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О разложениях по произведениям решений двух систем Дирака

Методом контурного интегрирования получены формулы обращения для разложения вектор-функции $h(x) = (h_1, h_2) \in L_1^{(2)}(-\infty, \infty)$ по произведениям решений двух канонических систем Дирака, в которых коэффициенты убывают достаточно быстро при $|x| \to \infty$. При этом для простоты предполагается, что их дискретный спектр исчерпывается конечным числом простых невещественных собственных чисел.

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On the Expansions over the Products of Solutions of Two Dirac Systems

Using the contour integration method we obtain inversion formulae for the expansion of the vector-function $h(x) = (h_1, h_2) \in L_1^{(2)}(-\infty, \infty)$ over the products of solutions of two canonical Dirac systems, provided their coefficients fall off fast enough as $|x| \to \infty$. For simplicity we assume that their discrete spectra consist only of a finite number of simple non-real eigenvalues.

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1. Let us consider two one-dimensional Dirac systems:

$$\left[\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\frac{d}{dx}+\begin{pmatrix}0 & q_n(x)\\ -r_n(x) & 0\end{pmatrix}\right]\begin{pmatrix}y_{n,1}\\ y_{n,2}\end{pmatrix}=i\zeta\begin{pmatrix}y_{n,1}\\ y_{n,2}\end{pmatrix},\ n=1,2 \quad (1)$$

with complex-valued integrable on the whole axis $-\infty < x < \infty$ coefficients $q_n(x)$, $r_n(x)$. Let us denote by $\phi_n^{\pm}(x,\zeta)=(\phi_{n,1}^{\pm},\phi_{n,2}^{\pm})^T$, $\psi_n^{\pm}(x,\zeta)=(\psi_{n,1}^{\pm},\psi_{n,2}^{\pm})^T$ (T means transposition) the Jost solutions, determined by the conditions:

$$\lim_{x\to-\infty}\phi_n^{+}(x,\zeta)e^{i\zeta x}=\begin{pmatrix}1\\0\end{pmatrix},\ \lim_{x\to\infty}\psi_n^{+}(x,\zeta)e^{-i\zeta x}=\begin{pmatrix}0\\1\end{pmatrix},\ \zeta\in\bar{C}^{+}, \quad (2)$$

$$\lim_{x\to-\infty}\phi_n^{-}(x,\zeta)=e^{-i\zeta x}=\begin{pmatrix}0\\-1\end{pmatrix},$$

$$\lim_{x\to\infty}\psi_n^{-}(x,\zeta)e^{i\zeta x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \zeta\in\bar{C}^{-}, \quad (3)$$

where

$$C^{+}(C^{-})=\{\zeta\in C \mid \operatorname{Im}\zeta>0\ (\operatorname{Im}\zeta<0)\},\ \bar{C}^{\pm}=C^{\pm}\cup R$$

and let

$$a_n^{\pm}(\zeta)=W(\phi_n^{\pm},\psi_n^{\pm})\equiv\phi_{n,1}^{\pm}\psi_{n,2}^{\pm}-\phi_{n,2}^{\pm}\psi_{n,1}^{\pm}\ ,\zeta\in\bar{\mathbf{C}}^{\pm}\ . \qquad (4)$$

Formula of the type (4) means for each fixed $n=1,2$ two relations: the first is obtained by taking everywhere the upper sign (in our case +), the second, by taking the lower sign.

Let us define the product $y_p \circ y_q$ of the solutions

$$y_p=(y_{p,1}\ ,\ y_{p,2}\ )^T \qquad \text{and} \qquad y_q=(y_{q,1}\ ,\ y_{q,2}\ )^T$$

of the systems (1) with $n=p,q$ by

$$y_p\circ y_q\ (x,\zeta)=(\,y_{p,1}y_{q,1}\ ,\ y_{p,2}\,y_{q,2}\ )^T\ ,p,q=1,2 \qquad (5)$$

and let us denote

$$y_p\circ y_q\,(x,\zeta)=(y_{p,2}\,y_{q,2},\ -y_{p,1}\,y_{q,1}\ ). \qquad (6)$$

Using the solutions $\psi_n^{\pm}$ and $\phi_n^{\pm}$ we can construct two systems of vector-functions $\{\Psi\}$ and $\{\Phi\}$ respectively:

$$\Psi^{\pm}(x,\zeta)=\psi_1^{\pm}\circ\psi_2^{\pm}\ ,(\zeta\in\boldsymbol{R}\cup\sigma_1^{\pm}\cup\sigma_2^{\pm}\ ),$$

$$\dot{\Psi}^{\pm}\ (x,\zeta_k^{\pm}),\ \ (\zeta_k^{\pm}\in\sigma_{\pm}'\ ),$$

$$\Phi^{\pm}(x,\zeta)=\phi_1^{\pm}\circ\phi_2^{\pm},(\zeta\in \boldsymbol{R}\cup\sigma_1^{\pm}\cup\sigma_2^{\pm}),$$

$$\dot{\Phi}^{\pm}(x,\zeta_k^{\pm}),\ (\zeta_k^{\pm}\in\sigma'_{\pm}),$$

where the dot means the derivative with respect to ζ, and

$$\sigma_n^{\pm}=\{\zeta_{n,k}^{\pm}\in \mathbf{C}^{\pm}\ |a_n^{\pm}(\zeta_{n,k}^{\pm})=0\}_{k=1}^{N_n^{\pm}},\ \sigma'_{\pm}=\sigma_1^{\pm}\cap\sigma_2^{\pm}.$$

Let $h(x)=(h_1,h_2)^T$ be a vector-function from $L_1^{(2)}(-\infty,\infty)$ $(\int_{-\infty}^{\infty}(|h_1(x)|+|h_2(x)|)\,dx<\infty)$, and let

$$\Phi^{\pm}(h;\zeta)=\int_{-\infty}^{\infty}\tilde{\Phi}^{\pm}(x,\zeta)h(x)dx, \tag{7}$$

$$\dot{\Phi}_k^{\pm}(h)=\int_{-\infty}^{\infty}\tilde{\dot{\Phi}}^{\pm}(x,\zeta_k^{\pm})h(x)dx$$

be its expansion coefficients with respect to the system $\{\Phi\}$. The main result of this paper consists in

<u>Theorem 1.</u> Let two Dirac systems be given (1) with complex-valued, absolutely continuous coefficients $q_n(x)$ and $r_n(x)$, satisfying the condition

$$\int_{-\infty}^{\infty}\{(1+|x|)[|q_n(x)|+|r_n(x)|]+|\frac{d}{dx}q_n(x)|+ \\ +|\frac{d}{dx}r_n(x)|\}dx<\infty \tag{8}$$

and such that the functions $a_n^{\pm}(\zeta)$ (4) have a finite number of simple non-real zeroes $\zeta_{n,k}^{\pm}$, and $a_n^{\pm}(\zeta)\neq 0$ when $\zeta\in\boldsymbol{R}$.

Then the integrals (7) are absolutely convergent for any vector-function $h \in L_1^{(2)}(-\infty,\infty)$ and, if one introduces for $R<\infty$ $h_R(x)=h_R^+(x)+h_R^-(x)$ with

$$h_R^{\pm}(x)=\frac{\mp 1}{\pi}\int_{-R}^{R}\Psi^{\pm}(x,\zeta)\Phi^{\pm}(h;\zeta)\frac{d\zeta}{a^{\pm}(\zeta)}+$$

$$+2i\sum_{\zeta^{\pm}_{n,k}\in\sigma''_{\pm}}\frac{1}{\dot{a}^{\pm}_{n,k}}\Psi^{\pm}(x,\zeta^{\pm}_{n,k})\Phi^{\pm}_{n,k}(h) \qquad (9)$$

$$+4i\sum_{\zeta^{\pm}_{k}\in\sigma'_{\pm}}\frac{1}{\ddot{a}^{\pm}_{k}}\{\dot{\Psi}^{\pm}(x,\zeta^{\pm}_{k})\Phi^{\pm}_{k}(h)+$$

$$+\Psi^{\pm}(x,\zeta^{\pm}_{k})[\dot{\Phi}^{\pm}_{k}(h)-\frac{\dddot{a}^{\pm}_{k}}{3\ddot{a}^{\pm}_{k}}\Phi^{\pm}_{k}(h)]\},$$

where

$$a^{\pm}(\zeta)=a_1^{\pm}(\zeta)a_2^{\pm}(\zeta),\quad \dot{a}^{\pm}_{n,k}=\dot{a}^{\pm}(\zeta^{\pm}_{n,k})\ldots,$$

$$\sigma''_{\pm}=(\sigma_1^{\pm}\cup\sigma_2^{\pm})\setminus\sigma'_{\pm}\ ,$$

then uniformly with respect to x in every finite interval

$$\lim_{R\to\infty}|h_R(x)-\frac{1}{\pi}\int_{-R}^{R}\{\int_{-\infty}^{\infty}\begin{bmatrix}e^{2i\zeta(y-x)} & 0\\ 0 & e^{2i\zeta(x-y)}\end{bmatrix}\begin{pmatrix}h_1(y)\\ h_2(y)\end{pmatrix}dy\}d\xi|=0, \quad (10)$$

i.e., the expansion $h_R(x)$ is equiconvergent with the expansion of $h(x)$ in Fourier integral. In particular, if $h(x)$ has a bounded variation in the neighbourhood of x, then $\lim_{R\to\infty} h_R(x) = (1/2)(h(x+0)+h(x-0))$.

Remark. The inversion formula (10) under the additional condition

$$q_1(x)=q_2(x), \quad r_1(x)=r_2(x), \quad -\infty < x < \infty \tag{11}$$

has been proposed by Kaup[1]. It should be noted that the completeness problem for the products of solutions of two Sturm-Liouville problems on a finite interval, originating from the famous Borg paper on the uniqueness of the corresponding inverse problem, has been completely solved by Levitan[2]. The interest to the properties of the solutions of system (1) was highly increased after the pioneering Zakharov-Shabat paper[3] and the paper of Ablowitz, Kaup, Newell, and Segur[4] who discovered the significance of the direct and the inverse scattering problem for the Dirac operator for the solution of a number of important nonlinear evolution equations in mathematical physics.

The proof of theorem 1 we obtain here by constructions, similar to [5,6]. Let us introduce the 2x2 matrix:

$$\begin{aligned} G(x,y;\zeta) = 2i a^{-1}(\zeta)\{ & \psi_1\circ\psi_2(x,\zeta)\widetilde{\phi_1\circ\phi_2}(y,\zeta)\theta(x-y) + \\ & +[\psi_1\circ\phi_2(x,\zeta)\widetilde{\phi_1\circ\psi_2}(y,\zeta)+\phi_1\circ\psi_2(x,\zeta)\widetilde{\psi_1\circ\phi_2}(y,\zeta) - \\ & -\phi_1\circ\phi_2(x,\zeta)\widetilde{\psi_1\circ\psi_2}(y,\zeta)]\theta(y-x)\}, \end{aligned} \tag{12}$$

where $\phi_n=\phi_n^+$, $\psi_n=\psi_n^+$, $a=a^+$ when $\zeta\in\bar{\mathbb{C}}^+$, and $\phi_n=\phi_n^-$, $\psi_n=\psi_n^-$, $a=a^-$ when $\zeta\in\bar{\mathbb{C}}^-$.

It is well known that for fixed x the solutions $\phi_n^{\pm}(x,\zeta)$ and $\psi_n^{\pm}(x,\zeta)$, as well as the functions $a_n^{\pm}(\zeta)$, are regular functions of $\zeta \in \mathbf{C}^{\pm}$, continuous at $\zeta \in \bar{\mathbf{C}}^{\pm}$ and satisfy the estimates*:

$$|\phi^{\pm}_{n,j}(x,\zeta)e^{\pm i\zeta x}| \le K, \quad |\psi^{\pm}_{n,j}(x,\zeta)e^{\mp i\zeta x}| \le K, \tag{13}$$

$$\zeta \in \bar{\mathbf{C}}^{\pm}, \quad (n,j=1,2),$$

where the constant K is independent of x and ζ. This, together with the restrictions on $a_n^{\pm}(\zeta)$ gives us that the vector-functions $Q^{\pm}(x,\zeta)=\int_{-\infty}^{\infty} G^{\pm}(x,y;\zeta)h(y)dy$ for $h \in L_1^{(2)}$ are regular functions of $\zeta \in \mathbf{C}^{\pm}$ respectively, except for a finite number of poles, determined by the zeroes of $a^{\pm}(\zeta)$, and continuous for $\zeta=\xi \pm i0$. Let us consider the integral

$$I_R(x)=\frac{1}{2\pi i}\{\int_{\gamma_+} Q^+(x,\zeta)d\zeta - \int_{\gamma_-} Q^-(x,\zeta)d\zeta\}, \tag{14}$$

where the integration contour γ_+ runs along the upper side of the real axis $\mathbf{R}$ from $-R+i0$ to $+R+i0$ and is closed in $\mathbf{C}^+$ by the semicircle $\gamma_R^+ : \zeta = R\exp(i\phi)$, $(0 \le \phi \le \pi)$; γ_- is an analogous contour in $\bar{\mathbf{C}}^-$; γ_+ is directed counter clockwise, and γ_-, clockwise. The residue theorem, together with the relations:

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*The proofs of all those facts can be obtained as usual, using the integral equations, corresponding to (1)-(3) and may be found, e.g., in [4].

$$\phi_n^{\pm}(x,\zeta_{n,k}^{\pm}) = b_{n,k}^{\pm}\psi_n^{\pm}(x,\zeta_{n,k}^{\pm}), \quad \zeta_{n,k}^{\pm} \in \sigma_n^{\pm} , \qquad (15)$$

gives us that when $R > \max|\zeta_{n,k}^{\pm}|$ $I_R(x)$ equals the discrete part of $h_R(x)$. The absolute convergency for the integrals $\Phi^{\pm}(h;\zeta)$ in (7) follows directly from (13) and (15), and for $-\dot{\Phi}_k^{\pm}(h)$ one should also use the fact that, if condition (3) holds, then at $x \to \infty$ $\dot{\phi}^{\pm}(x, \zeta_k^{\pm}) = O(\exp(\mp i\zeta_k^{\pm}(x))$, and the asymptotics (2) and (3) are differentiable with respect to ζ.

Let us now calculate $\lim I_R(x)$ when $R \to \infty$ directly along the contour $\gamma_+ \cup \gamma_-$. Since at $\zeta \in \boldsymbol{R}$

$$\phi_n^{\pm}(x,\zeta) = \pm a_n^{\pm}(\zeta)\psi_n^{\mp}(x,\zeta) + b_n^{\pm}(\zeta)\psi_n^{\pm}(x,\zeta), \qquad (16)$$

with $b_n^{\pm}(\zeta) = \pm W(\phi_n^{\pm}, \psi_n^{\mp})$, and $a_n^{+}(\zeta)a_n^{-}(\zeta) + b_n^{+}(\zeta)b_n^{-}(\zeta) = 1$, then the expression

$$-\frac{1}{2\pi i}\int \{ \int_{-\infty}^{\infty} [G^{+}(x,y;\xi+i0) - G^{-}(x,y;\xi-i0)]h(y)dy\}d\xi$$

equals the integral part of $h_R(x)$.

(This can also be obtained using the solutions $\chi_n^{(p)}(x,\zeta)$ $(p=1,2)$ of equations (1), defined by the initial conditions $\chi_n^{(1)}(0,\zeta) = (1,0)^T$, $\chi_n^{(2)}(0,\zeta) = (0,1)^T$. Using the fact that $\chi_n^{(p)}(x,\zeta)$ are entire functions of $\zeta \in \mathbf{C}$, bounded with respect to x for $\zeta \subset \boldsymbol{R}$, we get that for

$y > x$ $G(x,y;\zeta)$ equals

$$2i \sum_{p,q=1,2} (-1)^{p+q} \chi_1^{(p)} \circ \chi_2^{(q)} (x,\zeta) \widetilde{\chi_1^{(3-p)} \circ \chi_2^{(3-q)}} (y,\zeta) +$$
$$+ 2i a^{-1}(\zeta) \Psi(x,\zeta) \tilde{\Phi}(y,\zeta)) .$$

Furthermore, from (8) it follows [4] that the leading term of the asymptotics of Jost solutions $\phi_n^{\pm}(x,\zeta)$ and $\psi_n^{\pm}(x,\zeta)$ at $|\zeta| \to \infty$, $(\zeta \in \bar{\mathbf{C}}^{\pm})$ are given by the r.h.s. of the relations (2) and (3) respectively: the next term is uniformly estimated with respect to $-\infty < x < \infty$ and $\zeta \in \bar{\mathbf{C}}^{\pm}$ and has order $O(1/\zeta)$. This, togerher with $a_n^{\pm} = 1 + O(1/\zeta)$ gives that at $|\zeta| \to \infty$

$$Q^{\pm}(x,\zeta) = \int_{-\infty}^{\infty} \Gamma^{\pm}(x,y;\zeta) h(y)\, dy +$$
$$+ O(\frac{1}{\zeta} \int_{-\infty}^{\infty} \Gamma^{\pm}(x,y;\zeta) h(y)\, dy) + O(\frac{1}{\zeta^2}) ,$$

where $\Gamma(x,y;\zeta) = G(x,y;\zeta)$ for $q_n(x) = r_n(x) \equiv 0$. The integrals along $\gamma_R^{\pm}$ from the second and third term here vanish when $R \to \infty$ uniformly with respect to x in any finite interval. For the third term is evident, and for the second one this can be established in a standard way (see, e.g., ref. [7], p.363) using Jordan's lemma.

Now, noting that

$$\{ \int_{\gamma_R^+} - \int_{\gamma_R^-} \} (\int_{-\infty}^{\infty} \Gamma(x,y;\zeta) h(y)\, dy)\, d\zeta =$$
$$= - \int_{-R}^{R} \{ \int_{-\infty}^{\infty} [\Gamma^+(x,y;\zeta) - \Gamma^-(x,y;\zeta)] h(y)\, dy \} d\zeta$$

we obtain the inversion formula (10). This completes the proof of the theorem.

From (10) and the classical uniqueness theorem for the Fourier integrals in $L_1(-\infty,\infty)$ we get:

Corollary. Provided the conditions of theorem 1 hold, the system $\{\Phi\}$ is complete in $L_1^{(2)}(-\infty, \infty)$.

Note also, that interchanging $\phi_n^{\pm}$ and $\psi_n^{\pm}$ in (7) and (12) we obtain in the same way as theorem 1.

Theorem 1'. Provided the conditions of theorem 1 hold, the inversion formula (10) holds with $g_R(x) = g_R^{+}(x) + g_R^{-}(x)$ (instead of $h_R(x)$), where:

$$
\begin{aligned}
g_R^{\pm}(x) = \frac{\pm 1}{\pi} \int_{-R}^{R} \Phi^{\pm}(x,\zeta)\Psi^{\pm}(h;\zeta)\frac{d\zeta}{a^{\pm}(\zeta)} - \\
- 2i \sum_{\zeta_{n,k}^{\pm}\in\sigma_{\pm}''} \frac{1}{\dot{a}_{n,k}^{\pm}} \Phi^{\pm}(x,\zeta_{n,k}^{\pm})\Psi_{n,k}^{\pm}(h) - \qquad (17) \\
- 4i \sum_{\zeta_k^{\pm}\in\sigma_{\pm}'} \frac{1}{\ddot{a}_k^{\pm}} \{\dot{\Phi}^{\pm}(x,\zeta_k^{\pm})\Psi_k^{\pm}(h) + \\
+ \Phi^{\pm}(x),\zeta_k^{\pm})[\dot{\Psi}_k^{\pm}(h) - \frac{\dddot{a}_k^{\pm}}{3\ddot{a}_k^{\pm}}\Psi_k^{\pm}(h)]\}.
\end{aligned}
$$

Remark. This construction of inversion formulae can be generalized to an arbitrary number $n \geq 1$ of systems of equations (1). Indeed, let us denote by

$$\overset{(1)}{\phi}{}_{\ell}^{\pm}(x,\zeta) = (\phi_{\ell,1}^{\pm}, \phi_{\ell,2}^{\pm})^T, \text{ and } \overset{(2)}{\phi}{}_{\ell}^{\pm}(x,\zeta) = (\psi_{\ell,1}^{\pm}, \psi_{\ell,2}^{\pm})^T$$

the Jost solutions of the ℓ-th $(1 \leq \ell \leq n)$ system (1), determined by the conditions (2), (3) and construct in analogy with (5), (6) the functions:

$$\Phi_{p_1\ldots p_n}(x,\zeta)=(\prod_{\ell=1}^{n}\phi_{\ell,1}^{(p_\ell)},\prod_{\ell=1}^{n}\phi_{\ell,2}^{(p_\ell)})^T,\quad p_\ell=1,2,$$

$$\tilde{\Phi}_{q_1\ldots q_n}(x,\zeta)=(\prod_{\ell=1}^{n}\phi_{\ell,2}^{(q_\ell)},(-1)^{n-1}\prod_{\ell=1}^{n}\phi_{\ell,1}^{(q_\ell)}),\ q_\ell=3-p_\ell .$$

Then the role of $G(x,y;\zeta)$ (12) will be played by the matrix:

$$-in\prod_{\ell=1}^{n}a_\ell^{-1}(\zeta)\{\sum_{p=p_1+\ldots+p_n\le(3/2)n}(-1)^p\,\Phi_{p_1\ldots p_n}(x,\zeta)\tilde{\Phi}_{q_1\ldots q_n}(y,\zeta)\theta(y-x)$$

$$-\sum_{p>(3/2)n}(-1)^p\,\Phi_{p_1\ldots p_n}(x,\zeta)\tilde{\Phi}_{q_1\ldots q_n}(y,\zeta)\theta(x-y)\}. \tag{18}$$

The corresponding expansion formulae can be proved analogically to theorem 1. Its explicit form in the general case $n>2$ is very involved and we will not write it here. We only note that for $n=1$ (18) defines the Green function for the system (1) and the corresponding expansion is well known /4/.

2. Here we will demonstrate an elementary application of formula (10). Let us relate to each system (1) its scattering data /4/

$$\{S_n\}:\rho_n^{\pm}=b_n^{\pm}(\zeta)/a_n^{\pm}(\zeta),\ (\zeta\in \mathbb{R});$$

$$\zeta_{n,k}^{\pm},\ C_{n,k}^{\pm}=b_{n,k}^{\pm}/\dot{a}_{n,k}^{\pm},\ (k=1,2,\ldots,N_n^{\pm}),$$

where $a_n^{\pm}(\zeta)$, $b_n^{\pm}(\zeta)$ are defined in (16), $b_{n,k}^{\pm}$ - in (15), $\dot{a}_{n,k}^{\pm}=\dot{a}_n^{\pm}(\zeta_{n,k}^{\pm})$.

Lemma 1. The expansion coefficients (7) of the vector-function $\Delta q(x) = (q_1(x) - q_2(x), r_1(x) - - r_2(x))^T$ are given by:

$$\rho_2^{\pm}(\zeta) - \rho_1^{\pm}(\zeta) = (a^{\pm}(\zeta))^{-1}\, \Phi^{\pm}(\Delta q;\zeta), \quad \zeta \in \mathbf{R},$$

$$(-1)^n C^{\pm}_{n,k} = (\dot{a}^{\pm}_{n,k})^{-1}\, \Phi^{\pm}_{n,k}(\Delta q), \quad \zeta^{\pm}_{n,k} \in \sigma''_{\pm},$$

$$\Phi^{\pm}_k(\Delta q) = 0, \quad C^{\pm}_{2,k} - C^{\pm}_{1,k} = 2(\ddot{a}^{\pm}_k)^{-1}\, \dot{\Phi}^{\pm}_k(\Delta q), \zeta^{\pm}_k \in \sigma'_{\pm}.$$

The proof one receives by integrating over x from $-\infty$ to $+\infty$ the identity

$$\frac{d}{dx} W(\phi_1^{\pm}(x,\zeta),\ \phi_2^{\pm}(x,\zeta)) = \tilde{\Phi}(x,\zeta)\,\Delta q(x)$$

and its derivative with respect to ζ, taking into consideration the asymptotics (2), (3) and the relations (15), (16). The lemma is proved.

Lemma 1 together with the completeness of the system $\{\Phi\}$ gives

Corollary (Uniqueness theorem in the inverse scattering problem). Provided the conditions of theorem 1 hold, the scattering data $\{S_n\}$ uniquely determine the coefficients $q_n(x)$ and $r_n(x)$ of the Dirac system (1).

3. In parallel with $\{S_n\}$, as scattering data of system (1) we can consider the set[4]:

$$\{T_n\}: \sigma_n^{\pm}(\zeta) = b_n^{\mp}(\zeta)/a_n^{\pm}(\zeta), (\zeta \in \mathbf{R});$$

$$\zeta^{\pm}_{n,k},\ M^{\pm}_{n,k} = (b^{\pm}_{n,k}\dot{a}^{\pm}_{n,k})^{-1}, \quad (k = 1, \ldots, N_n^{\pm}).$$

Just like above the completeness of the system $\{\Psi\}$ following from theorem 1′ gives that the set $\{T_n\}$ uniquely determines the Dirac system (1). The relation between $\{S_n\}$ and $\{T_n\}$ which is usually established through the well known dispersion relations leads, because of lemma 1, to a somewhat more general problem of relating $\{\Phi\}$ to $\{\Psi\}$. Here we will present a simple derivation of this relation. This problem in the case of (11) has been explored in [1,8].

Let us consider for $p,q=1,2,\ p\neq q$ the contour integrals

$$I_{p,q}(x,z)=\frac{1}{2\pi i}\left\{\int_{\gamma_+}\frac{\psi_q^+\circ\phi_p^+(x,\zeta)}{(\zeta-z)\,a_p^+(\zeta)}d\zeta+\int_{\gamma_-}\frac{\psi_p^-\circ\phi_q^-(x,\zeta)}{(\zeta-z)\,a_q^-(\zeta)}d\zeta\right\},\ (z\in\mathbf{C}),$$

where in the case of $z\in\boldsymbol{R}$, γ_+ encircle the point z along the semicircle $\gamma_\epsilon^+:|\zeta-z|=\epsilon,\ \mathrm{Im}\zeta\geq 0$, and $\gamma_\epsilon^-:|\zeta-z|=\epsilon,\ \mathrm{Im}\zeta\leq 0$. Calculating analogously to (14), $I_{p,q}(x,z)$ with $z=\zeta_{q,k}^+\in\sigma_+''$ and $I_{q,p}(x,z)$ with $\zeta_{q,k}^-\in\sigma_-''$ one gets

$$\Phi^\pm(x,\zeta_{q,k}^\pm)=a_p^\pm(\zeta_{q,k}^\pm)\,b_{q,k}^\pm\,A_{p,q}^\pm(x,\zeta_{q,k}^\pm),$$

where $A_{p,q}^+(x,z)=A_{p,q}(x,z)$, $A_{p,q}^-=-A_{p,q}$.

$$A_{p,q}(x,z)=\frac{1}{2\pi i}\int_{-\infty}\frac{1}{\zeta-z}\{\rho_p^+(\zeta)\Psi^+(x,\zeta)+\rho_q^-(\zeta)\Psi^-(x,\zeta)\}d\zeta-$$

$$-\sum_{\zeta_{p,j}^+\in\sigma_p^+}\frac{C_{p,j}^+\Psi^+(x,\zeta_{p,j}^+)}{\zeta_{p,j}^+-z}+\sum_{\zeta_{q,j}^-\in\sigma_q^-}\frac{C_{q,j}^-\Psi^-(x,\zeta_{q,j}^-)}{\zeta_{q,j}^--z}.$$

In the same way for $z=\zeta_k^\pm\in\sigma_\pm'$ we obtain

$$\frac{1}{b^{\pm}_{1,k}b^{\pm}_{2,k}}\dot{\Phi}^{\pm}(x,\zeta^{\pm}_{k}) = -\dot{\Psi}^{\pm}(x,\zeta^{\pm}_{k}) + \frac{1}{2}\left\{\frac{\ddot{a}^{\pm}_{1,k}}{\dot{a}^{\pm}_{1,k}} + \frac{\ddot{a}^{\pm}_{2,k}}{\dot{a}^{\pm}_{2,k}}\right\}\Psi^{\pm}(x,\zeta^{\pm}_{k}) +$$

$$+\frac{1}{C^{\pm}_{1,k}}A^{\pm}_{1,2}{}'(x,\zeta^{\pm}_{k}) + \frac{1}{C^{\pm}_{2,k}}A^{\pm}_{2,1}{}'(x,\zeta^{\pm}_{k}), \tag{19}$$

where ′ means, that in $A^{\pm}_{p,q}(x,\zeta^{\pm}_{k})$ the term corresponding to $\zeta^{\pm}_{k}$ is dropped. Obviously from (15) we have

$$\Phi^{\pm}(x,\zeta^{\pm}_{k}) = b^{\pm}_{1,k}b^{\pm}_{2,k}\Psi^{\pm}(x,\zeta^{\pm}_{k}), (\zeta^{\pm}_{k}\in\sigma'_{\pm}). \tag{20}$$

For real $z=\zeta$ the calculation of $I_{p,q}(x,\zeta)$ (in the limit $\epsilon\to 0$) together with the identity

$$2\psi^{+}_{p}\circ\psi^{-}_{q} = (a^{+}_{p})^{-1}\psi^{+}_{q}\circ\phi^{+}_{p} - (a^{-}_{q})^{-1}\psi^{-}_{p}\circ\phi^{-}_{q} - \rho^{+}_{q}\Psi^{+} + \rho^{-}_{p}\Psi^{-}$$

following from (16), gives the representations:

$$\Phi^{\pm}(x,\zeta) = \frac{1}{2}\left\{\frac{a^{\pm}_{2}(\zeta)}{a^{\mp}_{1}(\zeta)} + \frac{a^{\pm}_{1}(\zeta)}{a^{\mp}_{2}(\zeta)}\right\}\Psi^{\mp}(x,\zeta) =$$

$$+ a^{\pm}_{2}(\zeta)b^{\pm}_{1}(\zeta)A^{\pm}_{1,2}(x,\zeta) + a^{+}_{1}(\zeta)b^{+}_{2}(\zeta)A^{\pm}_{2,1}(x,\zeta), \tag{21}$$

where the integrals in $A_{p,q}(x,\zeta)$ are understood in the sense of Cauchy principle value.

4. Let us introduce in the case (11) the system $\{P,Q\}$:

$$P(x,\zeta)=\sigma^+(\zeta)\Phi^+(x,\zeta)+\sigma^-(\zeta)\Phi^-(x,\zeta),\quad (\zeta\in \boldsymbol{R}),$$

$$Q(x,\zeta)=\sigma^+(\zeta)\Phi^+(x,\zeta)-\rho^+(\zeta)\Psi^+(x,\zeta),\quad (\zeta\in \boldsymbol{R}),\tag{22}$$

$$P_k^{\pm}(x)=\Psi^{\pm}(x,\zeta_k^{\pm})\equiv(b_k^{\pm})^{-2}\,\Phi^{\pm}(x,\zeta_k^{\pm}),\ (\zeta_k^{\pm}\in\sigma^{\pm}),$$
$$Q_k^{\pm}(x)=i(\dot a_k^{\pm})^{-2}\,\{(b_k^{\pm})^2\,\dot\Psi^{\pm}(x,\zeta_k^{\pm})-\dot\Phi^{\pm}(x,\zeta_k^{\pm})\}.$$

From (16) it follows that together with (22) we have

$$P(x,\zeta)=\rho^+\Psi^++\rho^-\Psi^-,\ Q(x,\zeta)=\rho^-\Psi^--\sigma^-\Phi^-.$$

This gives for real ζ the following identity:

$$(a^+(\zeta))^{-2}\,\{\Phi^+(x,\zeta)\tilde\Psi^+(y,\zeta)-\Psi^+(x,\zeta)\tilde\Phi^+(y,\zeta)\}-$$

$$-(a^-(\zeta))^{-2}\{\Phi^-(x,\zeta)\tilde\Psi^-(y,\zeta)-\Psi^-(x,\zeta)\tilde\Phi^-(y,\zeta)\}=\tag{23}$$

$$=2(b^+(\zeta)\,b^-(\zeta))^{-1}\,\{Q(x,\zeta)\tilde P(y,\zeta)-P(x,\zeta)\tilde Q(y,\zeta)\}.$$

Let us consider for $h\in L_1^{(2)}$ the vector-function

$$s_R(x)=\frac{1}{2\pi}\int_{-R}^{R}\{Q(x,\zeta)P(h;\zeta)-P(x,\zeta)Q(h;\zeta)\}\frac{d\zeta}{b^+(\zeta)b^-(\zeta)}+$$

$$+\sum_{\zeta_k^{\pm}\in\sigma^+\cup\sigma^-}\{Q_k^{\pm}(x)\,P_k^{\pm}(h)-P_k^{\pm}(x)\,Q_k^{\pm}(h)\}.\tag{24}$$

From (23) we see that the integrand expression in (24) is a continuous vector-function of ζ and is meaning full also at the points, where $b^+(\zeta)b^-(\zeta)\neq 0$. Noting that $2s_R(x) = h_R(x) + g_R(x)$, where $h_R(x)$ is defined in (9), and $g_R(x)$ - in (17), and using the inversion formulae in theorems 1 and 1′, we obtain

Theorem 2. If the conditions of theorem 1 hold together with (11), then the inversion formula (10) holds with $s_R(x)$ (24) (instead of $h_R(x)$).

Corollary. For any vector-functions $h(x) = (h_1, h_2)$ and $g(x) = (g_1, g_2)$ integrable together with their first derivative the bilinear form:

$$\omega(g,h) \overset{\text{defn.}}{=\!=\!=\!=\!=} \int_{-\infty}^{\infty} (g_1(x)h_2(x) - g_2(x)h_1(x))\,dx =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{Q(g;\zeta)P(h;\zeta) - P(g;\zeta)Q(h;\zeta)\}\frac{d\zeta}{b^+(\zeta)b^-(\zeta)} + \qquad (25)$$

$$+ \sum_{\zeta_k^{\pm}\in\sigma^+\cup\sigma^-} \{Q_k^{\pm}(g)P_k^{\pm}(h) - P_k^{\pm}(g)Q_k^{\pm}(h)\}.$$

Proof. Relation (25) is obtained from the inversion formula (10) and from $2s_R(x) = h_R(x) + g_R(x)$ in a standard way, first proving it in the case of finite $h(x)$ and $g(x)$ and then extending to the general case. For this it is enough to know that if $h_j(x), h'_{j,x}(x) \in L_1(-\infty,\infty)$, $j = 1,2,$ then from (1)-(3) it follows that

$$2i\zeta\Psi^{\pm}(h;\zeta) = \Psi^{\pm}(H^+;\zeta),$$
$$2i\zeta\Phi^{\pm}(h;\zeta) = \Phi^{\pm}(H^-;\zeta), \qquad (26)$$

where

$$H^{\pm}(x) = (-h'_{1,x}(x) \pm 2q(x) I^{\pm}(x)\ , h'_{2,x}(x) \mp 2r(x) I^{\pm}(x))^{T},$$

$$I^{\pm}(x) = \int_{-\infty}^{\infty} (h_1(y)\, r(y) + h_2(y)\, q(y))\, \theta(\pm(x-y))\, dy\,.$$

From (23) and (26) it also follows that the integrand expression in (25) is an integrable function. The corollary is proved.

Remark. The system {P,Q} (written through (19)-(21) in terms of {Ψ}) appears in the Hamiltonian formalism of Faddeev-Zakharov [9] for evolution equations solvable via the inverse scattering method; in this formalism $\omega(g,h)$ (24) plays the role of the corresponding symplectic form. Here one should use the representation of the variational derivatives of the scattering data {S} and {T} through {Φ} and {Ψ} (see, e.g., ref. [8]).

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