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K.Schmüdgen

SOME REMARKS ON TOPOLOGIZATION OF UNBOUNDED OPERATOR ALGEBRAS

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Несколько замечаний о топологии алгебр неограниченных операторов

Рассматривается топологизация алгебр неограниченных операторов в гильбертовом пространстве. Цель состоит в том, чтобы исследовать соответствия между некоторыми топологиями (например, равномерная топология, порядковая топология) и дать явное описание этих топологий.

Показано, что для определенного класса таких алгебр равномерная топология и порядковая топология совпадают. Эти результаты получены с помощью прямой конструкции и оценки операторов в гильбертовом пространстве. Разные примеры иллюстрируют результаты и разграничивают область их применимости.

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Some Remarks on Topologization of Unbounded Operator Algebras

The problem is studied whether on not for a given Op*-algebra A the topologies τ_D and ρ resp. $\tau^{(D)}$ and λ agree on A. It is proved that this is true for certain Op*-algebras on domains of the form $D = \bigcap_{n \in N} D(S^n)$ S, a self-adjoint operator. Some examples of Op*-algebras with $\tau_D \neq \rho$ and $\tau^{(D)} \neq \lambda$ are discussed in detail.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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0. INTRODUCTION

Given an unbounded operator algebra (Op*-algebra) on a dense invariant domain in a Hilbert space, it is possible to define in a "natural way" various locally convex topologies generalizing the operatornorm topology of C^*-algebras. In this paper, we consider the topologies $\tau_{\mathcal{D}}$, $\tau^{(\mathcal{D})}$, ρ and λ. The topologies $\tau_{\mathcal{D}}$ and ρ are related to basic concepts in the theory of ordered vector spaces. $\tau_{\mathcal{D}}$ is connected with the notion of normality of a cone because the cone of all positive operators in the algebra is $\tau_{\mathcal{D}}$-normal. If we regard the hermitian part of an Op*-algebra as an ordered vector space, then ρ is just the order topology.

Our aim is to study the following problem:

> Under what conditions to an Op*-algebra $\mathcal{A}$ on a domain $\mathcal{D}$ the topologies $\tau_{\mathcal{D}}$ and ρ resp. $\tau^{(\mathcal{D})}$ and λ coincide on $\mathcal{A}$?

In section 3 we prove that for certain Op*-algebras $\mathcal{A}$ which are "rich enough" (for example, the algebra $L^+(\mathcal{D})$) on domains of the form $\mathcal{D} = \bigcap_{n \in N} \mathcal{D}(S^n)$, S a self-adjoint operator, the question has an affirmative answer, that is, $\tau_{\mathcal{D}} = \rho$ and $\tau^{(\mathcal{D})} = \lambda$ on $\mathcal{A}$. Section 4 contains three examples. We include an example of a closed Op*-algebra on a domain of the form $\mathcal{D} = \bigcap \mathcal{D}(S^n)$ for which the positive cone is not normal with respect to the order topology ρ (in particular, $\tau_{\mathcal{D}} \neq \rho$). By another example we see that there are Frechet domains $\mathcal{D}$ such that $\tau_{\mathcal{D}} \neq \rho$ and $\tau^{(\mathcal{D})} \neq \lambda$ on $L^+(\mathcal{D})$. Section 1 collects some definitions about unbounded operator algebras. Section 2 gives some elementary facts about bounded sets in Op*-algebras.

1. DEFINITIONS

Let $\mathcal{D}$ be a dense domain in a Hilbert space $\mathcal{H}$. Let $L^+(\mathcal{D}) := \{a \in \mathrm{End}\,\mathcal{D} \ : \ a\mathcal{D} \subseteq \mathcal{D} \ , a^*\mathcal{D} \subseteq \mathcal{D}\}$. $L^+(\mathcal{D})$ is a $*$-algebra with the usual multplication and the involution $a \longrightarrow a^+ := a^* \upharpoonright \mathcal{D}$.

An Op*-algebra is a *-subalgebra of $L^+(\mathcal{D})$ containing the identity map $I=I_{\mathcal{D}}$. The locally convex topology $t_{\mathcal{A}}$ on $\mathcal{D}$ generated by the family of seminorms $\|\phi\|_a:=\|a\phi\|$, $a\in\mathcal{A}$, $\phi\in\mathcal{D}$, is called the graph topology. If $\mathcal{A} = L^+(\mathcal{D})$, then we write t_+ instead of $t_{\mathcal{A}}$.

Let $\underline{\mathcal{D}}(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\bar a)$ where $\bar a$ means the closure of the operator a. The operators $\underline{a}:=\bar a \restriction \underline{\mathcal{D}}$ form an Op*-algebra $\underline{\mathcal{A}}$ on $\underline{\mathcal{D}}=\underline{\mathcal{D}}(\mathcal{A})$. $\underline{\mathcal{D}}[t_{\underline{\mathcal{A}}}]$ is the completion of $\mathcal{D}[t_{\mathcal{A}}]$. $\mathcal{A}$ is said to be closed on $\mathcal{D}$ if $\mathcal{A}=\underline{\mathcal{A}}$, i.e. $\mathcal{D}=\underline{\mathcal{D}}(\mathcal{A})$.

Let us define $\mathcal{A}_h:=\{a\in\mathcal{A} : a^+=a\}$, $\mathcal{A}_+:=\{a\in\mathcal{A} : \langle a\phi,\phi\rangle \geqslant 0 \ \ \forall\phi\in\mathcal{D}\}$ and $a\geqslant b$ iff $a-b\in\mathcal{A}_+$ for $a,b\in\mathcal{A}_h$. A linear functional f an $\mathcal{A}$ is called strongly positive if $f(a)\geqslant 0 \ \ \forall\, a\in\mathcal{A}_+$.

Now we turn to the topologization of Op*-algebras:
For each bounded set $\mathfrak{M}$ of $\mathcal{D}[t_{\mathcal{A}}]$ we define

$$p_{\mathfrak{M}}(a):=\sup_{\phi,\psi\in\mathfrak{M}}|\langle a\phi,\psi\rangle| \quad \text{and} \quad p^{\mathfrak{M}}(a):=\sup_{\phi\in\mathfrak{M}}\|a\phi\| .$$

The topolgies $\tau_{\mathcal{D}}$ and $\tau^{(\mathcal{D})}$ generated by the seminorms $\{p_{\mathfrak{M}}\}$ resp. $\{p^{\mathfrak{M}}\}$ were introdueced by Lassner (/4/,/5/). $\tau_{\mathcal{D}}$ is called the uniform topology on $\mathcal{A}$. $\mathcal{A}[\tau_{\mathcal{D}}]$ is a topological *-algebra. Since $\tau_{\mathcal{D}}$ can be given also by the equivalent system of seminorms $p'_{\mathfrak{M}}(a):=\sup_{\phi\in\mathfrak{M}}|\langle a\phi,\phi\rangle|$, $\mathcal{A}_+$ is $\tau_{\mathcal{D}}$-normal(/9/).

Further, for $a,x\in\mathcal{A}$ we define

$$\mathcal{N}_x:=\{a\in\mathcal{A} : |\langle a\phi,\phi\rangle| \leqslant c_{a,x}\|x\phi\|^2 \ \ \forall\phi\in\mathcal{D}\},$$

$$\mathcal{M}_x:=\{a\in\mathcal{A} : \|a\phi\| \leqslant c_{a,x}\|x\phi\| \ \ \forall\phi\in\mathcal{D}\},$$

$$\rho_x(a):=\sup_{\phi\in\mathcal{D}}\frac{|\langle a\phi,\phi\rangle|}{\|x\phi\|^2} , \quad \lambda_x(a):=\sup_{\phi\in\mathcal{D}}\frac{\|a\phi\|}{\|x\phi\|} \quad \text{where } \frac{c}{0}=+\infty \text{ for}$$

$c>0$ and $\frac{0}{0}=0$.

ρ and λ denote the inductive toplogies on $\mathcal{A}$ with respect to the normed spaces $\{(\mathcal{N}_x,\rho_x), x\in\mathcal{A}\}$ resp. $\{(\mathcal{M}_x,\lambda_x), x\in\mathcal{A}\}$. In this form ρ and λ are introduced by Arnal and Jurzak (/1/). Since the inductive limit of normed spaces is bornological, $\mathcal{A}[\rho]$ and $\mathcal{A}[\lambda]$ are bornological locally convex spaces. The topology ρ on the hermitian part $\mathcal{A}_h$ of $\mathcal{A}$ is the order topology of the vector space $\mathcal{A}_h$ ordered by the positive cone $\mathcal{A}_+$. This follows from /8/, p.232,6.3, because the norm ρ_x is just the order unit norm. For simplicity, we say that ρ is the order topology on $\mathcal{A}$.

Suppose $\|\phi\|_{x_n}$, $x_n\in\mathcal{A}$, $n\in\mathbb{N}$, is a directed system of seminorms generating $t_{\mathcal{A}}$ on $\mathcal{D}$. Then a zero-neighbourhood base for ρ and λ is given by the families of sets $\mathcal{U}_\alpha:=\text{aco}(\alpha_n U_{x_n}, n\in\mathbb{N})$ resp. $\mathcal{V}_\alpha:=\text{aco}(\alpha_n V_{x_n}, n\in\mathbb{N})$ where $\alpha=\{\alpha_n, n\in\mathbb{N}\}$ is an arbitrary sequence of positive numbers α_n. U_{x_n} and V_{x_n} denote the unit balls in the normed spaces $(\mathcal{N}_{x_n},\rho_{x_n})$ resp. $(\mathcal{M}_{x_n},\lambda_{x_n})$. We have $\tau_{\mathcal{D}}\leq\rho$ and $\tau^{(\mathcal{D})}\leq\lambda$.
$\mathcal{D}_\infty(S)$ means the (dense) domain defined by $\mathcal{D}_\infty(S):=\bigcap_{n\in\mathbb{N}}\mathcal{D}(S^n)$ whereby S is a self-adjoint operator in Hilbert space. For a domain $\mathcal{D}=\mathcal{D}_\infty(S)$ we define
$\sigma_1(\mathcal{D}):=\{t\in\mathcal{L}(\mathcal{H}) : \overline{ta}$ is of trace class for all $a\in L^+(\mathcal{D})\}$ and $\sigma_1(\mathcal{D})_+:=\{t\in\sigma_1(\mathcal{D}) : t\geq 0\}$. Since $\mathcal{D}=\mathcal{D}_\infty(S)$, it follows that $t\mathcal{H}\subseteq\mathcal{D}$ for $t\in\sigma_1(\mathcal{D})$ (see for example /11/, lemma 1.1,(1)).
By $\mathcal{F}$ we denote the set of all linear functionals $f(a) = \text{Tr}\ ta$, $a\in\mathcal{A}$, $t\in\sigma_1(\mathcal{D})_+$, on $\mathcal{A}$. Let $\sigma^{\mathcal{D}}$ and $\sigma^{\mathcal{F}}$ be the strong operator topologies on $\mathcal{A}$ given by the families of seminorms $\{\|a\|_\phi:=\|a\phi\|$, $\phi\in\mathcal{D}\}$ resp. $\{q_f(a):=f(a^+a)^{1/2}, f\in\mathcal{F}\}$.
Finally, τ_{st} always denotes the strongest locally convex topology on $\mathcal{A}$.
For the remainder of this paper we assume that the graph topologies of all Op*-algebras are metrizable.

2. BOUNDED SETS

Let $\mathcal{A}$ be an Op*-algebra on $\mathcal{D}$ with metrizable graph topology $t_{\mathcal{A}}$. Since the family of seminorms $\|\phi\|_a$, $a\in\mathcal{A}$, is directed, there are operators $x_n\in\mathcal{A}$, $x_1=I$, $n\in\mathbb{N}$, so that $\|x_n\phi\|\leq\|x_{n+1}\phi\| \ \ \forall\phi\in\mathcal{D}$, $n\in\mathbb{N}$, and that the seminorms $\|\phi\|_{x_n}$, $n\in\mathbb{N}$, define the topology $t_{\mathcal{A}}$.

LEMMA 1: For each subset $\mathfrak{N}$ of $\mathcal{A}$ the following assertions are equivalent:

(i) $\mathfrak{N}$ is $\tau_{\mathcal{D}}$-bounded.

(ii) There are a constant $C>0$ and a $n\in\mathbb{N}$ such that $|\langle a\phi,\phi\rangle| \leq C\|x_n\phi\|^2 \ \ \forall\phi\in\mathcal{D}, a\in\mathfrak{N}$.

(iii) $\mathfrak{N}$ is ρ-bounded.

Proof:
(i) $\longrightarrow$ (ii):
Assume that (ii) is not true. Then there are vectors $\phi_n\in\mathcal{D}$ and operators $a_n\in\mathfrak{N}$ such that $|\langle a_n\phi_n,\phi_n\rangle| \geqslant n\|x_n\phi_n\|^2$. By

normalizing the vectors we get $\| x_n \phi_n \| = 1$. The set $\mathfrak{N} := \{\phi_n, n\in N\}$ is $4_{\mathcal{A}}$-bounded because $\sup_{\phi\in\mathfrak{M}} \|\phi\|_{x_k} \leq \mathrm{Max}(\|x_k\phi_1\|, \ldots, \|x_k\phi_{k-1}\|, 1) < +\infty$. On the other side, one has $\sup_{a\in\mathfrak{N}} p_{\mathfrak{M}}(a) \geq |\langle a_n\phi_n, \phi_n\rangle| \geq n$ for all $n\in N$ which is a contradiction to (i).

(ii) $\longrightarrow$ (iii):

Clear, because (ii) means that the set $\mathfrak{N}$ is bounded in the normed space $(\mathcal{N}_{x_n}, \rho_{x_n})$.

(iii) $\longrightarrow$ (i):

Trivial because $\tau_{\mathcal{D}} \subseteq \rho$. //

Similarly, we have

LEMMA 2: The following properties of a set $\mathfrak{N} \subseteq \mathcal{A}$ are equivalent:

(i) $\mathfrak{N}$ is $\tau^{(\mathcal{D})}$-bounded.

(ii) There exist a constant $C > 0$ and a number $n\in N$ such that $\|a\phi\| \leq C\|x_n\phi\|$ $\forall\ \phi\in\mathcal{D}, a\in\mathfrak{N}$.

(ii) $\mathfrak{N}$ is λ-bounded.

Since $\mathcal{A}[\rho]$ and $\mathcal{A}[\lambda]$ are bornological spaces, we obtain the following corollary.

COROLLARY 3: Suppose $\mathcal{A}$ is an Op*-algebra on $\mathcal{D}$ with metrizable graph topology. Then:

(1) $\mathcal{A}[\tau_{\mathcal{D}}]$ $[\mathcal{A}[\tau^{(\mathcal{D})}]]$ is the bornological space associated with $\mathcal{A}[\rho]$ $[\mathcal{A}[\lambda]]$.

(2) $\mathcal{A}[\tau_{\mathcal{D}}]$ is bornological if and only if $\tau_{\mathcal{D}} = \rho$.

(3) $\mathcal{A}[\tau^{(\mathcal{D})}]$ is bornological if and only if $\tau^{(\mathcal{D})} = \lambda$.

3. THE MAIN RESULTS

THEOREM 1: Let S, $S \geq I$, be a self-adjoint operator in a Hilbert space with spectral resolution $S = \int_1^\infty t\, dE(t)$.

Let $\{M_n, n\in N\}$ be a monotonic sequence with $M_1 = 1$ and $\lim_{n\to\infty} M_n = +\infty$.

Suppose $\mathcal{A}$ is an Op*-algebra on $\mathcal{D} = \mathcal{D}_\infty(S)$ such that $S \in \mathcal{A}$ and $E(M_{n+1}) - E(1-0) \in \mathcal{A}$ $\forall$ $n\in N$.

Then we have $\tau_{\mathcal{D}} = \rho$ and $\tau^{(\mathcal{D})} = \lambda$ on $\mathcal{A}$.

Before giving the proof of the theorem, we mention some corollaries.

COROLLARY 2: Let $\mathcal{A}$ and $\mathcal{D}$ as in theorem 1. Then:

(1) $\mathcal{A}[\tau_{\mathcal{D}}]$ and $\mathcal{A}[\tau^{(\mathcal{D})}]$ are bornological spaces.

(2) The cone $\mathcal{A}_+$ is normal for the order topology ρ.

(3) Suppose in addition that S is the inverse of a completely continuous operator in $\mathcal{H}$. Then each ρ-continuous linear functional f on $\mathcal{A}$ is a trace functional, i.e. $f(a) = \mathrm{Tr}\, ta$, $a \in \mathcal{A}$ whereby $t \in \mathfrak{S}_1(\mathcal{D})$.

Proof:

Since the topologies ρ and λ are bornological, (1) follows from theorem 1. $\tau_{\mathcal{D}} = \rho$ implies (2) because $\mathcal{A}_+$ is $\tau_{\mathcal{D}}$-normal (/9/). We prove (3). Since S^{-1} is compact, $\mathcal{D}[4_{\mathcal{A}}]$ is a Frechet Montel space(see / /,section 3,remark 2). f is $\tau_{\mathcal{D}}$-continuous because $\tau_{\mathcal{D}} = \rho$ on $\mathcal{A}$ by theorem 1. Now the assertion follows from /11/,section 4,theorem 4. //

In the case $\mathcal{A} = L^+(\mathcal{D})$ theorem 1 gives

COROLLARY 3: If $\mathcal{D} = \mathcal{D}_\infty(S)$, S a self-adjoint operator in a Hilbert space, then $\tau_{\mathcal{D}} = \rho$ and $\tau^{(\mathcal{D})} = \lambda$ on $L^+(\mathcal{D})$.

COROLLARY 4: Suppose $\mathcal{A}$ is an Op*-algebra on the domain $\mathcal{D}_\infty(S)$. Suppose that $S \in \mathcal{A}$. Then each strongly positive linear functional f on is $\tau_{\mathcal{D}}$-continuous.

Proof:

By the closed graph theorem, we get $4_{\mathcal{A}} = 4_+$ on $\mathcal{D}$. Hence $\mathcal{A}$ is cofinal in $L^+(\mathcal{D})$ and f can be extended to a strongly positive linear functional $\tilde{f}$ on $L^+(\mathcal{D})$. Since strongly positive linear functionals are always continuous in the order topology ρ and $\tau_{\mathcal{D}} = \rho$ on $L^+(\mathcal{D})$ by corollary 3, $\tilde{f}$ is $\tau_{\mathcal{D}}$-continuous on $L^+(\mathcal{D})$. Because $4_{\mathcal{A}} = 4_+$, the uniform topology of $\mathcal{A}$ agrees with the topology induced by the uniform topology of $L^+(\mathcal{D})$. Therefore, f is $\tau_{\mathcal{D}}$-continuous on $\mathcal{A}$. //

Now we pass to some preliminaries for the proof of theorem 1. Let $\gamma = \{\gamma_k, k\in N\}$ be a sequence of real numbers γ_k with $0 < \gamma_k \leq 1/k!$ $\forall$ $k\in N$. We define a real function $h_\gamma(z)$ by $h_\gamma(z) := \sup_{k\in N} \gamma_k z^k$ for $z\in\mathbb{R}_1$, $z \geq 1$.

LEMMA 5: Let $\beta=\{\beta_k, k\in N\}$ be a sequence of positive real numbers. Then there exist monotonic sequences $\{n_k,k\in N\},\{m_k,k\in N\}$ of natural numbers n_k, m_k where $m_1=1$ and a real sequence $\gamma=\{\gamma_k,k\in N\}$ such that:

(i) $0<\gamma_k\le\beta_k$, and $\gamma_k\le 1/k!$ $\forall\ k\in N$.

(ii) $h_\gamma(z)=\gamma_{n_k} z^{n_k}$ for $M_{m_k}\le z\le M_{m_{k+1}}$, $k\in N$.

Moreover, (i) and (ii) imply

(iii) $\sup_{z\ge 1} z^k h_\gamma(z)^{-1} < +\infty$ for each $k\in N$.

Proof:

Let $\delta=\{\delta_k,k\in N\}$ be an arbitrary real sequence satisfying $0<\delta_k\le 1/k!$ $\forall k\in N$. Since $\lim_{k\to\infty}\delta_k z^k=0$, for each $z\ge1$ the supremum in the definition of $h_\delta(z)$ will be attained for some $k\in N$ (depending on z). Further, the set of all $k\in N$ such that $h_\delta(z)=\delta_k z^k$ is a finite set for each (fixed) $z\ge1$. From this facts it follows the existence of a monotonic sequence $\{r_k,k\in N\}$ of natural numbers and of a real sequence $\{L_k\equiv L_k(\delta),k\in N\}$ (both depending on δ !) such that $1\le L_k<L_{k+1}, L_1=1$, and $h_\delta(z)=\delta_{r_k}z^{r_k}$ for $L_k\le z\le L_{k+1}$.

After these preliminary observations we shall inductively define the sequences $\{n_k\},\{m_k\},\{\gamma_k\}$.

First let $\delta_k=\mathrm{Min}(\beta_k,1/k!)$. Since $\lim_{k\to\infty}M_k=+\infty$, there is an index m_1 so that $M_{m_1}\ge L_1(\delta)$. Suppose that $M_{m_1}\subseteq[L_{i_1},L_{i_1+1}]$. By making $\delta_2,\dots,\delta_{r_{i_1+1}}$ sufficiently small we can get $M_{m_1}=L_1(\delta)$. Put $\gamma_1=\delta_1$ and $n_1=r_1(\equiv r_1(\delta))$. We continue this procedure with the new sequence δ. Take m_2 such that $M_{m_2}\ge L_2(\delta)$. Let $M_{m_2}\subseteq[L_{i_2},L_{i_2+1}]$. Now we change $\delta_2,\dots,\delta_{r_{i_2+1}}$ in such a way that $M_{m_2}=L_2(\delta)$ and define $\gamma_2=\delta_2$, $n_2=r_2$. By induction this construction proves the first part of lemma 5.

Next we show (iii). Fix $k\in N$. We take a $i\in N$ such that $n_i\ge k$. Then we have $h_\gamma(z)\ge\gamma_k z^k$ for all $z\ge M_{m_i}$. Hence $z^k h_\gamma(z)^{-1}\le\gamma_k^{-1}$ for all $z\ge M_{m_i}$. Since the continuous function $h_\gamma(z)^{-1}z^k$ on $[1,M_{m_i}]$ is bounded, this implies (iii).

Proof of Theorem 1:

We want to show that $\tau_{\mathcal D}=\rho$ on $\mathcal A$. Since always $\tau_{\mathcal D}\le\rho$, it is enough to prove that $\rho\le\tau_{\mathcal D}$. Let $\mathcal U_\alpha=\mathrm{aco}(\alpha_k U_{S^k},k\in N)$, $\alpha_k>0$ $\forall\ k\in N$, be a fixed 0-neighbourhood for ρ in $\mathcal A$.

First we inductively define a new sequence $\beta=\{\beta_k,k\in N\}$ by

$$\beta_1=\sqrt{2^{-3}\alpha_1}\ ,\ \beta_k=\mathrm{Min}\big(2^{-(k+2)}\alpha_k\beta_1^{-1},2^{-(k+3)}\alpha_k\beta_2^{-1},\dots,2^{-(k+k)}\alpha_k\beta_{k-1}^{-1},\sqrt{2^{-(2k+1)}\alpha_k}\ \big).$$

Then we have $\beta_k\beta_m\le\alpha_m 2^{-(k+m+1)}$ $\forall\ k\le m,\ k,m\in N$. (1)

Next we take sequences $\{\gamma_k\},\{n_k\},\{m_k\}$ having the properties described in lemma 5. For simplicity we write $h(z)$ instead of $h_\gamma(z)$. Let $\mathfrak M=h(S)^{-1}B$ where B denotes the unit ball of the Hilbert space.

Since $\mathcal D=\mathcal D_\infty(S)$ and $S\in\mathcal A$, the seminorms $\|\phi\|_{S^k}=\|S^k\phi\|$, $k\in N$, already generate the topology $t_{\mathcal A}$ on $\mathcal D$. Hence we conclude from lemma 5,(iii), that $\mathfrak M$ is a bounded subset of $\mathcal D[t_{\mathcal A}]$.

Now let $\mathcal W_{\mathfrak M}=\{x\in\mathcal A: p_{\mathfrak M}(x)\le1\}$. Our proof is complete if we have shown $\mathcal W_{\mathfrak M}\subseteq\mathcal U_\alpha$. Take an element $a\in\mathcal W_{\mathfrak M}$. Then

$p_{\mathfrak M}(a)\equiv\sup_{\xi,\eta\in h(S)^{-1}B}|\langle a\xi,\eta\rangle|\le 1$ and so $|\langle a\xi,\eta\rangle|\le\|\xi\|\,\|\eta\|$ $\forall\ \xi,\eta\in h(S)^{-1}\mathcal H$. Putting $\xi=h(S)^{-1}\phi$, $\eta=h(S)^{-1}\psi$ we get

$|\langle a\phi,\psi\rangle|\le\|h(S)\phi\|\ \|h(S)\psi\|$ $\forall\ \phi,\psi\in\mathcal D(h(S))$. (2)

Let $P_k=E(M_{m_{k+1}})-E(M_{m_k}-0)$ and let $F_k=P_1+\dots+P_k$ for $k\in N$. Our assumptions imply that $P_k\in\mathcal A$ and $F_k\in\mathcal A$ $\forall\ k\in N$.

STATEMENT I: $P_k a P_r\in 2^{-(n_k+n_r+1)}\alpha_{n_r}U_{S^{n_r}}$ for $k\le r,\ k,r\in N$.

Proof: For $k\le r$, $\phi\in\mathcal D$, we have

$$|\langle P_k a P_r\phi,\phi\rangle|=|\langle aP_r\phi,P_k\phi\rangle|\le\|h(S)P_r\phi\|\ \|h(S)P_k\phi\|\underset{(ii)}{=}\gamma_{n_r}\|S^{n_r}P_r\phi\|\,\gamma_{n_k}\|S^{n_k}P_k\phi\|\underset{(i)}{\le}\beta_{n_r}\beta_{n_k}\|S^{n_r}\phi\|^2\underset{(1)}{\le}\alpha_{n_r}2^{-(n_r+n_k+1)}\|S^{n_r}\phi\|^2$$

which proves the statement.

We assumed that $S \geq I$. Therefore $\mathcal{A} = \bigcup_{k\in N} \mathcal{N}_{S^k}$. Hence there is a number $l \in N$ such that $a \in \mathcal{N}_{S^l}$. We choose an index m_s such that

$$M_{m_s} \geq 24\, \rho_{S^l}(a)\, \alpha_{l+1}^{-1} \qquad (3)$$

which is possible because $\lim_{k\to\infty} M_{m_k} = +\infty$. For brevity let us write $\rho_i(a)$ and U_i instead of $\rho_{S^i}(a)$ resp. U_{S^i}, $i \in N$.

STATEMENT II: $(I-F_s)a(I-F_s)+F_s a(I-F_s)+(I-F_s)aF_s \in 1/2\,\alpha_{l+1} U_{l+1}$.

Proof: By the spectral theorem, it follows that

$$\| S^l (I-F_s)\phi\| \leq M_{m_s}^{-1} \| S^{l+1}(I-F_s)\phi\| \quad \forall\, \phi\in\mathcal{D} . \qquad (4)$$

Clearly, $|\langle a\phi,\phi\rangle| \leq \rho_l(a)\,\|S^l\phi\|^2$ implies $|\langle a\phi,\psi\rangle| \leq 4\rho_l(a)\,\|S^l\phi\|\,\|S^l\psi\|$ $\forall\, \phi,\psi\in\mathcal{D}$ by polarization. Moreover, $\|S^l F_s\phi\| \leq \|S^{l+1}\phi\|$ because $S\geq I$. Using this facts, we obtain

$$|\langle F_s a(I-F_s)\phi,\phi\rangle| = |\langle a(I-F_s)\phi, F_s\phi\rangle| \leq 4\rho_l(a)\|S^l(I-F_s)\phi\|\,\|S^l F_s\phi\|$$

$$\underset{(4)}{\leq} 4\rho_l(a)\, M_{m_s}^{-1}\,\|S^{l+1}(I-F_s)\phi\|\,\|S^{l+1}\phi\| \underset{(3)}{\leq} 1/6\,\alpha_{l+1}\|S^{l+1}\phi\|^2$$

and

$$|\langle (I-F_s)a(I-F_s)\phi,\phi\rangle| \leq \rho_l(a)\,\|S^l(I-F_s)\phi\|^2 \leq$$

$$\rho_l(a)\, M_{m_s}^{-2}\,\|S^{l+1}(I-F_s)\phi\|^2 \leq 1/6\,\alpha_{l+1}\|S^{l+1}\phi\|^2 .$$

Therefore, $F_s a(I-F_s) \in 1/6\,\alpha_{l+1}U_{l+1}$, $(I-F_s)a(I-F_s)\in 1/6\,\alpha_{l+1}U_{l+1}$ and similarly, $(I-F_s)aF_s \in 1/6\,\alpha_{l+1}U_{l+1}$. This completes the proof of statement II.

By statements I and II, we conclude that

$a = \sum_{k,r=1}^{s} P_k a P_r + (I-F_s)a(I-F_s)+F_s a(I-F_s)+(I-F_s)aF_s$ is in $\mathcal{U}_\alpha = \mathrm{aco}(\alpha_k U_{S^k}, k\in N)$ because $\sum_{k,r=1}^{s} 2^{-(n_k+n_r+1)} + 1/2 < 1$.

Thus we proved that $\tau_{\mathcal{D}} = \rho$.

In a similar way, it can be shown that $\tau^{(\mathcal{D})} = \lambda$ on $\mathcal{A}$. //

The preceding arguments can be used to get further results about the comparison of topologies. We mention an example.

PROPOSITION 6: Let $\mathcal{A}$, $\mathcal{D} = \mathcal{D}_\infty(S)$ and $\{M_n\}$ as in theorem 1. Suppose in addition that $\mathcal{D}[t_{\mathcal{A}}]$ is a nuclear space. Then the topologies $\sigma^{\mathcal{F}}$ and λ coincide on $\mathcal{A}$.

Proof:

We first prove that $\lambda \subseteq \sigma^{\mathcal{F}}$.

Since $\mathcal{D}[t_{\mathcal{A}}]$ is a nuclear space and $S\in\mathcal{A}$, a certain power S^{-2r}, $r\in N$, is a nuclear operator. We can choose an orthonormal base $\{\phi_k, k\in N\}$ of $\mathcal{H}$ consisting of eigenvectors of S. Let $\{\mu_k, k\in N\}$ be the sequence of the corresponding eigenvalues. Since $S\geq I$, $\mu_k \geq 1$ $\forall\, k\in N$.

Consider an arbitrary 0-neighbourhood $\mathcal{V}_\alpha = \mathrm{aco}(\alpha_k V_{S^k}, k\in N)$ of λ whereby $\alpha_k > 0$ $\forall\, k\in N$.

Let $\gamma = \{\gamma_k := \sqrt{C^{-1}2^{-k-r}}\,\alpha_{k+r},\ k\in N\}$ where $C := \sum_{i=1}^\infty \mu_i^{-2r}$.

Notice that $C<\infty$ by the nuclearity of S^{-2r}.

We define $t := h_\gamma(S)^{-2}$.

STATEMENT I: $t \in \mathfrak{S}_1(\mathcal{D})_+$.

Proof: Let $a\in L^+(\mathcal{D})$. By the closed graph theorem, there is a $n\in N$ such that $\|a\phi\| \leq C_a\|S^n\phi\|$ $\forall\, \phi\in\mathcal{D}$. Then we have

$$\sum_{k=1}^\infty \|a\sqrt{t}\,\phi_k\|^2 \leq C_a \sum_{k=1}^\infty \|S^n h_\gamma(S)^{-1}\phi_k\|^2 = C_a\sum_{k=1}^\infty \mu_k^{2n}\, h_\gamma(\mu_k)^{-2}$$

$$= C_a \sum_{k=1}^\infty \mu_k^{-2r}\left[\mu_k^{2n+2r}\, h_\gamma(\mu_k)^{-2}\right] < +\infty \text{ because } \sum_{k=1}^\infty \mu_k^{-2r} < +\infty$$

and $\sup_{k\in N} \mu_k^{2n+2r}\, h_\gamma(\mu_k)^{-2} < +\infty$ by lemma 5,(iii). Therefore, $a\sqrt{t}$ is a Hilbert-Schmidt operator. Putting $a=I$, we see that $\sqrt{t}$ is of Hilbert-Schmidt class. Hence, the closure of $at = a\sqrt{t}\sqrt{t}$ is of trace class, that is, $t\in\mathfrak{S}_1(\mathcal{D})_+$. //

Let $f(a) = \mathrm{Tr}\, ta$ for $a\in\mathcal{A}$.

STATEMENT II: If $f(a^+a)\leq 1$ for $a\in\mathcal{A}$, then $a\in\mathcal{V}_\alpha$.

Proof: Suppose that $f(a^+a) \equiv \sum_{k=1}^\infty h_\gamma(\mu_k)^{-2}\,\|a\phi_k\|^2 \leq 1$.

Let $a\in\mathcal{M}_{S^1}$. Now we repeat some arguments used in the proof of

theorem 1. Let $P_k = E(M_{m_{k+1}}) - E(M_{m_k} - 0)$ and let $F_k = P_1 + \dots + P_k$.
We choose m_s so large that $M_{m_s} \geq 2\lambda_1(a)\,\alpha_{l+1}^{-1}$. Then

$$\|a(I-F_s)\phi\| \le \lambda_l(a)\,\|S^l(I-F_s)\phi\| \le M_{m_s}^{-1}\lambda_l(a)\,\|S^{l+1}(I-F_s)\phi\| \le$$
$$1/2\,\alpha_{l+1}\|S^{l+1}\phi\| \quad \text{, i.e. } a(I-F_s) \in 1/2\,\alpha_{l+1}\,V_{S^{l+1}} . \qquad (5)$$

Let N_k be the set of all $i \in N$ with $\mu_i \in [M_{m_k}, M_{m_{k+1}}]$. Then $P_k\phi = \sum_{i\in N_k}\langle\phi,\phi_i\rangle\phi_i$. Using lemma 5,(ii), and $f(a^+a)\le 1$, we get

$$\|aP_k\phi\|^2 \le \Big(\sum_{i\in N_k}|\langle\phi,\phi_i\rangle|\,\|a\phi_i\|\Big)^2 \le \Big(\sum_{i\in N_k}\mu_i^{-2r}\Big)\Big(\sum_{i\in N_k}|\langle\phi,\phi_i\rangle|^2\mu_i^{2r}\|a\phi_i\|^2\Big)$$
$$\le c\sum_{i\in N_k}|\langle\phi,\phi_i\rangle|^2\mu_i^{2r}\,\|h_\gamma(S)\,\phi_i\|^2 = c\,\gamma_{n_k}^2\sum_{i\in N_k}|\langle\phi,\phi_i\rangle|^2\mu_i^{2r+2n_k}$$
$$= c\,\gamma_{n_k}^2\,\|S^{r+n_k}P_k\phi\|^2 \le 2^{-r-n_k}\,\alpha_{r+n_k}^2\,\|S^{r+n_k}\phi\|^2 ,$$
$$aP_k \in 2^{-r-n_k}\,\alpha_{r+n_k}\,V_{S^{r+n_k}} . \qquad (6)$$

Since $a = \sum_{k=1}^{s} aP_k + a(I-F_s)$, (5) and (6) together give us $a \in V_\alpha$. Thus, we proved that $\lambda \le \sigma^{\mathcal F}$.
It remains to show that $\sigma^{\mathcal F} \le \lambda$. Let $t \in \mathfrak S_1(\mathcal D)_+$. We have $\mathrm{Tr}\,tx = \sum_i \delta_i\langle x\psi_i,\psi_i\rangle$ whereby $\{\psi_i\}$ is an orthonormal system of eigenvectors corresponding to the positive eigenvalues δ_i of t. Notice that $\psi_i = t(\delta_i^{-1}\psi_i) \in \mathcal D$ because $t\mathcal H \subseteq \mathcal D$.
Now let $a \in \mathcal M_{S^l}$. Then

$$f(a^+a) = \mathrm{Tr}\,ta^+a = \sum_i \delta_i\|a\psi_i\|^2 \le \sum_i \delta_i\lambda_l(a)^2\|S^l\psi_i\|^2$$
$$= \lambda_l(a)^2 f(S^{2l}).$$

Since λ is the inductive topology on $\mathcal A$ with respect to the family of normed spaces $\{(\mathcal M_{S^k}, \lambda_k), k\in N\}$,this implies that $\sigma^{\mathcal F}\le\lambda$, completing the proof of proposition 6.
//

4. THREE EXAMPLES

In this section we examine three examples. Examples 2 and 3 may be regarded as counter-examples to our main problem.

EXAMPLE 1:
Our first examples gives two classes of Op*-algebras for which our question has an affirmative answer. To see this, we reformulate the main results of /10/.

THEOREM: Let $\mathcal A$ be a countable generated Op*-algebra on $\mathcal D$.
(1) If $\tau_{\mathcal D} = \tau_{st}$ or $\rho = \tau_{st}$, then we have $\tau_{\mathcal D} = \rho = \tau_{st}$.
(2) If $\tau^{(\mathcal D)} = \tau_{st}$ or $\lambda = \tau_{st}$, then $\tau^{(\mathcal D)} = \lambda = \tau_{st}$.
(3) Suppose $\mathcal A$ is closed on $\mathcal D$.
If $\sigma^{\mathcal D} = \tau_{st}$ or $\tau^{(\mathcal D)} = \tau_{st}$ or $\lambda = \tau_{st}$, then $\sigma^{\mathcal D} = \tau^{(\mathcal D)} = \lambda = \tau_{st}$.

Proof:
(1): According to theorem 1 in /10/, we know that $\tau_{\mathcal D} = \tau_{st}$ if and only if all vector spaces $\mathcal N_x, x \in \mathcal A$, are finite dimensional. Since $\mathcal A[\rho]$ is the inductive limit of the spaces $\mathcal N_x$, the latter is equivalent to $\rho = \tau_{st}$. This proves (1).
(3): Since $\sigma^{\mathcal D} \le \tau^{(\mathcal D)} \le \lambda \le \tau_{st}$, it is sufficient to show that $\lambda = \tau_{st}$ implies $\sigma^{\mathcal D} = \tau_{st}$. If $\lambda = \tau_{st}$, then the vector space $\mathcal M_x$ is finite dimensional for each $x \in \mathcal A$ because $\mathcal A[\lambda]$ is the inductive limit of the spaces $\mathcal M_x, x \in \mathcal A$. In view of theorem 2 in /10/, this is equivalent to $\sigma^{\mathcal D} = \tau_{st}$.
(2): Suppose that $\lambda = \tau_{st}$ on $\mathcal A$. Then, $\lambda = \tau_{st}$ on $\underline{\mathcal A}$. By (3), we get $\sigma^{\mathcal D} = \tau_{st}$ on $\underline{\mathcal A}$. Hence for each seminorm p on $\mathcal A$ there exist vectors $\phi^1,\dots,\phi^k \in \underline{\mathcal D}$ such that $p(a) \le \sum_i \|\underline{a}\,\phi^i\|\ \forall a \in \mathcal A$ (In / /, it was even shown that one vector suffices). Since $\underline{\mathcal D}$ is the completion of $\mathcal D[t_{\mathcal A}]$, there are sequences $\{\phi_n^i, n\in N\}$ $i=1,\dots,k$, $\phi_n^i \in \mathcal D$, with $\phi^i = t_{\mathcal A}\text{-}\lim_n \phi_n^i$. $\mathfrak M := \{\phi_n^i\}$ is a $t_{\mathcal A}$-bounded set. Further, $p(a) \le p^{\mathfrak M}(a) \equiv \sup_{i,n}\|a\phi_n^i\|$. Consequently, $\tau^{(\mathcal D)} = \tau_{st}$. //

Our theorem can be rephrased by saying that for countable generated "very unbounded " Op*-algebras (which means that the vector spaces $\mathcal N_x$ resp. $\mathcal M_x$ are finite dimensional for all $x \in \mathcal A$) we have $\tau_{\mathcal D} = \rho = \tau_{st}$ resp. $\tau^{(\mathcal D)} = \lambda = \tau_{st}$.

EXAMPLE 2:

Let $\mathcal{A} = \underline{E}_{\otimes}$ be the tensor algebra over a nuclear Frechet space $E[\tau]$ with a continuous involution. For example, one can take the Schwartz space $\mathcal{S}(\mathbb{R}_n)$ for $E[\tau]$. $\mathcal{A}$ is a $*$-algebra in a natural way. In /12/.it was shown that the $*$-algebra $\mathcal{A}$ can be realized as an Op$*$-algebra on a domain $\mathcal{D} = \mathcal{D}_\infty(S)$, S a certain self-adjoint operator in a Hilbert space, such that $\tau_{\mathcal{D}} = \tau^{(\mathcal{D})} = \rho = \lambda$ and $\mathcal{A}_{\mathcal{A}} = \mathcal{A}_+$. For the definitions and facts about tensor algebras used in this example we refer to /6/ or to /12/.

We claim that $\rho = \lambda = \tau_{\otimes}$ for this realization of $\mathcal{A}$ as an Op$*$-algebra. Indeed, by corollary 3 in section 2 $\mathcal{A}[\rho]$ is the associated bornological space for $\mathcal{A}[\tau_{\mathcal{D}}]$. Since $\mathcal{A}[\tau_{\otimes}]$ is bornological and $\tau_{\otimes}$ and τ_∞ have the same bounded sets, $\tau_{\mathcal{D}} = \tau_\infty$ implies $\rho = \tau_{\otimes}$. Similarly, we get $\lambda = \tau_{\otimes}$.

Suppose now that $E[\tau]$ is not normable (since $E[\tau]$ is nuclear, this is equivalent to the requirement that E is not finite dimensional). Then we have $\tau_{\otimes} \neq \tau_\infty$. Consequently, $\rho \neq \tau_{\mathcal{D}}$ and $\lambda \neq \tau^{(\mathcal{D})}$.

From this example we can learn a little bit more.

1. Because $\mathcal{A}[\tau_{\otimes}]' \neq \mathcal{A}[\tau_\infty]'$, $\mathcal{A}[\rho]$ and $\mathcal{A}[\tau_{\mathcal{D}}]$ have different dual spaces and ρ is not the Mackey topology to $\tau_{\mathcal{D}}$ in general.
2. In our example, the cone $\mathcal{A}_+$ is not normal with respect to the order topology ρ . Since $\rho = \tau_{\otimes}$, this follows from the known fact that the smaller cone $P(\mathcal{A}) := \left\{ \sum_1 x_i^+ x_i, x_i \in \mathcal{A} \right\}$ of all finite sums of squares is not $\tau_{\otimes}$-normal (/6/).
3. Moreover, our example can be used to give an order-bounded linear functional which is not a linear combination of (strongly) positive linear functionals. The only example of this kind which the author has found in the standard literature on ordered vector spaces is due to Namioka (cf./2/,p.30). Namioka's construction makes use of the spaces L_p, $0 < p < 1$.

Let $E[\tau] = \mathcal{S}(\mathbb{R}_1)$. Then $\mathcal{A} \equiv \underline{E}_{\otimes} = \sum_{n=0}^{\infty} \underline{E}_n$ (direct sum) where $\underline{E}_0 = C_1$ and $\underline{E}_n = \mathcal{S}(\mathbb{R}_n)$ for $n \in N$. We define a linear functional f on $\mathcal{A}$ by putting $f(x) := x_0 + \sum_{n=1}^{\infty} \frac{\partial^{n \cdot n}}{\partial t_1^n \ldots \partial t_n^n} x_n(0,\ldots,0)$ for $x = (x_n) \in \mathcal{A}$, $x_n \in \underline{E}_n$. Obviously, f is $\tau_{\otimes}$-continuous. Hence, f is

order-bounded because the order topology ρ coincides with $\tau_{\otimes}$. f is not a linear combination of $\mathcal{A}_+$-positive linear functionals because it is not in the linear hull of $P(\mathcal{A})$-positive linear functionals (/13/,section 4,theorem 5,(iii)) and $P(\mathcal{A}) \subseteq \mathcal{A}_+$.

Thus, we have seen that there are Op$*$-algebras $\mathcal{A}$ with $\mathcal{A}_{\mathcal{A}} = \mathcal{A}_+$ on domains $\mathcal{D} = \mathcal{D}_\infty(S)$ such that $\tau_{\mathcal{D}} \neq \rho$ and $\tau^{(\mathcal{D})} \neq \lambda$. On the other side, if the Op$*$-algebra $\mathcal{A}$ on $\mathcal{D} = \mathcal{D}_\infty(S)$ satisfies the assumptions of theorem 1, then $\tau_{\mathcal{D}} = \rho$ and $\tau^{(\mathcal{D})} = \lambda$ on $\mathcal{A}$ (in particular, this is true for $L^+(\mathcal{D})$). In our next example, we construct a Frechet domain $\mathcal{D}[\mathcal{A}_+]$ such that $\tau_{\mathcal{D}} \neq \rho$ and $\tau^{(\mathcal{D})} \neq \lambda$ even on $L^+(\mathcal{D})$.

EXAMPLE 3:

Let $a^{(k)}$, $k \in N$, be the infinite matrix $a^{(k)} = (a_{ij}^{(k)}) = (y_1^{(k)}, \ldots, y_{k-1}^{(k)}; k^k e, k^k e, \ldots)$ where $y_j^{(k)} = (1, 2^k, 3^k, 4^k, \ldots)$, $e = (1,1,1,\ldots)$, $j = 1,\ldots,k-1$, $k \in N$. Writing each matrix as a sequence $a^{(k)}$ corresponds to a diagonal operator a_k in the Hilbert space l_2. We use the following notations:

$a_{k}^{n} := a_{k_1}^{n_1} \ldots a_{k_r}^{n_r}$ for $k = (k_1,\ldots,k_r)$, $n = (n_1,\ldots,n_r)$, $k_i, n_i \in N$, $r \in N$, and $|k| = k_1 + \ldots + k_r$.

Let $\mathcal{D} = \bigcap_{k,n} \mathcal{D}(a_k^n)$. Let $\mathcal{A}$ be the set of all complex matrices $a = (a_{ij})$ whose elements can be estimated by the elements of a certain operator a_k^n , i.e. $|a_{ij}| \leq C\, a_{ij}^{(k_1)\, n_1} \ldots a_{ij}^{(k_r)\, n_r}$ where $C > 0$ is a constant. $\mathcal{A}$ corresponds to an Op$*$-algebra of diagonal operators on $\mathcal{D}$. Clearly, all a_k^n are in $\mathcal{A}$.

STATEMENT: Let $\mathcal{B}$ be an Op$*$-algebra on $\mathcal{D}$ with $\mathcal{B} \supseteq \mathcal{A}$. Then we have $\tau_{\mathcal{D}} \neq \rho$ and $\tau^{(\mathcal{D})} \neq \lambda$ on $\mathcal{B}$.

Proof:

We prove only that $\tau_{\mathcal{D}} \neq \rho$.

Take the 0-neighbourhood $\mathcal{U}_\alpha = \mathrm{aco}(\alpha_{k,n}\, U_{a_k^n}; k, n)$ for the topology ρ where $\alpha_{k,n} = |k|^{|k||n|}$. Let $\mathcal{M}$ be an arbitrary bounded set of $\mathcal{D}[\mathcal{A}_{\mathcal{B}}]$. Then $\sup_{\phi \in \mathcal{M}} \| a_{k+1} \phi \|^2 = \sum_{i,j} |\phi_{ij}|^2\, a_{ij}^{(k+1)2} =: C_k^2 < +\infty$.

In particular, this implies that $\sup_{\phi \in \mathcal{M}} |\phi_{kj}|\, j^k \leq C_k$ for each $j \in N$.

We choose $j=j_k$ so large that $c_k^2 j_k^{-2k} \leq 2^{-(k+1)}$ for all $k \in N$.

Put $a_{ij} = 2$ for $(i,j)=(k,j_k)$ and $a_{ij} = 0$ otherwise.
We claim that $p_{\mathfrak{M}}(a) \leq 1$ and $a \notin \mathcal{U}_\alpha$ for $a=(a_{ij}) \in \mathcal{B}$.

In fact, we have $p_{\mathfrak{M}}(a) = \sup_{\phi \in \mathfrak{M}} \sum_{i,j} a_{ij} |\phi_{ij}|^2 = \sup_{\phi \in \mathfrak{M}} \sum_{k=1}^{\infty} 2 |\phi_{kj_k}|^2$
$\leq \sum_{k=1}^{\infty} 2\, c_k^2 j_k^{-2k} \leq \sum_{k=1}^{\infty} 2 \cdot 2^{-(k+1)} = 1.$

Now consider an arbitrary element $b = \sum_{i=1}^{r} \mu_i b_i \in \mathcal{U}_\alpha$ where

$b_i \in \alpha_{k_i, n_i} U_{a_{k_i}}^{n_i}$ and $\sum_{i=1}^{r} |\mu_i| \leq 1$. The absolute values of the

matrix elements in the m.-row of b are not larger than
$\sum_{i=1}^{r} |\mu_i|\, |k_i|^{|k_i||n_i|} \alpha_{k_i, n_i} \leq 1$ if we take $m := \operatorname{Max}_i |k_i|$.
Therefore, $a \notin \mathcal{U}_\alpha$.
Thus, there is no bounded subset $\mathfrak{M}$ of $\mathcal{D}[t_{\mathcal{B}}]$ such that
$\mathcal{W}_{\mathfrak{M}} := \{x \in \mathcal{B} : p_{\mathfrak{M}}(x) \leq 1\} \subseteq \mathcal{U}_\alpha$. This completes the proof.
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