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Градуированные и фильтрованные топологические $*$-алгебры. Замыкание положительного конуса

В работе показано, что для некоторых локально-выпуклых топологий на фильтрованных $*$-алгебрах конус всех финитных сумм квадратов совпадает с конусом всех бесконечных сходящихся сумм квадратов, подобно случаю алгебры $S_{\otimes}$ основных функций. Результаты применены к тензорным алгебрам и симметризованным алгебрам над ядерными пространствами Фреше с инволюцией и к конечнопорожденным $*$-алгебрам таким, как алгебра многочленов Вейля и обертывающая алгебра.

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Graded and Filtrated Topological $*$-Algebras. The Closure of the Positive Cone

It is shown that for certain graded locally convex topologies on a filtrated $*$-algebra the closure of the cone of all finite sums of squares is precisely the cone of all infinite (convergent) sums of squares, similar to the case of the test function algebra $S_{\otimes}$. The result applies to tensor algebras and symmetrized tensor algebras over involutive nuclear Frechet spaces and to some finitely generated $*$-algebras such as polynomial algebras, the Weyl algebra and enveloping algebras.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

This paper is a continuation of /6/. We shall use some of the results, definitions and notations from /6/. In order to make the paper as self-contained as possible we repeat the essential definitions below.

The main object of this note is to prove that for certain graded locally convex topologies τ on a filtrated $*$-algebra $\mathcal{A}$ the closure of the cone $\mathcal{P}(\mathcal{A}):=\left\{\sum_{i=0}^{r} x_i^+ x_i, x_i \in \mathcal{A}, r \in N\right\}$ of all finite sums of squares coincides with the cone $\mathcal{P}_{\infty}(\mathcal{A})_{\tau}:=\left\{\sum_{i=0}^{\infty} x_i^+ x_i, x_i \in \mathcal{A}\right\}$ of all τ-convergent infinite sums of squares, similar to the case of the test function algebra $\mathcal{S}_{\otimes}$ (/1/,/2/). The proof will be given by two steps. Firstly, we show that $\overline{\mathcal{P}(\mathcal{A})}^{\tau} \cap \mathcal{A}^{2n} = \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}^{\tau}$ $\forall$ $n \in N$. This allows to reduce the problem to $\mathcal{A}^{2n}[\tau]$. Secondly, we prove that each element $x \in \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}^{\tau}$ can be represented as an infinite sum of squares. The first step will be done in Sec. 2. For this part we only assume that the topology τ is a graded locally convex topology generated by seminorms which satisfy condition (a). In Sec. 3 we carry out the second step. Here we use the main idea from Borchers' proof of the corresponding result for $\mathcal{S}_{\otimes}$. For the second step we assume among others that $\mathcal{A}^n[\tau]$ is a nuclear Frechet space.

Now we collect the basic definitions and notations.

Suppose a $*$-algebra $\mathcal{A}$ with unit element 1 is the direct sum of $*$-invariant vector spaces $\mathcal{A}_k, k\in N$, where $\mathcal{A}_o := C\cdot 1$. Let $\mathcal{A}^n = \sum_{k=o}^{n} \mathcal{A}_k$. We say $\mathcal{A}$ is a graded [filtrated] $*$-algebra if $\mathcal{A}_n \cdot \mathcal{A}_k \subseteq \mathcal{A}_{n+k}$ $[\mathcal{A}^n \cdot \mathcal{A}^k \subseteq \mathcal{A}^{n+k}]$ $\forall$ $n,k\in N$. Denote by $x_k = P_k x$ the canonical projection of $x\in\mathcal{A}$ into $\mathcal{A}_k$. Concerning the notation of elements (for example x_n^i) we adopt the following convention throughout the paper. Upper letters such as i are always indices. They nowhere refer to powers of elements. Lower letters such as n are either indices (in Sec. 3) or the components of the elements (in Sec. 2).

Let $\{q_j, j\in \mathcal{J}\}$ be a family of seminorms on the filtrated $*$-algebra $\mathcal{A} = \sum \mathcal{A}_k$. Put $q_j^+(x) := \max(q_j(x), q_j(x^+))$. The locally convex topology (briefly, l.c.t.) on $\mathcal{A}$ defined by the seminorms

$q_{j,\gamma}(x) := \sum_k \gamma_k q_j^+(P_k x)$, $j\in\mathcal{J}$, $\gamma = \{\gamma_k\}$ an arbitrary positive real sequence, is called the graded l.c.t. generated by $\{q_j, j\in\mathcal{J}\}$.

We say a seminorm q_j satisfies condition (a) or (b) if there are constants $C_{j,k,r,s}$ resp. $C_{j,r}$ so that

$$\text{(a)}\quad q_j\Big(\sum_{i=o}^{m} P_k(x_r^{i+} x_s^i)\Big) \leq C_{j,k,r,s}\, q_j\Big(\sum_{i=o}^{m} P_{2r}(x_r^{i+} x_r^i)\Big)^{1/2} q_j\Big(\sum_{i=o}^{m} P_{2s}(x_s^{i+} x_s^i)\Big)^{1/2}$$

$$\text{(b)}\quad q_j\Big(\sum_{i=o}^{m} P_{2r}(x_r^{i+} x_r^i)\Big) \leq C_{j,r}\, q_j\Big(\sum_{i=o}^{m+n} P_{2r}(x_r^{i+} x_r^i)\Big)$$

for all $x_r^i \in \mathcal{A}_r$, $x_s^i \in \mathcal{A}_s$, $r,s,k,m,n\in N$, $j\in\mathcal{J}$.

We denote by $\overline{\mathcal{M}}^{\tau}$ or simply by $\overline{\mathcal{M}}$ the closure of the set $\mathcal{M}$ w.r.t. the topology τ .

2. The Proof of $\overline{\mathcal{P}(\mathcal{A})} \cap \mathcal{A}^{2n} = \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}$

PROPOSITION 1: Let $\mathcal{A} = \sum \mathcal{A}_k$ be a filtrated $*$-algebra and $\{q_j\}$ a family of seminorms on $\mathcal{A}$ which satisfy condition (a). Let τ be the graded l.c.t. on $\mathcal{A}$ generated by $\{q_j\}$.

Then $\overline{\mathcal{P}(\mathcal{A})}^{\tau} \cap \mathcal{A}^{2n} = \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}^{\tau}$ $\forall n\in N$.

We start with a technical lemma about quadratic forms stated in a convenient form.

LEMMA 2: Let $Q_1(t)$ be a real quadratic form in the finite real sequence $t=(t_o, t_1, \ldots)$ so that $Q_1(t)=0$ if $t_{n+1}=t_{n+2}=\ldots=0$. Let $\eta = \eta(z_o,\ldots,z_n)$ be a real function of n+1 real variables and $L_{k,r,s}$ real numbers with $L_{k,r,s} = L_{k,s,r}$.

Then there exists a sequence $\{\delta_{2k}, k\in N\}$ of positive numbers such that the quadratic form

$$Q(t) := \sum_k \delta_{2k}\Big(t_k^2 - \sum_{\substack{(r,s)\neq(k,k)\\ r+s \geq 2k}} L_{k,r,s} t_r t_s \Big) - Q_1(t)\ \eta(\delta_o, \ldots, \delta_{2n})$$

is positive definite in the finite real sequence $t=(t_o, t_1, \ldots)$.

Proof:

The lemma will be shown by a slight modification of the induction argument used in the proof of lemma 3.2 in /6/. Putting $Q_o(t)=0$ we begin just as in lemma 3.2 and construct positive numbers $\delta_o, \ldots, \delta_{2n}$ such that the form $\sum_k \delta_{2k}\Big(t_k^2 - \sum_{\substack{(r,s)\neq(k,k)\\ r+s \geq 2k}} L_{k,r,s}\, t_r t_s\Big)$ is positive definite in $t=(t_o, t_1, \ldots t_n, 0, \ldots)$. Then we set $\eta = \eta(\delta_o, \ldots, \delta_{2n})$ and continue the induction procedure with the quadratic form $Q_o(t) := \eta\ Q_1(t)$. //

Proof of Proposition 1:

It suffices to prove that $\overline{\mathcal{P}(\mathcal{A})} \cap \mathcal{A}^{2n} \subseteq \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}$ because the converse inclusion is trivial. Let $x \in \overline{\mathcal{P}(\mathcal{A})} \cap \mathcal{A}^{2n}$ and $\varepsilon > 0$. Consider a fixed seminorm $q_{j,\gamma}$ whereby q_j satisfies (a). Without loss of generality we may assume that $\gamma_k = 1 \ \forall k \in \mathbb{N}$, $q_j = q_j^+$ and $C_{j,k,r,s} = C_{j,k,s,r}$. We put $L_{k,r,s} = C_{j,k,r,s}$, $\eta = \eta(z_o, \dots, z_n) = \max\left(1, \sum_{i=o}^{n} 4/\varepsilon \cdot z_k q_j(x_{2k})\right)$ and

$$Q_1(t) = \sum_{k \geq n+1} \Big[\sum_{\substack{r+s \geq k \\ r,s \geq n+1}} C_{j,k,r,s} t_r t_s + \sum_{\substack{r+s \geq k \\ n \geq s, r \geq n+1}} C_{j,k,r,s} t_r t_s \Big].$$

According to lemma 1 there is a positive sequence $\{\delta_{2k}, k \in \mathbb{N}\}$ so that

$$\sum_k \delta_{2k}\Big(t_k^2 - \sum_{\substack{(r,s) \neq (k,k) \\ r+s \geq 2k}} L_{k,r,s} t_r t_s\Big) \geq \eta(\delta_o, \dots, \delta_{2n}) Q_1(t) \quad (1)$$

for all finite real sequences t.

Since the topology τ is graded, the seminorm $q_{j,\delta}(a) := \sum_k \delta_{2k} q_j(a_{2k})$ is τ-continuous. Since $x \in \overline{\mathcal{P}(\mathcal{A})}$, there is an element $y = \sum_{i=o}^{m} y^{i+} y^i \in \mathcal{P}(\mathcal{A})$ such that $q_{j,\gamma}(x-y) \leq \varepsilon/2$ and $q_{j,\delta}(x-y) \leq \varepsilon/4$. Let $a^i := (y_o^i, \dots, y_n^i, 0, \dots)$ and $c := y - \sum_{i=o}^{m} a^{i+} a^i$. Let us assume for a moment that we have shown $q_{j,\gamma}(c) \leq \varepsilon/2$. Then $q_{j,\gamma}\left(x - \sum_{i=o}^{m} a^{i+} a^i\right) \leq q_{j,\gamma}(x-y) + q_{j,\gamma}(c) \leq \varepsilon$. Since $\sum_{i=o}^{m} a^{i+} a^i \in \mathcal{A}^{2n}$, this implies that $x \in \overline{\mathcal{P}(\mathcal{A}) \cap \mathcal{A}^{2n}}$.

Thus it remains to prove that $q_{j,\gamma}(c) \leq \varepsilon/2$.

Firstly, putting $t_k = q_j\left(\sum_i P_{2k}(y_{2k}^{i+} y_{2k}^i)\right)^{1/2} \ \forall k \in \mathbb{N}$ and using (a) we obtain

$$q_{j,\gamma}(c) = \sum_{k \geq n+1} q_j(P_k c) = \sum_{k \geq n+1} q_j\Big(\sum_{\substack{r+s \geq k \\ r,s \geq n+1}} P_k(y_r^{i+} y_s^i) + \sum_{\substack{r+s \geq k \\ n \geq s, r \geq n+1}} P_k(y_r^{i+} y_s^i + y_s^{i+} y_r^i)\Big)$$

$$\leq \sum_{k \geq n+1} \Big[\sum_{\substack{r+s \geq k \\ r,s \geq n+1}} C_{j,k,r,s}\, q_j\Big(\sum_i P_{2r}(y_r^{i+} y_r^i)\Big)^{1/2} q_j\Big(\sum_i P_{2s}(y_s^{i+} y_s^i)\Big)^{1/2}$$

$$+ \sum_{\substack{r+s \geq k \\ n \geq s, r \geq n+1}} 2C_{j,k,r,s}\, q_j\Big(\sum_i P_{2r}(y_r^{i+} y_r^i)\Big)^{1/2} q_j\Big(\sum_i P_{2s}(y_s^{i+} y_s^i)\Big)^{1/2} \Big]$$

$$\equiv Q_1(t). \quad (2)$$

By the triangle inequality and (a),(1) it follows that

$$q_{j,\delta}(y) = \sum_k \delta_{2k} q_j(P_{2k} y) = \sum_k \delta_{2k} q_j\Big(\sum_{r+s \geq 2k} \sum_i P_{2k}(y_r^{i+} y_s^i)\Big) \underset{(a)}{\geq}$$

$$\sum_k \delta_{2k} \Big[q_j\Big(\sum_i P_{2k}(y_k^{i+} y_k^i)\Big) - \sum_{\substack{(r,s) \neq (k,k) \\ r+s \geq 2k}} C_{j,k,r,s}\, q_j\Big(\sum_i P_{2r}(y_r^i y_r^i)\Big)^{1/2} \cdot q_j\Big(\sum_i P_{2s}(y_s^i y_s^i)\Big)^{1/2} \Big]$$

$$\equiv \sum_k \delta_{2k}\Big(t_k^2 - \sum_{\dots} C_{j,k,r,s} t_r t_s\Big) \underset{(1)}{\geq} \eta(\delta_o, \dots, \delta_{2n}) Q_1(t). \quad (3)$$

Combining (2) and (3) and taking into account that by construction $\eta(\delta_o, \dots, \delta_{2n})^{-1} \leq 1$ and $\eta(\delta_o, \dots, \delta_{2n})^{-1} \cdot \sum_{k=o}^{n} \delta_{2k} q_j(x_{2k}) \leq \varepsilon/4$, we get

$$q_{j,\gamma}(c) \leq \eta^{-1} q_{j,\delta}(y) \leq \eta^{-1} q_{j,\delta}(x) + \eta^{-1} q_{j,\delta}(x-y) \leq \varepsilon/4 + q_{j,\delta}(x-y) \leq \varepsilon/2.$$

This completes the proof of proposition 1. //

The following corollary is an immediate consequence of proposition 1.

COROLLARY 3: Let $\mathcal{A} = \sum \mathcal{A}_k$ and τ as in prop. 1. Suppose in addition that $\mathcal{A}^n[\tau]$ is a metrizable space for each $n \in \mathbb{N}$.

Then $\overline{\mathcal{P}(\mathcal{A})}^{\tau}$ coincides with the τ-sequence closure of $\mathcal{P}(\mathcal{A})$.

Note that this corollary is not trivial, since the topology τ is not metrizable on $\mathcal{A}$.

3. The Proof of $\overline{\mathcal{P}(\mathcal{A})} = \mathcal{P}_\infty(\mathcal{A})$

The main result of this paper is

THEOREM 1: Let $\mathcal{A} = \sum \mathcal{A}_k$ be a filtrated $*$-algebra and $\{q_j\}$ a family of seminorms on $\mathcal{A}$ satisfying (a) and (b). Let τ be the graded l.c.t. generated by $\{q_j\}$. Suppose $\mathcal{A}^n[\tau]$ is a nuclear Frechet space for each $n\in N$. Suppose that the multiplication $\mathcal{A}^n[\tau] \ni y \longrightarrow xy \in \mathcal{A}^{2n}[\tau]$ is continuous for all $x\in\mathcal{A}^n$. -

Then we have $\overline{\mathcal{P}(\mathcal{A})}^{\tau} = \mathcal{P}_\infty(\mathcal{A})_\tau$.

Our proof is based on the following lemma which seems to be of interest in itself. This lemma was advised to the author by K.-D. Kürsten.

LEMMA 2: Let $E[\tau]$ be a metrizable locally convex space with a continuous involution $x \longrightarrow x^+$ and $E\hat{\otimes}E$ the completion of $E\otimes E$ w.r.t. the ε-topology $\tau\otimes_\varepsilon\tau$. Let $\mathcal{P} = \left\{\sum_{i=0}^{r} x_i^+\otimes x_i, x_i\in E, r\in N\right\}$ and $\mathcal{P}_\infty = \left\{\sum_{i=0}^{\infty} x_i^+\otimes x_i, x_i\in E\right\}$ where the convergence is meant in the ε-topology.

Then $\mathcal{P}_\infty$ is the $\tau\otimes_\varepsilon\tau$-closure of $\mathcal{P}$ in $E\hat{\otimes}E$.

Proof:

First let us recall the concept of the ultra product of Hilbert spaces (see e.g./3/). Let $\{\mathcal{H}_n, n\in N\}$ be a sequence of Hilbert spaces and $\mathcal{U}$ an ultra filter on N containing all sets $N_k := \{n\in N: n \geq k\}$. Let $\mathcal{L} = \{(x_n): x_n\in\mathcal{H}_n, \|(x_n)\| := \sup_{n\in N}\|x_n\| < \infty\}$ and $\mathcal{N} = \{(x_n)\in\mathcal{L}: \lim_{\mathcal{U}}\|x_n\| = 0\}$. The factor space $(\mathcal{H}_n)_{\mathcal{U}} :=$ endowed with the scalar product $\langle (x_n),(y_n)\rangle := \lim_{\mathcal{U}}\langle x_n, y_n\rangle$ is a Hilbert space which is called the ultra product of the family $\{\mathcal{H}_n\}$ w.r.t. the ultra filter $\mathcal{U}$.

Suppose now that $x\in E\hat{\otimes}E$ is the $\tau\otimes_\varepsilon\tau$-limit of $y_n = \sum_{i=0}^{r_n} y_{ni}^+\otimes y_{ni}$, $y_{ni}\in E$. Let $\mathcal{H}_n = l_2$ and $e_{ni} = \{\delta_{ik}, k\in N\}$ be the unit vector base of $\mathcal{H}_n$. Further, let $F_n(f) = \sum_{i=0}^{r_n} f(y_{ni})e_{ni} \in \mathcal{H}_n$ for each $f\in E'$. Since $\sup_n \|F_n(f)\|^2 = \sup_n \sum_{i=0}^{r_n}|f(y_{ni})|^2 = \sup_n (f^+\otimes f)(y_n) < \infty$, this induces a map $E'\ni f \longrightarrow F(f) = (F_n(f)) \in (\mathcal{H}_n)_{\mathcal{U}}$ for which

$\langle F(f),F(g)\rangle = \lim_{\mathcal{U}}\langle F_n(f),F_n(g)\rangle = \lim_{\mathcal{U}} \sum_{i=0}^{r_n} f(y_{ni})\overline{g(y_{ni})} = \lim_{n\to\infty}\sum_{i=0}^{r_n} f(y_{ni})\overline{g(y_{ni})} = (g^+\otimes f)(x)$. Since $\|F(f)-F(g)\|^2 = \langle F(f),F(f)\rangle - \langle F(f),F(g)\rangle - \langle F(g),F(f)\rangle + \langle F(g),F(g)\rangle$, this implies that the map $E'[\sigma] \ni f \longrightarrow F(f)\in(\mathcal{H}_n)_{\mathcal{U}}$ is norm-continuous. Here σ is the weak topology w.r.t. the dual pair (E,E'). Let $\{p_n, n\in N\}$ be a sequence of $*$-invariant seminorms on E which define the topology τ. By the Alaoglu-Bourbaki Theorem the polars $U^o_{p_n}$ are compact subsets of $E'[\sigma]$. Because the map $f\longrightarrow F(f)$ is continuous, the image of each set $U^o_{p_n}$ is norm-compact in the Hilbert space $(\mathcal{H}_n)_{\mathcal{U}}$. Therefore the set $\left\{\bigcup_{n\in N}\bigcup_{f\in U^o_{p_n}} F(f)\right\}$ is contained in a separable closed subspace $\mathcal{H}$ of $(\mathcal{H}_n)_{\mathcal{U}}$. Let $\{e_i, i\in N\}$ be an orthobase of $\mathcal{H}$ and $x_i(f) := \langle F(f), e_i\rangle$. By the continuity of the map $f\longrightarrow F(f)$, $x_i(f)$ is a σ-continuous linear functional on E', that is, $x_i\in E$.

It remains to prove that $x = \sum_{i=0}^{\infty} x_i^+\otimes x_i$. We have

$$p_n\otimes_\varepsilon p_n\Big(x - \sum_{i=0}^{k} x_i^+\otimes x_i\Big) = \sup_{f,g\in U^o_{p_n}} \Big| g\otimes f(x) - \sum_{i=0}^{k} g(x_i^+)f(x_i)\Big| =$$

$$\sup_{\dots}\Big|\langle F(f),F(g^+)\rangle - \sum_{i=0}^{k} f(x_i)\overline{g^+(x_i)}\Big| = \sup_{\dots}\Big|\sum_{i=k+1}^{\infty} f(x_i)\overline{g^+(x_i)}\Big|$$

$$\leq \sup_{f,g\in U^o_{p_n}}\Big(\sum_{i=k+1}^{\infty}|f(x_i)|^2\Big)^{1/2}\Big(\sum_{i=k+1}^{\infty}|g(x_i)|^2\Big)^{1/2}.$$

Since $\{F(f), f\in U^o_{p_n}\}$ is a compact set in the Hilbert space $\mathcal{H}$, the rigth-hand side can be made arbitrary small by taking k large enough. This proves lemma 2. //

Proof of Theorem 1:

Statement I: The map $\mathcal{A}^n \hat{\otimes} \mathcal{A}^n[\tau\otimes_\pi\tau] \ni \sum_i x_i\, y_i \xrightarrow{M} \sum_i x_i y_i \in \mathcal{A}^{2n}[\tau]$ is continuous.

Proof: It is sufficient to show the continuity for finite sums. Let q be a τ-continuous seminorm on $\mathcal{A}$. Since the right multiplications (by assumption) and the involution (by definition) are τ-continuous, the left multiplications are τ-continuous. Since $\mathcal{A}^{2n}[\tau]$ is a Frechet space, the multiplication is a continuous map of $\mathcal{A}^n[\tau] \times \mathcal{A}^n[\tau]$ into $\mathcal{A}^{2n}[\tau]$, that is, there exists a τ-continuous seminorm p on $\mathcal{A}$ so that $q(xy) \leq p(x)p(y)\ \forall\ x,y\in\mathcal{A}^n$. If $u=\sum_i x_i\otimes y_i = \sum_j a_j\otimes b_j$, then $v=\sum_i x_i y_i = \sum_j a_j b_j$. Further, $q(v)\leq\sum_i p(x_i)p(y_i)$ which implies $q(v)\leq p\otimes_\pi p(u)$. //

Statment II: If $\{u_n := \sum_{i=0}^{r_n} x_{ni}^+ \cdot x_{ni}, n\in N\}$ is bounded in $\mathcal{A}^{2n}[\tau]$,

then $v_n = \{\sum_{i=0}^{r_n} x_{ni}^+ \otimes x_{ni}, n\in N\}$ is bounded in $\mathcal{A}^n\otimes\mathcal{A}^n[\tau\otimes_\varepsilon\tau]$.

Proof: Since the seminorms $\{q_j\}$ satisfy (a) and (b), τ is a normal topology on $\mathcal{A}$ by theorem 3.1 of /6/. Therefore τ can be given by seminorms of the form $p_{\mathfrak{M}}(a) = \sup_{f\in\mathfrak{M}} |f(a)|$ where $\mathfrak{M}$ is a certain weakly bounded set of continuous linear functionals f on $\mathcal{A}[\tau]$. Using lemma 3.2 of /6/ and the Cauchy-Schwarz inequality, we get

$$p_{\mathfrak{M}}\otimes_\varepsilon p_{\mathfrak{M}}(v_n) = \sup_{f\in\mathfrak{M}} \sum_{i=0}^{r_n} |f(x_{ni})|^2 \leq \sup_{f\in\mathfrak{M}} \sum_{i=0}^{r_n} f(x_{ni}^+ x_{ni}) f(1) = p_{\mathfrak{M}}(1)p_{\mathfrak{M}}(u_n).$$ //

We now complete the proof of the theorem. By proposition 1 of the preceding section, it suffices to show that $\overline{\mathcal{P}(\mathcal{A})\cap\mathcal{A}^{2n}}^{\tau} \subseteq \mathcal{P}_\infty(\mathcal{A})_\tau$. Let $\{u_n = \sum_{i=0}^{r_n} x_{ni}^+ x_{ni}\}$ be a sequence of $\mathcal{P}(\mathcal{A})\cap\mathcal{A}^{2n}$ converging to $u\in\mathcal{A}^{2n}$. Then the set $\{v_n = \sum_{i=0}^{r_n} x_{ni}^+ \otimes x_{ni}, n\in N\}$ is bounded in $\mathcal{A}^n\otimes\mathcal{A}^n[\tau\otimes_\varepsilon\tau]$ by statement II. Since $\mathcal{A}^n[\tau]$ was assumed to be a nuclear Frechet space, it follows that $\tau\otimes_\varepsilon\tau = \tau\otimes_\pi\tau$ on $\mathcal{A}^n\otimes\mathcal{A}^n$ and that $\mathcal{A}^n\hat{\otimes}\mathcal{A}^n[\tau\otimes_\varepsilon\tau]$ is again a nuclear Frechet space (/4/). Furthermore, the bounded subset of this nuclear space has a convergent subsequence $v_{n'} \longrightarrow v\in\mathcal{A}^n\hat{\otimes}\mathcal{A}^n$ (/4/). Now, by lemma 2, v can be represented as a $\tau\otimes_\varepsilon\tau$-convergent sum $\sum_{i=0}^{\infty} x_i^+\otimes x_i, x_i\in\mathcal{A}^n$. By the continuity of M, we see that $u_{n'} = M(v_{n'}) \longrightarrow u = M(v)$, that is, $u\in\mathcal{P}_\infty(\mathcal{A})_\tau$. //

Remarks:

1. Suppose the locally convex space E in lemma 2 is finite dimensional. By choosing a base for the linear space E, consisting of hermitian vectors, the elements of $\mathcal{P}$ are in one-to-one correspondence with the positive semi-definite matrices. Hence $\mathcal{P} = \mathcal{P}_\infty$. In fact, each element of $\mathcal{P}_\infty$ can be represented as a sum $\sum_{i=0}^{n} x_i^+ x_i$ with $n+1 \leq \dim E$.

2. If in theorem 1 all vector spaces $\mathcal{A}^n$ of the filtrated $*$-algebra $\mathcal{A} = \sum\mathcal{A}_k$ are finite dimensional, then, of course, the graded l.c.t. τ coincides with the strongest l.c.t. τ_{st} on $\mathcal{A}$. In this case, $\mathcal{P}(\mathcal{A})$ is τ_{st}-closed. This follows immediately from the above proof combined with the preceding remark.

4. Applications

THEOREM 1: Let $E[\tau]$ be a nuclear Frechet space with continuous involution $x \longrightarrow x^+$ and let $\mathcal{A} = \underline{E}_\otimes^{\mathcal{G}}$, $\mathcal{G} = \{G_n\}$, the graded $*$-algebra defined in / 6 /, Sec. 5. Let $\mathfrak{f}$ be a l.c.t. on $\mathcal{A}$ so that $\tau_\infty \subseteq \mathfrak{f} \subseteq \tau_\otimes$.
Then $\overline{\mathcal{P}(\mathcal{A})}^{\mathfrak{f}} = \mathcal{P}_\infty(\mathcal{A})_{\tau_\otimes}$, i.e. the $\mathfrak{f}$-closure of $\mathcal{P}(\underline{E}_\otimes^{\mathcal{G}})$ is precisely the set of all $\tau_\otimes$-convergent infinite sums $\sum_{i=0}^{\infty} x_i^+ x_i$, $x_i \in \underline{E}_\otimes^{\mathcal{G}}$. In particular, this is true for the completed tensor algebra $\underline{E}_\otimes$.

Proof:
Note first that $\mathcal{P}_\infty(\mathcal{A})_{\tau_\infty} = \mathcal{P}_\infty(\mathcal{A})_{\tau_\otimes}$, since the topologies τ_∞ and $\tau_\otimes$ have the same convergent sequences. Because $\mathcal{P}_\infty(\mathcal{A})_{\tau_\otimes} \subseteq \overline{\mathcal{P}(\mathcal{A})}^{\mathfrak{f}}$, it is sufficient to prove that $\overline{\mathcal{P}(\mathcal{A})}^{\tau_\infty} = \mathcal{P}_\infty(\mathcal{A})_{\tau_\infty}$. According to theorem 5.1 of / 6 /, τ_∞ is generated by seminorms fulfilling (a) and (b). Now theorem 3.1 applies. / /

THEOREM 2: Suppose $\mathcal{A}$ is one of the following $*$-algebras:

- the free polynomial algebra in n hermitian indeterminants,
- the polynomial algebra in n commuting hermitian indeterminants,
- the Weyl algebra, i.e. the $*$-algebra generated by the canonical commutation relations,
- the universal enveloping algebra of a finite dimensional Lie algebra.

Then $\mathcal{P}(\mathcal{A})$ is closed in the strongest l.c.t. on $\mathcal{A}$.

Proof:
Each of these $*$-algebras has a natural filtration $\mathcal{A} = \sum \mathcal{A}_k$ for which all vector spaces $\mathcal{A}^n$ are finite dimensional. In / 7 / it was shown that $\mathcal{P}(\mathcal{A})$ is normal w.r.t. the strongest l.c.t. τ_{st}

on $\mathcal{A}$. From these proofs one can see that the topology τ_{st} is generated by seminorms which satisfy (a) and (b). Thus the assertion follows from theorem 3.1 and remark 3.2. / /

Concluding Remarks:

1. The case of the tensor algebra $\underline{\mathcal{S}}_\otimes$ over the Schwartz space $\mathcal{S}(R_d)$ was already treated in / 1 / (the second step in the terminology of our paper) and in / 2 / (the first step). In this special case, our proof seems to be simpler. For enveloping algebras the assertion of theorem 2 was shown in / 7 /.

2. Without the assumption that τ can be defined by seminorms $\{q_j\}$ which satisfy (a) and (B) (hence, $\mathcal{P}(\mathcal{A})$ is τ-normal) the assertions of theorems 3.1 and 4.2 are no longer true. We include a simple counter-example. Let $\mathcal{A}$ be the $*$-algebra of all polynomials in a generator x for which $x=x^+$ and $x^2=0$. Of course, $\mathcal{A}$ is a graded $*$-algebra. The norm topology τ of the two dimensional vector space $\mathcal{A}$ is a graded l.c.t. fulfilling all other assumptions of theorems 3.1 and 4.2. $\mathcal{P}_\infty(\mathcal{A})_\tau$ can be identified with the set $\{(\alpha,\beta)\in R_2 : \alpha > 0\} \cup \{(0,0)\}$. Clearly, $\mathcal{P}(\mathcal{A})$ is not τ-normal and $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{P}_\infty(\mathcal{A})_\tau \neq \overline{\mathcal{P}(\mathcal{A})}^\tau \equiv \{(\alpha,\beta)\in R_2 : \alpha \geq 0\}$.

3. For topologies τ weaker than graded topologies (for example, when the seminorms $q_{j,\gamma}$ are not continuous for all positive sequences γ) $\overline{\mathcal{P}(\mathcal{A})}^\tau = \mathcal{P}_\infty(\mathcal{A})_\tau$ is false. Let $\mathcal{A}$ be the polynomial algebra in two real variables, $\mathcal{A}_+ := \{p \in \mathcal{A} : p(t_1,t_2) \geq 0 \ \forall (t_1,t_2) \in R_2\}$ and τ the topology of uniform convergence on compact subsets of the real plane. Then we have $\overline{\mathcal{P}(\mathcal{A})}^\tau = \mathcal{A}_+$ (/ 7 /, p.). Since there exist polynomials $p \in \mathcal{A}_+$ which are not infinite sums of squares in the pointwise convergence, it follows that $\mathcal{P}_\infty(\mathcal{A})_\tau \neq \overline{\mathcal{P}(\mathcal{A})}^\tau \equiv \mathcal{A}_+$.

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