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TWO THEOREMS ABOUT TOPOLOGIES ON COUNTABLE GENERATED OP*-ALGEBRAS

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Две теоремы о счетно-порожденных OP*-алгебрах

Найдена характеристика счетно-порожденных (замкнутых) OP* -алгебр A, для которых равномерная топология τ_D (относительно сильная операторная топология τ_D) совпадает с сильнейшей локально-выпуклой топологией на A.

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Two Theorems About Topologies on Countable Generated OP*-Algebras

We characterize the countable generated (closed) Op*-algebras A for which the uniform topology τ_D (resp. the strong operator topology τ_D) concides with the strongest locally convex topology on A.

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0. INTRODUCTION

The present paper deals with the topologization of unbounded operator algebras (Op*-algebras) in Hilbert space. We consider two possible topologies, the so-called uniform topology $\tau_{\mathcal{D}}$ introduced in /2/ and the strong operator topology $\sigma^{\mathcal{D}}$. We characterize the countable generated [closed] Op*-algebras $\mathcal{A}$ for which $\tau_{\mathcal{D}}$ [resp. $\sigma^{\mathcal{D}}$] agrees with the strongest locally convex topology on $\mathcal{A}$. Our main theorems contain some known result for concrete Op*-algebras (/2/,/3/,/4/,/6/).

In /5/, theorem 1 was used in proving that for certain Op*-algebras $\mathcal{A}$ (for example, the Op*-algebra of all differential operators with polynomial coefficients on the Schwartz space $\mathcal{S}(R_n)$) all linear functionals f on $\mathcal{A}$ are trace functionals, i.e. they can be given by $f(a) = \mathrm{Tr}\, ta$, $a \in \mathcal{A}$, t an appropriate nuclear operator.

1. DEFINITIONS AND NOTATIONS

First we repeat some basic definitions and facts about unbounded operator algebras from /2/. Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. An Op*-algebra $\mathcal{A}$ on $\mathcal{D}$ is a *-algebra of unbounded operators leaving the domain $\mathcal{D}$ invariant. We assume that the identity map is in $\mathcal{A}$ and denote it by 1. The graph topology $t_{\mathcal{A}}$ on $\mathcal{D}$ is the locally convex topology defined by the seminorms $\|\phi\|_a := \|a\phi\|$, $\phi \in \mathcal{D}$, $a \in \mathcal{A}$. For each bounded subset $\mathfrak{M}$ of $\mathcal{D}[t_{\mathcal{A}}]$ we put $p_{\mathfrak{M}}(a) = \sup_{\phi,\psi \in \mathfrak{M}} |\langle a\phi, \psi\rangle|$. The uniform topology $\tau_{\mathcal{D}}$ on $\mathcal{A}$ is generated by the family $\{p_{\mathfrak{M}}\}$ of these seminorms. $\mathcal{A}[\tau_{\mathcal{D}}]$ is always a topological *-algebra. The strong operator topology $\sigma^{\mathcal{D}}$ on $\mathcal{A}$ is given by the seminorms $\|a\|_\phi := \|a\phi\|$, $\phi \in \mathcal{D}$, $a \in \mathcal{A}$.

Let $\underline{\mathcal{D}}(\mathcal{A}) := \bigcap_{a \in \mathcal{A}} \mathcal{D}(\bar{a})$. The operators $\underline{a} := \bar{a} \upharpoonright \underline{\mathcal{D}}$ form an Op*-algebra $\underline{\mathcal{A}}$ on $\underline{\mathcal{D}} = \underline{\mathcal{D}}(\mathcal{A})$ which will be called the closed extension of $\mathcal{A}$. $\mathcal{A}$ is said to be closed if $\mathcal{A} = \underline{\mathcal{A}}$, i.e. $\mathcal{D} = \underline{\mathcal{D}}(\mathcal{A})$.

An Op*-algebra $\mathcal{A}$ is closed on $\mathcal{D}$ if and only if the space $\mathcal{D}[t_{\mathcal{A}}]$ is complete.

Furthermore we use the following notations throughout the paper (adapted from /1/):

$\mathcal{N}_x := \{a \in \mathcal{A} : |\langle a\phi, \phi\rangle| \leq C_{a,x} \|x\phi\|^2 \quad \forall \phi \in \mathcal{D}\}$,

$\mathcal{M}_x := \{a \in \mathcal{A} : \|a\phi\| \leq C_{a,x} \|x\phi\| \quad \forall \phi \in \mathcal{D}\}$,

$\rho_x(a) := \sup_{\phi \in \mathcal{D}} \frac{|\langle a\phi, \phi\rangle|}{\|x\phi\|^2}$, $\lambda_x(a) := \sup_{\phi \in \mathcal{D}} \frac{\|a\phi\|}{\|x\phi\|}$ for $a, x \in \mathcal{A}$.

Here we make the convention that $\frac{C}{0} = +\infty$ for $C > 0$ and $\frac{0}{0} = 0$.

Clearly, $\mathcal{N}_x$ and $\mathcal{M}_x$ are vector spaces.

By τ_{st} we always denote the strongest locally convex topology on $\mathcal{A}$.

3. THE RESULTS

THEOREM 1: For each countable generated Op*-algebra $\mathcal{A}$ on $\mathcal{D}$ the following are equivalent:

(1.1) For all operators $x \in \mathcal{A}$ the vector space $\mathcal{N}_x$ is finite dimensional.

(1.2) There are operators $x_n \in \mathcal{A}, n \in \mathbb{N}$, such that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_{x_n}$ and the vector spaces $\mathcal{N}_{x_n}$ are finite dimensional.

(1.3) $\tau_{\mathcal{D}} = \tau_{st}$.

THEOREM 2: Let $\mathcal{A}$ be a countable generated Op*-algebra on $\mathcal{D}$. Consider the following conditions:

(2.1) For all operators $x \in \mathcal{A}$ the vector space $\mathcal{M}_x$ is finite dimensional.

(2.2) There are operators $x_n \in \mathcal{A}, n \in \mathbb{N}$, such that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_{x_n}$ and the vector spaces $\mathcal{M}_{x_n}$ are finite dimensional.

(2.3) $\sigma^{\mathcal{D}} = \tau_{st}$.

Then we have (2.3) $\longrightarrow$ (2.2) $\longleftrightarrow$ (2.1). If $\mathcal{A}$ is a closed Op*-algebra on $\mathcal{D}$, then all three conditions are equivalent.

The proofs of theorems 1 and 2 will be given in sections 4 and 5. Here we note a corollary only.

COROLLARY 3: If $\mathcal{A}$ is a countable generated Op*-algebra on $\mathcal{D}$ and if $\tau_{\mathcal{D}} = \tau_{st}$ on $\mathcal{A}$, then $\sigma^{\mathcal{D}} = \tau_{st}$ on $\underline{\mathcal{A}}$.

Proof:

Let $x \in \underline{\mathcal{A}}$. Since $\tau_{\mathcal{D}} = \tau_{st}$, theorem 1 implies that the space $\mathcal{N}_{x^+x+1}$ is finite dimensional. Moreover, $\mathcal{M}_x \subseteq \mathcal{N}_{x^+x+1}$ by the Cauchy-Schwarz inequality. Therefore, $\mathcal{M}_x$ is finite dimensional. Since all operators $a \in \mathcal{A}$ are $t_{\mathcal{A}}$-continuous on $\mathcal{D}$ and $\mathcal{D}$ is $t_{\mathcal{A}}$-dense in $\underline{\mathcal{D}}$, it is clear that $a \in \mathcal{M}_x$ if and only if $\underline{a} \in \mathcal{M}_{\underline{x}}$. Hence, $\mathcal{M}_{\underline{x}}$ is a finite dimensional vector space. Thus, condition (2.1) is fulfilled and we have $\sigma^{\mathcal{D}} = \tau_{st}$. //

Remark:

In /1/, Op*-algebras satisfying condition (1.1) are called hyperfinite.

4. SOME EXAMPLES

In this section we mention some examples of Op*-algebras satisfying the assumptions of our theorems.

Example 1: Let $\mathcal{A}_1$ be the Op*-algebra $\mathcal{P}(T)$ of all polynomials in a symmetric linear operator T on a dense invariant domain $\mathcal{D}_1$ in a Hilbert space. Suppose that the operator T is not bounded on $\mathcal{D}_1$.

Example 2: Denote by $\mathcal{A}_2$ the Op*-algebra generated by the position and momentum operators $q_j = t_j$, $p_j = i\frac{\partial}{\partial t_j}$, $j=1,\dots,n$, on the domain $\mathcal{D}_2 := C_0^\infty(\mathbb{R}_n)$. In other words, $\mathcal{A}_2$ is the *-algebra of all differential operators with polynomial coefficients.

Now we pass to a more general class of examples which give a greater variety of Op*-algebras fulfilling our conditions.

Example 3: Let G be a Lie group and $\mathcal{E}(G)$ be the universal enveloping algebra of the Lie algebra of G. Suppose $\mathcal{U}$ is a (fixed) neighbourhood of the unit element in G. If we realize $\mathcal{E}(G)$ as an algebra of left invariant differential operators acting on the Lie group G with the domain $\mathcal{D}_3 := C_0^\infty(\mathcal{U})$ in the Hilbert space $L_2(\mathcal{U}, \mu)$, μ the right Haar measure of G, then we obtain an Op*-algebra $\mathcal{A}_3$ depending on G.

With this notations we have the following theorem.

THEOREM 4: The uniform topologies on the Op*-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and the strong operator topologies on $\underline{\mathcal{A}}_1, \underline{\mathcal{A}}_2, \underline{\mathcal{A}}_3$ coincide with the strongest locally convex topologies τ_{st} on $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ resp. $\underline{\mathcal{A}}_1, \underline{\mathcal{A}}_2, \underline{\mathcal{A}}_3$.

Remarks:

1. Most of this results are already known. For $\mathcal{A}_1$ both assertions were first proved in /3/ (for the uniform topology partial results were obtained in /2/). In the case $\mathcal{A}_2$ both statements are shown in /6/. For $\mathcal{A}_3$ the assertion concerning the uniform topology was proved in /4/. The methods applied in the proofs for $\mathcal{A}_2$ and $\mathcal{A}_3$ in /6/ and /4/ are different from the method used in the present paper. Some basic arguments of the proofs are drawn from the proofs of theorems B and C in /2/.

2. By considering unitary representations of Lie groups (more precisely, the associated represenations of the enveloping algebras) algebras of differential operators (for example, the Op*-algebra generated by $a = t^{-1}$ and $p^2 = -\frac{d^2}{dt^2}$ on $\mathcal{D} = C_0^\infty(0,1)$), sequence spaces, etc., it is not difficult to construct further examples satisfying the conditions of theorems 1 and 2.

LEMMA 5: Suppose that $\mathcal{A}$ is an Op*-algebra on $\mathcal{D}$, $a, x \in \mathcal{A}$ and $a \in \mathcal{N}_x$. Then there is a constant $K_{a,x}$ such that

$$\|a\phi\|^2 \leq K_{a,x} \|x\phi\| \; \|xa\phi\| \quad \forall \phi \in \mathcal{D}. \tag{1}$$

Proof:

Let $\mathcal{U}_x := \{\phi \in \mathcal{D} : \|x\phi\| \leq 1\}$. Since $|\langle a\phi, \phi\rangle| \leq C_{a,x} \|x\phi\|^2 \; \forall \phi \in \mathcal{D}$ by $a \in \mathcal{N}_x$, we have $\sup_{\phi \in \mathcal{U}_x} |\langle a\phi, \phi\rangle| \leq C_{a,x}$.

Using $\langle a\phi, \psi\rangle = 1/4 \{\langle a(\phi+\psi), \phi+\psi\rangle - \langle a(\phi-\psi), \phi-\psi\rangle - i\langle a(\phi+i\psi), \phi+i\psi\rangle + i\langle a(\phi-i\psi), \phi-i\psi\rangle\}$ it follows $\sup_{\phi,\psi \in \mathcal{U}_x} |\langle a\phi, \psi\rangle| \leq 4\, C_{a,x}$ because the elements $1/2(\phi+\psi), \ldots, 1/2(\phi-i\psi)$ are in the absolutely convex set $\mathcal{U}_x$. Hence, we get $|\langle a\phi, \psi\rangle| \leq 4C_{a,x} \|x\phi\| \, \|x\psi\| \quad \forall \phi, \psi \in \mathcal{D}$. Putting $\psi = a\phi$, this gives (1). //

Now let us turn to the proof of theorem 4.

By corollary 3, we only have to prove the assertions about the uniform topologies. In view of theorem 1, it is sufficient to show that condition (1.1) is fulfilled for $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, that is, the vector spaces $\mathcal{N}_x$ are finite dimensional for all operators x of the Op*-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Here we only carry out the proof of this fact in the case of $\mathcal{A}_2$. For $\mathcal{A}_1$ and $\mathcal{A}_3$ condition (1.) could be verified by repeating parts of the arguments used in /2/ and /4/.

Proof for the Op*-algebra $\mathcal{A}_2$:

For simplicity in notation we restrict us to the case n = 1. The subset $\{y_{ij} := q^i p^j, i \in N, j \in N\}$ is a Hamel base for $\mathcal{A}_2$. Take a fixed vector $\phi \in \mathcal{D}_2$, $\phi \neq 0$, with supp $\phi \subseteq [0,1]$ and put $\phi_{\alpha,\beta}(t) := \beta\, \phi(\beta(t-\alpha))$ for $\alpha, \beta \in \mathbb{R}_1$. A simple calculation leads to

$$\alpha^i \beta^j \|p^j\phi\| \leq \|y_{ij}\phi_{\alpha,\beta}\| \leq (\alpha+1)^i \beta^j \|p^j\phi\| \tag{2}$$

where $\|p^j\phi\| \neq 0 \;\; \forall j \in N$.

Let $a = \sum_{i,j} \alpha_{ij} y_{ij}$, $x = \sum_{i,j} \beta_{ij} y_{ij}$ and $xa = \sum_{i,j} \gamma_{ij} y_{ij}$. The degree of an element a is defined by $d(a) := \mathrm{Max}\{i+j : \alpha_{ij} \neq 0\}$. Let (k,l) be the lexicographic largest tupel for which this maximum will be attained, i.e. $k = \mathrm{Max}\{i : \alpha_{ij} \neq 0 \text{ for } j = d(a)-i \geq 0\}$ and $l = d(a)-j$. Denote by (r,s) the corresponding tupel for the element x.

Now we show that $a \in \mathcal{N}_x$ implies $d(a) \leq 2d(x)+1$. In particular, this means that $\mathcal{N}_x$ is a finite dimensional vector space.

We assume that $d(a) \geq 1$ and $d(x) \geq 1$ (otherwise the assertion is trivial).

Let $\beta = \alpha^{1 - \frac{1}{k+l+r+s}}$. Then $\alpha^i \beta^j = \alpha^{i+j-\frac{j}{k+l+r+s}} \quad \forall i, j \in N$. (3)

We write, $f(\alpha) = \mathcal{O}(\alpha^m)$, $m \in R_1$, for a function $f(\alpha)$ if $|f(\alpha)\alpha^{-m'}|$ is bounded for sufficiently large α if and only if $m' \geq m$. m is called the order of the function f with respect to α.

From (2) and (3) it follows that $\|y_{kl}\phi_{\alpha,\beta}\| = \mathcal{O}(\alpha^{k+l-\frac{l}{k+l+r+s}})$ and for all other elements y_{ij} with $\alpha_{ij} \neq 0$ the functions $\|y_{ij}\phi_{\alpha,\beta}\|$ have smaller orders with respect to α. By the triangle inequality, this implies $\|a\phi_{\alpha,\beta}\|^2 = \mathcal{O}(\alpha^{2(k+l-\frac{l}{k+l+r+s})})$. (4)

Further we have $\|x\phi_{\alpha,\beta}\| = \mathcal{O}(\alpha^{r+s-\frac{s}{k+l+r+s}})$. (5)

By the commutation rules it is clear that $\gamma_{ij} = 0$ for $j > l+s$ and $\gamma_{k+r,l+s} \equiv \alpha_{kl}\beta_{rs} \neq 0$. Hence, if there is a $i \in N$ such that $\gamma_{ij} \neq 0$, then $\frac{j}{k+l+r+s} < 1$. Using this two facts, we get

$$\|xa\,\phi_{\alpha,\beta}\| = \mathcal{O}(\alpha^{k+l+r+s-\frac{l+s}{k+l+r+s}}). \tag{6}$$

Putting (4),(5) and (6) into (1), it follows

$2(k+l - \frac{l}{k+l+r+s}) \leq r+s - \frac{s}{k+l+r+s} + k+l+r+s - \frac{l+s}{k+l+r+s}$.

Therefore, $d(a) = k+l \leq 2r+2s+1 = 2d(x)+1$ which finishes the proof. //

4. PROOF OF THEOREM 1

(1.1) $\longrightarrow$ (1.2): Trivial.

(1.2) $\longrightarrow$ (1.1):

Let $x \in \mathcal{A}$. Since $\mathcal{A} = \bigcup_{n\in N} \mathcal{N}_{x_n}$, there exists a number $n\in N$ such that $x^+x \in \mathcal{N}_{x_n}$. This implies $\mathcal{N}_x \subseteq \mathcal{N}_{x_n}$. Hence $\mathcal{N}_x$ is finite dimensional.

(1.3) $\longrightarrow$ (1.1):

We suppose that $\tau_{\mathcal{D}} = \tau_{st}$. Let us assume that (1.1) is not true, i.e. there is an element $x\in\mathcal{A}$ such that $\mathcal{N}_x$ contains an infinite set $\{a_n, n\in N\}$ of linear independent operators $a_n \in \mathcal{A}$. We supply elements $b_m \in \mathcal{A}$ such that the system $\{a_n, b_m\}$ is a Hamel base of $\mathcal{A}$. Without restriction of generality we may assume that $\rho_x(a_n) \le 1/4$ $\forall\, n\in N$. Then $|\langle a\phi,\psi\rangle| \le \|x\phi\|\,\|x\psi\|$ $\forall\, \phi,\psi\in\mathcal{D}$ by polarization (cf. lemma 5 in section 3). For each positive sequence $\gamma = \{\gamma_n\}$ we define the seminorm $p_\gamma(a) = \sum_n \gamma_n|\alpha_n|$ for $a = \sum_n \alpha_n a_n + \sum_m \beta_m b_m$.
Since $\tau_{\mathcal{D}} = \tau_{st}$, there is a bounded subset $\mathfrak{M}$ of $\mathcal{D}[t_{\mathcal{A}}]$ such that $p_\gamma(a) \le p_{\mathfrak{M}}(a)$ $\forall\, a\in\mathcal{A}$. Putting $a = \alpha_1 a_1 + \dots + \alpha_k a_k$ we get
$p_\gamma(a) \equiv \sum_n \gamma_n|\alpha_n| \le p_{\mathfrak{M}}(a) \equiv \sup_{\phi,\psi\in\mathfrak{M}} |\langle(\alpha_1a_1+\dots+\alpha_ka_k)\phi,\psi\rangle| \le$
$\sum_n |\alpha_n| (\sup_{\phi,\psi\in\mathfrak{M}} \|x\phi\|\,\|x\psi\|) = C\sum_n|\alpha_n|$ whereby $C = \sup_{\phi,\psi\in\mathfrak{M}}\|x\phi\|\,\|x\psi\|$ $< +\infty$. Since $\alpha_1,\dots,\alpha_k,\dots$ are arbitrary complex numbers, this is a contradiction if $\sup_{n\in N}\gamma_n = +\infty$. //

Now we turn to the main part in the proof of theorem 1.

(1.1) $\longrightarrow$ (1.3):

First we note a simple lemma. We shall need it only for finite dimensional Hilbert spaces $\mathcal{H}_1$.

LEMMA 1: Let $\mathcal{H}_1$ be a Hilbert subspace of $\mathcal{H}$ with $\mathcal{H}_1 \subseteq \mathcal{D}$. Let P_1 be the orthogonal projection on $\mathcal{H}_1$ and $\mathcal{D}_1 := (1-P_1)\mathcal{D} \equiv \mathcal{D} \ominus \mathcal{H}_1$. Suppose $\mathcal{A}$ is an Op*-algebra on $\mathcal{D}$, $a,x\in\mathcal{A}$ and $\|\phi\| \le \|x\phi\|$ for all $\phi\in\mathcal{D}$.
If $\rho_x(a) \equiv \sup_{\phi\in\mathcal{D}} \frac{|\langle a\phi,\phi\rangle|}{\|x\phi\|^2} = +\infty$, then $\sup_{\psi\in\mathcal{D}_1}\frac{|\langle a\psi,\psi\rangle|}{\|x\psi\|^2} = +\infty$.

Proof:
Since the operators $a, a^+, x \in \mathcal{A}$ have dense defined adjoint operators in $\mathcal{H}$, their restrictions to $\mathcal{H}_1$ are closed and hence bounded by the closed graph theorem. Thus $\|a\eta\| \le C\|\eta\|$, $\|a^+\eta\| \le C\|\eta\|$, $\|x\eta\|\le C\|\eta\|$ $\forall\, \eta\in\mathcal{H}_1$.
Let us assume that $\sup_{\psi\in\mathcal{D}_1} \frac{|\langle a\psi,\psi\rangle|}{\|x\psi\|^2} = C_1 < +\infty$, i.e. $|\langle a\psi,\psi\rangle| \le C_1\|x\psi\|^2$ $\forall\,\psi\in\mathcal{D}_1$. For each $\phi\in\mathcal{D}$, $\phi=\psi+\eta$, $\psi\in\mathcal{D}_1$, $\eta\in\mathcal{H}_1$, we get $|\langle a\phi,\phi\rangle| = |\langle a(\psi+\eta),\psi+\eta\rangle| \le$
$|\langle a\psi,\psi\rangle| + |\langle a\eta,\psi\rangle| + |\langle\psi,a^+\eta\rangle| + |\langle a\eta,\eta\rangle| \le$
$C_1\|x\psi\|^2 + 2C\|\eta\|\|\psi\| + C\|\eta\|^2 \le C_1\|x(\phi-\eta)\|^2 + 2C\|\phi\|^2 + C\|\phi\|^2 \le$
$C_1(\|x\phi\| + C\|\eta\|)^2 + 3C\|\phi\|^2 \le [C_1(1+C)^2+3C]\,\|x\phi\|^2$ because $\|\phi\|\le\|x\phi\|$. Therefore $\rho_x(a) < +\infty$ which is a contradiction. //

Now suppose that condition (1.1) is fulfilled. To prove that $\tau_{\mathcal{D}} = \tau_{st}$, we need some preparations and notations. Let us take a sequence $\{x_n, n\in N\}$ of operators $x_n \in\mathcal{A}$ such that

(i) $\|\phi\| \le \|x_n\phi\| \le \|x_{n+1}\phi\|$ $\forall\,\phi\in\mathcal{D}$, $n\in N$,

(ii) $\mathcal{N}_{x_n} \subsetneqq \mathcal{N}_{x_{n+1}}$ and (iii) $\mathcal{A} = \bigcup_{n\in N}\mathcal{N}_{x_n}$.

It is very easy to see that such a sequence exists. The vector space $\mathcal{A}$ has a countable Hamel basis $\{y_n, n\in N\}$. Let $z_n = 1 + y_1^+y_1 + \dots + y_n^+y_n$. Then $y_i \in \mathcal{N}_{z_n}$ for $i\le n$. Take $x_1 = z_1$. Because $\mathcal{N}_{x_1}$ is finite dimensional, there is a number, hence a smallest number $n_2\in N$ such that $y_{n_2} \notin \mathcal{N}_{x_1}$. Putting $x_2 = x_1^2 + z_{n_2}^2 + 1$ we have $\mathcal{N}_{z_{n_2}} \cup \mathcal{N}_{x_1} \subseteq \mathcal{N}_{x_2}$, $\mathcal{N}_{x_1} \ne \mathcal{N}_{x_2}$ and $y_i \in \mathcal{N}_{x_2}$ $\forall\, i=1,\dots,n_2$.
Continuing this procedure, we get a sequence $\{x_n\}$ with desired properties.

Since each vector space $\mathcal{N}_{x_n}$, $n\ge 2$, is finite dimensional, $\mathcal{N}_{x_n}$ can be decomposed as a direct sum of $\mathcal{N}_{x_{n-1}}$ and a certain vector space $\mathcal{A}_n \subseteq \mathcal{N}_{x_n}$. Let $\mathcal{A}_1 = \mathcal{N}_{x_1}$. Then $\mathcal{A} = \sum_n \mathcal{A}_n$ (direct sum of vector spaces). Let d_n be the dimension of $\mathcal{A}_n$ and let $a_1^n, a_2^n, \dots, a_{d_n}^n$ be a basis of $\mathcal{A}_n$. $\mathcal{A}_n$ is $*$-invariant because $\mathcal{N}_{x_n}$ obviously is $*$-invariant. Without loss of generality we suppose that the operators a_i^n are symmetric (which is possible since $\mathcal{A}_n$ is $*$-invariant) and $\varrho_{x_n}(a_i^n) = 1$ for $i=1,\dots,d_n$. By $a_\alpha^n = \alpha_1 a_1^n + \dots + \alpha_{d_n} a_{d_n}^n$, $\alpha = (\alpha_1,\dots,\alpha_{d_n})$, we shall denote the elements of $\mathcal{A}_n$. Further we use the norm $\| a_\alpha^n \| := \sum_{i=1}^{d_n} |\alpha_i|$ on $\mathcal{A}_n$. Let S_n be the unit sphere in this norm. For each sequence $\gamma = \{\gamma_n, n\in N\}$ of positive real numbers γ_n we define the seminorm $p_\gamma(a) := \sum_n \gamma_n \| a_\alpha^n \|$, $a = \sum_n a_\alpha^n \in \mathcal{A}$, on $\mathcal{A}$. Clearly, all seminorms of this kind give the strongest locally convex topology τ_{st} on $\mathcal{A}$. Let us take a fixed sequence $\gamma = \{\gamma_n\}$.

<u>STATEMENT 2:</u> For each $n\in N$ there exists a finite set of vectors $\psi_1^n,\dots,\psi_{r_n}^n$ having the following properties:

(a) $$\max_{i=1,\dots,r_n} |\langle a_\alpha^n \psi_i^n, \psi_i^n\rangle| \geq \| a_\alpha^n \| \Big\{\gamma_n + 1 + \sum_{k,m<n} \max_{j,l,s} |\langle a_s^n \psi_j^k, \psi_l^m\rangle|$$

(b) $\|x_k \psi_i^n\| \leq 2^{-n}$ $\forall$ $k<n$ and $i=1,\dots,r_n$.

(c) $\langle a\psi_i^n, \psi_j^m\rangle = 0$ $\forall$ $a\in\mathcal{N}_{x_m}$, $1\leq n<m$, $i=1,\dots,r_n$, $j=1,\dots,r_m$.

Proof:
We choose the sequence $\psi_1^n,\dots,\psi_{r_n}^n$ by induction on n. We postpone the proof that $\psi_1^1,\dots,\psi_{r_1}^1$ exist because it requires parts of the following arguments.
Suppose for $k=1,\dots,n-1$ sequences ψ_i^k, $i=1,\dots,r_k$, are chosen such that the conditions (a),(b),(c) are satisfied. Let $\mathcal{H}_1$ be the linear span of all elements $a\psi_i^k$ where $a\in\mathcal{N}_{x_n}$, $k=1,\dots,n-1$, $i=1,\dots,r_k$. $\mathcal{H}_1$ is finite demensional because $\mathcal{N}_{x_n}$ is finite dimensional. Put $\mathcal{D}_1 = \mathcal{D}\ominus\mathcal{H}_1$. Let C_n be a fixed positive number. If $a_\alpha^n\in S_n$, then $a_\alpha^n \notin \mathcal{N}_{x_{n-1}}$ by construction. This means that $\varrho_{x_{n-1}}(a_\alpha^n) = +\infty$. In view of lemma 1, this implies

$$\sup_{\psi\in\mathcal{D}_1} \frac{|\langle a_\alpha^n \psi,\psi\rangle|}{\|x_{n-1}\psi\|^2} = +\infty .$$ Hence there exists a vector $\psi_\alpha^n \in\mathcal{D}_1$ (depending on a_α^n) such that $|\langle a_\alpha^n \psi_\alpha^n, \psi_\alpha^n\rangle| > C_n \|x_{n-1}\psi_\alpha^n\|^2$. (1)

Since inequality (1) remains valid if we multiply ψ_α^n with a factor, we may assume that $\|x_{n-1}\psi_\alpha^n\| = 2^{-n}$. (2)
By $U(a_\alpha^n)$ we denote the set of all elements $a_\beta^n\in S_n$ with $|\langle a_\beta^n \psi_\alpha^n, \psi_\alpha^n\rangle| \equiv |\beta_1\langle a_1^n\psi_\alpha^n,\psi_\alpha^n\rangle + \dots + \beta_{d_n}\langle a_{d_n}^n\psi_\alpha^n,\psi_\alpha^n\rangle| > C_n 2^{-n}$.
Clearly, $U(a_\alpha^n)$ is an open subset of the sphere S_n. Furthermore, $a_\alpha^n\in U(a_\alpha^n)$ according to (1) and (2). By the Heine-Borel theorem the open cover $\{U(a_\alpha^n)\}$ of S_n has a finite subcover $\{U(a_{\alpha_i}^n)\}$, $i=1,\dots,r_n$. Put $\psi_i^n = \psi_{\alpha_i}^n$, $i=1,\dots,r_n$. Then we have

$$\max_{i=1,\dots,r_n} |\langle a_\alpha^n \psi_i^n, \psi_i^n\rangle| > C_n 2^{-n} \quad \text{for all } a_\alpha^n\in S_n.$$
By norming elements $a_\alpha^n\in\mathcal{A}_n$ it follows

$$\max_i |\langle a_\alpha^n\psi_i^n, \psi_i^n\rangle| \geq C_n 2^{-2n} \| a_\alpha^n \| \quad \text{for each } a_\alpha^n\in\mathcal{A}_n.$$

Putting now $C_n = 2^{2n}\Big\{\gamma_n + 1 + \sum_{k,m<n} \max_{j,l,s} |\langle a_j^n\psi_l^k, \psi_s^m\rangle|\Big\}$,

this is just condition (a). Because $\|x_l\phi\| \leq \|x_{n-1}\phi\|$ $\forall\phi\in\mathcal{D}$, $l\leq n-1$, condition (b) is fulfilled by (2). The vectors ψ_i^n, $i=1,\dots,r_n$, are in $\mathcal{D}_1 = \mathcal{D}\ominus\mathcal{H}_1$ by construction. Hence, (c) is also true. Consequently, the induction hypothesis is proved.

We have to say some words about the construction of $\psi_1^1,\dots,\psi_{r_1}^1$. In this case we only have to check condition (a), i.e.

$$\max_i |\langle a_\alpha^1\psi_i^1, \psi_i^1\rangle| \geq \| a_\alpha^1 \| (\gamma_1 + 1).$$
This can be done by using the covering argument of the preceding proof. Now the proof of statement 2 is complete. //

Next we regard the following subset $\mathfrak{M}$ of the domain $\mathcal{D}$:

$$\mathfrak{M} := \Big\{ \eta = \sum_{n=1}^q \varepsilon_n \psi_{i_n}^n : q\in N,\ \varepsilon_n\in\mathbb{C}_1, |\varepsilon_n| = 1 \Big\} .$$

Let us verify this assertion. If $\lambda_a(x_n)=+\infty$ for a certain operator $a \in \mathcal{A}_n$, then the new sequence $x_1,\dots,x_{n-1},a,x_n,\dots$ gives a "finer" decomposition of $\mathcal{A}$ which satisfies (i) and (ii). (ii) is obvious. Since $a \notin \mathcal{M}_{x_{n-1}}$ implies $\mathcal{M}_{x_{n-1}} \neq \mathcal{M}_a$ and $\lambda_a(x_n)=+\infty$ implies $\mathcal{M}_a \neq \mathcal{M}_{x_n}$, (i) is also true. According to (2.1), all vector spaces $\mathcal{M}_x$, $x \in \mathcal{A}$, are finite dimensional. Consequently, by an induction argument this procedure can be continued until (iii) is fulfilled.

Without loss in generality, we can assume that $\lambda_{x_{n+1}}(x_n) \leq 1$.

Further we use the following notations from the proof of theorem 1: $a_1^n, d_n, a_\alpha^n, \| a_\alpha^n \|, S_n$.

STATEMENT 1: $C_n := \sup_{a_\alpha^n \in S_n} \lambda_n(x_n) < +\infty$ for each $n \in N$.

Proof:

Assume that the contrary is true. Then there exist sequences $a_{\alpha_k}^n \in S_n$, $k \in N$, (for brevity we write a_k instead of $a_{\alpha_k}^n$ and x for x_n) and $\phi_k \in \mathcal{D}$, $k \in N$, such that $\|a_k \phi_k\| \geq k \|x \phi_k\|$. We may assume that $\|x \phi_k\| = 1 \ \forall k \in N$ (otherwise we multiply by a suitable factor). Then we have $\lim_{k\to\infty} \| a_k \phi_k \| = 0$. By the compactness of the unit sphere S_n there is a subsequence of $\{a_k\}$ converging to an element $a \in S_n$. For simplicity suppose that $\lim_{k\to\infty} \| a_k - a \| = 0$.

Let $a_k - a = \alpha_{1k} a_1^n + \dots + \alpha_{d_n k} a_{d_n}^n$. Then

$$\|(a_k-a)\phi_k\| \leq \sum_{i=1}^{d_n} |\alpha_{ik}| \, \|a_i^n \phi_k\| \leq \sum_{i=1}^{d_n} |\alpha_{ik}| \, \|x \phi_k\| = \| a_k - a \|$$

$\longrightarrow 0$ for $k \to \infty$. By $\|a\phi_k\| \leq \|(a_k-a)\phi_k\| + \|a_k \phi_k\|$ this gives $\lim_{k\to\infty} \|a \phi_k\| = 0$.

On the other hand, we have $\lambda_a(x) < +\infty$ by condition (iii). In particular, this implies that $1 = \|x\phi_k\| \leq \lambda_a(x) \|a \phi_k\|$. This is a contradiction to $\lim_{k\to\infty} \|a\phi_k\| = 0$. //

An immediate consequence of statement 1 is

STATEMENT 2: There are constants $C_n > 0$, $n \in N$, with

$$\| a_\alpha^n \| \, \|x_n \phi\| \leq C_n \|a_\alpha^n \phi\| \quad \forall \phi \in \mathcal{D}, \ a_\alpha^n \in \mathcal{A}_n, \ n \in N.$$

Let $\gamma = \{\gamma_n\}$ be a sequence of positive numbers and q_γ be the seminorm on $\mathcal{A}$ defined by $q_\gamma(a) = \{\sum_n \gamma_n \| a_\alpha^n \|^2\}^{1/2}$ for $a = \sum_n a_\alpha^n \in \mathcal{A}$. Our goal is to prove that $\sigma^{\mathcal{D}} = \tau_{st}$ on $\mathcal{A}$. Since all seminorms q_γ define the topology τ_{st}, it is enough to show that for each sequence γ there exist a vector $\phi \in \mathcal{D}$ (depending on γ) such that $q_\gamma(a) \leq \|a\phi\| \ \forall a \in \mathcal{A}$. Now fix a positive sequence $\gamma = \{\gamma_n\}$. The next step of our construction is

STATEMENT 3: There exist a sequence $\{\delta_n, n \in N\}$ of positive numbers and a sequence $\{\phi_n, n \in N\}$ of vectors $\phi_n \in \mathcal{D}$ satisfying the following conditions:

(a) $\|x_n \phi_n\| = \sqrt{\delta_n^2 + \gamma_n C_n^2} + 1 + \sum_{i=1}^{n-1} \|x_n \phi_i\| \quad \forall n \in N.$

(b) $\|x_n \phi_n\| \leq 2\delta_n \quad \forall n \in N.$

(c) $\|x_k \phi_n\| \leq 2^{-n} \quad \forall k < n, \ k, n \in N.$

(d) The determinants $D_n = D_n(\delta_1,\dots,\delta_n)$ are positive.

$$D_n = \begin{vmatrix} C_1^{-2}\delta_1^2 & -16\,\delta_1\delta_2 & \dots & -16\,\delta_1\delta_n \\ -16\,\delta_1\delta_2 & C_2^{-2}\delta_2^2 & & -16\,\delta_2\delta_n \\ & & & \\ -16\,\delta_1\delta_n & -16\,\delta_2\delta_n & \dots & C_n^{-2}\delta_n^2 \end{vmatrix}$$

Proof:

In the case $n=1$ we take a positive number δ_1 with $2\delta_1 \geq \sqrt{\delta_1^2 + \gamma_1 C_1^2} + 1$ and a vector $\phi_1 \in \mathcal{D}$ with $\|x_1\phi_1\| = \sqrt{\delta_1^2 + \gamma_1 C_1^2} + 1$.

Now suppose that $\delta_1,\dots,\delta_{n-1}$ and $\phi_1,\dots,\phi_{n-1}$ are already chosen so that (a) - (d) are fulfilled. Let us consider the determinant D_n. D_n is a quadratic polynomial in δ_n. The coefficient of the quadratic term is just equal to $C_n^{-2} D_{n-1}$ which is positive by induction assumption. Hence δ_n may be taken so large that $D_n > 0$ and $2\delta_n \geq \sqrt{\delta_n^2 + \gamma_n C_n^2} + 1 + \sum_{i=1}^{n-1} \|x_n \phi_i\| := M_n$.

Further, we assumed that $\mathcal{M}_{x_{n-1}} \neq \mathcal{M}_{x_n}$, i.e. $\lambda_{x_{n-1}}(x_n) = +\infty$. Thus there is a vector $\phi_n \in \mathcal{D}$ such that $\|x_n \phi_n\| \geq M_n 2^n \|x_{n-1}\phi_n\|$. After a suitable norming of ϕ_n we obtain $\|x_n\phi_n\| = M_n$. Consequently, $\|x_{n-1}\phi_n\| \leq 2^{-n}$. Since $\|x_l \phi_n\| \leq \|x_{n-1}\phi_n\| \leq 2^{-n}$ for $l \leq n-1$, the conditions (a) - (d) are satisfied for $\delta_1,\dots,\delta_n$ and $\phi_1,\dots,\phi_n$. By induction, statement 3 is proved. //

From $\mathcal{A} = \bigcup_{n\in N} \mathcal{M}_{x_n}$ it is clear that the topology $t_{\mathcal{A}}$ on $\mathcal{D}$ can be given by the seminorms $\|\phi\|_{x_n} := \|x_n\phi\|$, $n\in N$. Therefore condition (c) of statement 3 implies that the sequence $\psi_n := \sum_{i=1}^{n} \phi_i$ is a Cauchy sequence in $\mathcal{D}[t_{\mathcal{A}}]$. Since the Op*-algebra $\mathcal{A}$ was assumed to be closed on $\mathcal{D}$, the space $\mathcal{D}[t_{\mathcal{A}}]$ is complete, Consequently, the sequence $\{\psi_n, n\in N\}$ is converging to an element $\phi = \sum_{i=1}^{\infty} \phi_i \in \mathcal{D}$.

STATEMENT 4: For all $a\in\mathcal{A}$ we have $\|a\phi\| \geq q_{\gamma}(a)$.

Proof:

Applying (a) and (c) we obtain

$$\|x_n\phi\| \geq \|x_n\phi_n\| - \sum_{i=1}^{n-1}\|x_n\phi_i\| - \sum_{i=n+1}^{\infty}\|x_n\phi_i\| \geq \sqrt{\delta_n^2+\gamma_n c_n^2},$$

i.e. $\|x_n\phi\|^2 - \gamma_n c_n^2 \geq \delta_n$. (1)

In a similar way, (a),(b) and (c) give us

$$\|x_n\phi\| \leq \|x_n\phi_n\| + \sum_{i=1}^{n-1}\|x_n\phi_i\| + \sum_{i=n+1}^{\infty}\|x_n\phi_i\| \leq$$

$$\|x_n\phi_n\| + \sum_{i=1}^{n-1}\|x_n\phi_i\| + \sum_{i=n+1}^{\infty} 2^{-i} \leq 2\|x_n\phi_n\| \leq 4\delta_n. \quad (2)$$

Now we make use of condition (d). It implies that the quadratic form $Q(t) := \sum_n t_n\bar{t}_n c_n^{-2}\delta_n^2 - 16\sum_{n\neq m} t_n\bar{t}_m \delta_n\delta_m$ is positive definite. In particular, this means that

$$\sum_n \|a_\alpha^n\|^2 c_n^{-2}\delta_n^2 - 16\sum_{n\neq m}\|a_\alpha^n\|\,\|a_\alpha^m\|\,\delta_n\delta_m \geq 0.$$

By the estimations (1) and (2) we get

$$\sum_n \|a^n_\alpha\|^2 c_n^{-2}(\|x_n\phi\|^2-\gamma_n c_n^2) - \sum_{n\neq m}\|a^n_\alpha\|\,\|a^m_\alpha\|\,\|x_n\phi\|\,\|x_m\phi\| \geq 0. \quad (3)$$

The triangle inequality combined with $\|a_i^n\phi\| \leq \|x_n\phi\|$ $\forall$ $i=1,\dots,d_n$, leads to $\|a^n_\alpha\phi\| \leq \|a^n_\alpha\|\,\|x_n\phi\|$.

Further we have $c_n^{-2}\|a^n_\alpha\|^2\|x_n\phi\|^2 \leq \|a^n_\alpha\phi\|^2$ by statement 2.

Putting the two last inequalities into (3) it follows that

$$\sum_n\|a^n_\alpha\phi\|^2 - \sum_n\|a^n_\alpha\|^2\gamma_n - \sum_{n\neq m}\|a^n_\alpha\phi\|\,\|a^m_\alpha\phi\| \geq 0.$$

Therefore

$$\|a\phi\|^2 - q_\gamma(a)^2 = \Big\langle\sum_n a^n_\alpha\phi, \sum_m a^m_\alpha\phi\Big\rangle - \sum_n\|a^n_\alpha\|^2\gamma_n \geq$$

$$\sum_n\|a^n_\alpha\phi\|^2 - \sum_{n\neq m}\|a^n_\alpha\phi\|\,\|a^m_\alpha\phi\| - \sum_n\|a^n_\alpha\|^2\gamma_n \geq 0$$

which completes the proof.

6. CONCLUDING REMARKS

The preceding proofs of our theorems 1 and 2 show that the multiplication in the Op*-algebra $\mathcal{A}$ was used only to ensure that the families of vector spaces $\{\mathcal{N}_x, x\in\mathcal{A}\}$ and $\{\mathcal{M}_x, x\in\mathcal{A}\}$ are directed. In fact, our proofs yield the following more general results.

Let $\mathcal{A}$ be a vector space of linear operators on a dense domain $\mathcal{D}$ in a Hilbert space (we don't assume that the operators map $\mathcal{D}$ into itself). Suppose, $\{x_n, n\in N\}$ is a sequence of linear operators defined on $\mathcal{D}$ so that $x_1 = 1$ and $\|x_n\phi\| \leq \|x_{n+1}\phi\|$ $\forall\phi\in\mathcal{D}$, $n\in N$. By the seminorms $\|\phi\|_{x_n} := \|x_n\phi\|$, $n\in N$, we define a locally convex topology t_+ on $\mathcal{D}$. Let $\mathcal{N}_{x_n} := \{a\in\mathcal{A} : |\langle a\phi,\phi\rangle| \leq c_{a,n}\|x_n\phi\|^2\ \forall\phi\in\mathcal{D}\}$ and $\mathcal{M}_{x_n} := \{a\in\mathcal{A} : \|a\phi\| \leq c_{a,n}\|x_n\phi\|\ \forall\phi\in\mathcal{D}\}$.

THEOREM 1': Suppose that for each $a\in\mathcal{A}$ the operator a^* is defined on $\mathcal{D}$ and $a^+ := a^*\upharpoonright\mathcal{D} \in\mathcal{A}$. Suppose $\mathcal{A} = \bigcup_{n\in N}\mathcal{N}_{x_n}$. The uniform topology $\tau_{\mathcal{D}}$ on $\mathcal{A}$ will be defined by the seminorms $p_{\mathfrak{M}}(a) = \sup_{\phi,\psi\in\mathfrak{M}} |\langle a\phi,\psi\rangle|$ taken for all bounded subsets $\mathfrak{M}$ of the locally convex space $\mathcal{D}[t_+]$. Then, $\tau_{\mathcal{D}} = \tau_{st}$ if and only if all vector spaces $\mathcal{N}_{x_n}$, $n\in N$, are finite dimensional.

THEOREM 2': Suppose $\mathcal{A} = \bigcup_{n\in N}\mathcal{M}_{x_n}$ and the space $\mathcal{D}[t_+]$ is complete. Let $\sigma^{\mathcal{D}}$ be the locally convex topology on $\mathcal{A}$ generated by the seminorms $\|a\|_\phi := \|a\phi\|$, $\phi\in\mathcal{D}$. Then, $\sigma^{\mathcal{D}} = \tau_{st}$ if and only if all vector spaces $\mathcal{M}_{x_n}$ are finite dimensional.

Notice that the assumption $\mathcal{A} = \bigcup_{n\in N}\mathcal{N}_{x_n}$ implies that $p_{\mathfrak{M}}(a) < +\infty$ $\forall a\in\mathcal{A}$. If $a\in\mathcal{N}_{x_n}$, then $|\langle a\phi,\psi\rangle| \leq 4\rho_{x_n}(a)\|x_n\phi\|\|x_n\psi\|$ by polarization; hence $p_{\mathfrak{M}}(a) \leq 4\rho_{x_n}(a)\sup_{\phi,\psi\in\mathfrak{M}}\|x_n\phi\|\|x_n\psi\| < +\infty$ because $\mathfrak{M}$ is bounded in $\mathcal{D}[t_+]$.

REFERENCES

1. D.Arnal,J.-P.Jurzak. Topological aspects of algebras of unbounded operators, J.Funct.Analysis ,1977.

2. G.Lassner. Topological algebras of operators, Rep.Math.Phys. 3(1972),279-293

3. K.Schmüdgen. Uniform topologies and strong operator topologies on polynom algebras and on the algebra of CCR, Rep.Math.Phys. 10(1976),369-384

4. - , Uniform topologies on enveloping algebras, Preprint, KMU-MPh-1(1977)

5. - , On trace representation of linear functionals on unbounded operator algebras, Preprint,KMU-MPh-4(1977), to appear

6. B.Timmermann,W.Timmermann. Über einige Topologien auf der Algebra der CCR, Wiss.Zeitschrift,KMU Leipzig 1978

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