t_{\mathfrak{A}}$-equicontinuous subsets in $\mathfrak{D}'$.

Lemma 2.6

Let $\mathfrak{A}$ be a closed Op $*$-algebra and K the unit ball in $\mathfrak{D}$. Then the bipolars $A(K)^{00}$, $A \in \mathfrak{A}$, form a basis of the bounded subsets in $\mathfrak{D}'$ if and only if the domain $\mathfrak{D}[t_{\mathfrak{A}}]$ is reflexive.

Proof: $\mathfrak{D}[t_{\mathfrak{A}}]$ is reflexive if and only if it is barrelled ref. /2/ Lemma 13). The barrels in $\mathfrak{D}$ are exactly the polars of the $\sigma(\mathfrak{D})$-bounded subsets in $\mathfrak{D}'$ (ref. /9/ ch. IV, prep. 1). On account of the semi-reflexivity of $\mathfrak{D}[t_{\mathfrak{A}}]$ the weakly and the strongly bounded subsets in $\mathfrak{D}'$ coincide. Therefore, every barrel absorbs a $t_{\mathfrak{A}}$-neighbourhood of 0 if and only if every bounded subset in $\mathfrak{D}'$ is absorbed from a $t_{\mathfrak{A}}$-equicontinuous subset. By Lemma 2.5 this is exactly the case if the sets $A(K)^{00}$ form a basis of the bounded subsets in $\mathfrak{D}'$.

Corollary 2.7

Let $\mathfrak{A}$ be a closed Op*-algebra and suppose $\mathfrak{D}[t_{\mathfrak{A}}]$ is reflexive. Then the t'-closures in $\mathfrak{D}$ of the sets $A(K)$, $A\in\mathfrak{A}$, form a basis of the t'-bounded subsets in $\mathfrak{D}$.

Proof: topology t' is consistent with the duality $(\mathfrak{D},\mathfrak{D}')$ since by Lemma 1.1 $\mathfrak{D}[t_{\mathfrak{A}}]$ is semireflexive, and $\mathfrak{D}$ is dense in $\mathfrak{D}'[t']$ (ref.[2], Folgerung 12). The sets $A(K)$, $A\in\mathfrak{A}$, are convex, therefore, $A(K)^{00}|_{\mathfrak{D}} = \overline{A(K)}^{\sigma(\mathfrak{D},\mathfrak{D})} = \overline{A(K)}^{t'}$.

Corollary 2.8

Let $\mathfrak{A}$ be a closed Op*-algebra and suppose $\mathfrak{D}[t_{\mathfrak{A}}]$ is reflexive. Then the topologies $r^{\mathfrak{D}}$ and $r_0^{\mathfrak{D}}$ are dual to each other, consequently, the lines in (*) break off.

Proof: The sets $\overline{A(K)}^{t'}$ form a basis of the t'-bounded subsets in $\mathfrak{D}$. Since every operator $A\in\mathfrak{A}$ is (t',t')-continuous (Lemma 2.1), we have for every $r_0^{\mathfrak{D}}$ semi-norm $||A||^{M}_{\mathfrak{N}} = \sup_{\phi\in M,\psi\in\mathfrak{N}} |<\phi, A\psi>| = ||A||^{M}_{\mathfrak{N}t'}$ for all $M\in\mathfrak{M}_{\mathfrak{A}}$ and $\mathfrak{N}\in\mathfrak{M}^{\mathfrak{A}}$, and the Corollary follows by analogous considerations as in the proof of Lemma 2.4.

Finally we consider the maximal Op*-algebra $\mathcal{L}^{+}(\mathfrak{D})$. We write $t = t_{\mathcal{L}^{+}}$. Let $\mathcal{L}(\mathfrak{D},\mathfrak{D}')$ be the linear space of all continuous linear mappings of $\mathfrak{D}[t]$ into $\mathfrak{D}'[t']$. Further we write $\mathcal{L}(\mathfrak{D}) = \mathcal{L}(\mathfrak{D},\mathfrak{D})$ and $\mathcal{L}(\mathfrak{D}') = \mathcal{L}(\mathfrak{D}',\mathfrak{D}')$. In the case that $\mathfrak{D}[t]$ is reflexive we can give an interesting description of $\mathcal{L}^{+}(\mathfrak{D})$ by the following Lemma [8]:

Lemma 2.9

Let $\mathfrak{D}[t]$ be a reflexive space. Then

(i) if $A\in\mathcal{L}(\mathfrak{D},\mathfrak{D}')$ then the adjoint operator $A^{+}\in\mathcal{L}(\mathfrak{D},\mathfrak{D}')$ is uniquely defined by $<A\phi,\psi> = \overline{<A^{+}\psi,\phi>}$ and $A\to A^{+}$ is an involution on $\mathcal{L}(\mathfrak{D},\mathfrak{D}')$,

(ii) $\mathcal{L}(\mathfrak{D})$, $\mathcal{L}(\mathfrak{D}')\subset\mathcal{L}(\mathfrak{D},\mathfrak{D}')$ and $\mathcal{L}(\mathfrak{D})^{+} = \mathcal{L}(\mathfrak{D}')$,

(iii) $\mathcal{L}^{+}(\mathfrak{D})$ is a subspace of $\mathcal{L}(\mathfrak{D})$ and it is

$$\mathcal{L}^{+}(\mathfrak{D}) = \mathcal{L}(\mathfrak{D})\cap\mathcal{L}(\mathfrak{D}').$$

3. THE PROBLEM OF REFLEXIVITY OF THE DOMAIN $\mathfrak{D}[t_{\mathfrak{A}}]$

Till now we have stated that in the case of a closed Op*-algebra $\mathfrak{A}$ the domain $\mathfrak{D}[t_{\mathfrak{A}}]$ is always complete and semireflexive, whereas the strong dual $\mathfrak{D}'[t']$ is barrelled and bornological (ref.[2], Folg. 9, 11). The example 3 of ref.[2] shows that for sufficient meagre closed algebras $\mathfrak{A}$ $\mathfrak{D}[t_{\mathfrak{A}}]$ does not need to be barrelled and, therefore, reflexive, and $\mathfrak{D}'[t']$ must not be semireflexive. There arises the question, whether at the very last for the maximum Op*-algebra $\mathcal{L}^{+}(\mathfrak{D})$ $\mathfrak{D}[t]$ becomes a reflexive space. We show by a counterexample that even this in general is not true.

Example:

Let $\mathcal{H}$ be the non-separable Hilbert space of all quadratic summable number-sequences which potency is more than countable, and $\mathfrak{D}$ the subspace of all finite number-sequences:

$$\mathcal{H} = \ell^2 = \{\psi\ ;\ \psi = \sum_{\alpha\in I} x_\alpha \phi_\alpha ,\ \sum_{\alpha\in I} |x_\alpha|^2 < \infty ,\quad \text{card}\, I = \varphi > \aleph_0 ;$$

$$(\phi_\alpha)_{\alpha\in I} \text{ is a complete orthonormal basis } \}$$

$$\mathfrak{D} = d = \{\psi ; \psi = \sum_{\text{finite}} x_\alpha \phi_\alpha\} = \sum_{\alpha\in I} K_\alpha ,\quad K_\alpha = C .$$

Then the maximum Op*-algebra $\mathfrak{L}^+(d)$ is isomorphic with the set of all matrices whose rows and columns have the potency φ, but in each row and column only a finite number of elements different from zero. Obviously, d is a closed domain. $\mathfrak{L}^+(d)$ generates on d the algebra-topology t which is given by the seminorms

$$\|\psi\|_A = \Big(\sum_{\alpha\in I} \Big|\sum_{\beta\in I} \mu_{\alpha\beta} x_\beta\Big|^2\Big)^{1/2} ,\ \psi\in d ,\ A = (\mu_{\alpha\beta})_{\alpha\beta\in I} \in \mathfrak{L}^+(d).$$

It is easy to see that the topology t is given already by the diagonal-operators:

$$\|\psi\|_B = \Big(\sum_{\alpha\in I} |b_\alpha|^2 |x_\alpha|^2\Big)^{1/2},\quad \psi\in d ,\quad B = (b_\alpha)_{\alpha\in I} .$$

Furthermore we consider on d the topologies of the locally convex direct sum $\tau_\oplus$ and of the topological direct sum τ. Let $\{U^\alpha_\beta\}$ be a neighbourhood base of 0 in every K_α, then $\tau_\oplus$ is defined by the base of neighbourhoods $\{\underset{\alpha\in I}{\text{aco}}\, U^\alpha_\beta\}$, τ on the other hand by $\{\sum_{\alpha\in I} \oplus U^\alpha_\beta\}$ (ref. /4/ §18, 5). We have $\tau_\oplus \geq \tau$, and in the countable case both topologies coincide. But in our example $\tau_\oplus$ is different from τ (ref. /4/, §18, 5(8)). Since the sets $U^\alpha_\beta = \{x_\beta \in C : |x_\beta| \leq \epsilon_\beta,\ \epsilon_\beta > 0\}$ form a neighbourhood base of zero in every one-dimensional space $K_\alpha = C$, the topology $\tau_\oplus$ is given by

$$p_{(\lambda_\alpha)}(\psi) = \sum_{\alpha\in I} |\lambda_\alpha|\,|x_\alpha| ,\qquad \psi\in d ,$$

where the $(\lambda_\alpha)_{\alpha\in I}$ are elements of the space of all number-sequences with the potency φ d'. It is easily to show that $\tau_\oplus$ coincides with the strongest locally convex topology $\beta(d,d')$. The topology τ of the topological direct sum has the seminorms

$$p_{(\rho_\alpha)}(\psi) = \sup_{\alpha\in I} |\rho_\alpha|\,|x_\alpha| ,\quad \psi\in d ,\quad (\rho_\alpha)_{\alpha\in I} \in d' .$$

<u>Lemma 3.1</u>

We have $\tau_\oplus > t > \tau$ on d .

Proof: By (ref. /1/ Lemma 3.3) in the contable case the t-seminorms

$$p_{(b_\alpha)}(\psi) = \Big(\sum_{\alpha\in I} |b_\alpha|^2 |x_\alpha|^2\Big)^{1/2},\ (b_\alpha)_{\alpha\in I} \in d'$$

define the topology of the locally convex direct sum, otherwise not. Clearly, the topology t is stronger than τ:

$$\{\psi = (x_\alpha)\in d;\ \sup_{\alpha\in I} |\rho_\alpha|\,|x_\alpha| \leq 1\} \supset \{\psi = (x_\alpha)\in d ;\ (\sum_{\alpha\in I} |\rho_\alpha|^2 |x_\alpha|^2) \leq 1\}$$

On the other hand, the t-neighbourhood $U^t_1 = \{\psi = (x_\alpha)\in d;\ \sum_{\alpha\in I} |x_\alpha|^2 \leq 1\}$, does not contain any τ-neighbourhood $U^\tau_{(\rho_\alpha)} = \{\psi = (x_\alpha)\in d;\ \sup_{\alpha\in I} |\rho_\alpha|\,|x_\alpha| \leq 1\}$, $(\rho_\alpha)\in d'$. In fact, we have $\rho_\alpha \neq 0$ for all $\alpha\in I$. Since card $I > \aleph_0$ we can find real numbers $\epsilon_1 > \epsilon_0 > 0$ such that $\epsilon_1 > |\rho_\alpha| > \epsilon_0$ holds for more than countable many indices $\alpha\in I$. Therefore, in $U^\tau_{(\rho_\alpha)}$ always elements $\psi = (x_\alpha)$ exists with an arbitrary great $\sum_{\alpha\in I} |x_\alpha|^2$, i.e., $\psi \notin U^t_1$.

Now we can formulate the main result of this section.

<u>Lemma 3.2</u>

The domain d[t] is not reflexive.

Proof: It is sufficient to show that the topology t is different from $\beta(d, d[t]')$. We show that this strong topology coincides with $r_{\oplus}$. The dual space of $d[r_{\oplus}]$ is the space d' of all number-sequences of the potency φ, but the dual space of $d[r]$ is the subspace d'_0 of all number-sequences with only countable many components different from zero (ref. /4/, §22,5 (5)). Furthermore, we can state that the dual space of $d[t]$ coincides with d' too. The bounded subsets in the dual spaces d', resp. d'_0 are the sets of number-sequences which components are bounded. Therefore, the bounded subsets of d' are contained in the weak completions of bounded subsets of d'_0. Consequently, we have

$$r_{\oplus} = \beta(d, d') = \beta(d, d'_0) = \beta(d, d[t]') > t,$$

i.e., $d[t]$ is not reflexive.

Moreover, the example shows that the strong dual of a semireflexive space must not be quasi-complete, since we have

$$d[t]' = d'_0 \neq \overline{d'_0} = \overline{d'_0}^{\sigma} = d'.$$

(We denote by $\overline{d'_0}$, resp., $\overline{d'_0}^{\sigma}$ the quasi-complete, resp., weakly quasi-complete hull of d'_0).

Corollary 3.3

In the non-separable case closed domains $\mathfrak{D}$ exist with

(i) $\mathfrak{D}[t]$ is not reflexive (and not barrelled, not bornological either),

(ii) $\mathfrak{D}'[t']$ is not semireflexive and even not quasi-complete.

The problem is unsolved, whether for separable Hilbert spaces any closed domain $\mathfrak{D}[t]$ becomes reflexive.

Finally we investigate the situation which we had in our example 3 for arbitrary closed domains. We denote by $r(\mathfrak{D}')$ the Mackey topology on $\mathfrak{D}$ with respect to the dual pair $(\mathfrak{D}, \mathfrak{D}')$. Then by (ref. /4/ §23,8 (1)) the following holds:

Let $\mathfrak{D}[r]$ be semireflexive and $\overline{\mathfrak{D}'}$ the quasi-complete hull of the strong dual space $\mathfrak{D}'[\beta(\mathfrak{D})]$. Then $\mathfrak{D}[r(\overline{\mathfrak{D}'})]$ is semireflexive too, and its strong dual space is the quasi-complete hull of the strong dual of $\mathfrak{D}[r]$.

Therefore, we can always increase the topology of an arbitrary semi-reflexive space so much that the space remains semireflexive, but its strong dual space becomes quasi-complete (or even complete). If we demand besides that the quasi-complete hull of the strong dual space is weakly quasi-complete too:

$$\text{(A)} \qquad \overline{\mathfrak{D}'} = \overline{\mathfrak{D}'}^{\sigma}$$

then $\mathfrak{D}[r(\overline{\mathfrak{D}'})]$ even is reflexive (ref. /2/ Lemma 1.13). Consequently, if $\mathfrak{D}[r]$ is semireflexive, but not reflexive, r is the Mackey topology, and if (A) holds, than we obtain for the dual spaces the sequences

$$\mathfrak{D} = \mathfrak{D}'' = \mathfrak{D}^{IV} = \ldots = \mathfrak{D}^{2N} = \ldots$$

$$\mathfrak{D}' \subseteq \mathfrak{D}''' \subseteq \mathfrak{D}^{V} \subseteq \ldots \subseteq \mathfrak{D}^{2N+1} \subseteq \ldots \subseteq \overline{\mathfrak{D}'}$$

and an increasing sequence of topologies

$$\beta(\mathfrak{D}, \mathfrak{D}'), \beta(\mathfrak{D}, \mathfrak{D}'''), \ldots, \beta(\mathfrak{D}, \mathfrak{D}^{2N+1}), \ldots, \beta(\mathfrak{D}, \overline{\mathfrak{D}'}),$$

for which $\mathfrak{D}$ is semireflexive (ref. /4/ §23,8). Just now we have seen that the topology $\beta(\mathfrak{D}, \overline{\mathfrak{D}'})$ even is reflexive. In our example we had

$$d[t]' = d'_0 \neq d''' = (d'_0)'' = d[r_{\oplus}]' = d' = \overline{d'_0}.$$

i.e., $d[t''] = d[\beta(d[t]')]$ is already reflexive, and the sequences break off. It is an open problem, whether these sequences break off always, and it is unknown, whether spaces exist at all, for which only a higher dual space than the third coincides with the quasi-complete hull of the strong dual. Sufficient for $\mathfrak{D}''' = \widetilde{\mathfrak{D}'}$ is the following condition

(B) Every $\beta(\mathfrak{D}''',\mathfrak{D})$-bounded subset $\mathfrak{B} \subset \mathfrak{D}'''$ is contained in the completion of a $\beta(\mathfrak{D}',\mathfrak{D})$-bounded subset $\mathfrak{B}_1 \subset \mathfrak{D}'$: $\mathfrak{B} \subset \widetilde{\mathfrak{B}_1}[\beta(\mathfrak{D})]$.

Then these sets have the same polars, and the topologies $\beta(\mathfrak{D},\mathfrak{D}')$ and $\beta(\mathfrak{D},\mathfrak{D}''')$ coincide. Therefore, they are already reflexive. Finally we give some positive statements concerning the reflexivity in the case of a closed domain. If $\mathfrak{D}[t]$ is not reflexive, i.e., $t < \beta(\mathfrak{D},\mathfrak{D}') =: t''$ holds, then we have $\mathfrak{D}' \subsetneq \mathfrak{D}'''$.

<u>Lemma 3.4</u>

Let $\mathfrak{D}$ be a closed domain and t''' the strong topology $\beta(\mathfrak{D}''',\mathfrak{D})$. Then $t'''|_{\mathfrak{D}'} = t'$ holds

Proof: t''', resp., t' are the topologies of uniform convergence on all t''-, resp., t-bounded subsets in $\mathfrak{D}$. $\mathfrak{D}[t]$ is sequentially complete, therefore, the bounded and the strongly bounded subsets coincide (ref. /4/ §20, 11 (8)).

<u>Lemma 3.5</u>

Let $\mathfrak{D}$ be closed with the property (A). Then $\mathfrak{D}[t'']$ is semireflexive.

Proof: The quasi-complete hull $\widetilde{\mathfrak{D}'}$ contains $(\mathfrak{D}')''$ (ref. /4/, §23,2 (3)). The assertion follows by the mentioned statement

by /4/, since besides $(\mathfrak{D},\mathfrak{D}')$ the dual pair $(\mathfrak{D},\widetilde{\mathfrak{D}'})$ is semireflexive, too. Therefore, $(\mathfrak{D},\mathfrak{D}''')$ is semireflexive.

<u>Lemma 3.6</u>

Let $\mathfrak{D}$ be closed with the properties (A) and (B). Then $\mathfrak{D}[t'']$ is reflexive.

Proof: On account of Lemma 3.5 it is sufficient to show that the strong topology $t^{(4)} := \beta(\mathfrak{D},\mathfrak{D}''')$ coincides with $t'' = \beta(\mathfrak{D},\mathfrak{D}')$. But this follows by (B).

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Received by Publishing Department
on April 24 1979.