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B.Timmermann

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Тиммерман Б. **E5 - 10796**

Равномерная и сильная топологии на представлениях алгебры полиномов

Доказано, что при достаточно общих предположениях на операторы $A_1, \ldots, A_n$ (неограниченные, симметричные) и на область $\mathfrak{D}$ на реализации $\mathcal{P}(A_1, \ldots, A_n)$ алгебры полиномов $\mathcal{P}(x_1, \ldots, x_n)$ сильнейшая локально выпуклая топология τ_{st} совпадает как с равномерной топологией $\tau_{\mathfrak{D}}$, так и с сильной топологией τ_s. Для $n=2$ мы приведем некоторые более конкретные условия для выполнения этих общих предположений.

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Timmermann B. **E5 - 10796**

The Uniform and the Strong Topology on Realizations of the Algebra of Polynomials

It is shown that under quite general assumptions on the operators $A_1, \ldots, A_n$ (unbounded, symmetric) and on the domain $\mathfrak{D}$ on the realization $\mathcal{P}(A_1, \ldots, A_n)$ of the algebra of polynomials $\mathcal{P}(x_1, \ldots, x_n)$ the strongest locally convex topology τ_{st} coincides with the uniform topology $\tau_{\mathfrak{D}}$ as well as with the strong operator topology τ_s. In the case $n=2$ some conditions are given so, that these general assumptions are fulfilled.

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0. In an earlier paper /4/ we regarded the algebra of polynomials of one variable. For this case we have given a quite general condition on the unbounded symmetric operator A and on the domain $\mathcal{D}$ so, that on the representation $\mathcal{P}(A)$ of $\mathcal{P}(x)$ the strongest locally convex topology τ_{st} coincides with the strong topology τ_s. Now we consider the algebra of polynomials of n commuting variables. In /5/ it is shown that the strongest locally convex topology τ_{st} on $\mathcal{P}(x_1, \ldots, x_n)$ is a uniform topology (see definition below) as well as a strong operator topology, i.e., there is a realization of $\mathcal{P}(x_1, \ldots, x_n)$ as an operator algebra $\mathcal{A}(\mathcal{D}) = \mathcal{P}(A_1, \ldots, A_n)$ and on this algebra the strongest locally convex topology τ_{st} coincides with the uniform topology $\tau_{\mathcal{D}}$ and with the strong topology τ_s. The proof in /5/ is abstract in the sense that the universal representation is used.

In this paper we give (similary as in /4/) conditions on the operators $A_1, \ldots, A_n$ and on the domain $\mathcal{D}$ which provide the same result.

1. Let us repeat some definitions and properties (/2/). Let $\mathcal{D}$ be a unitary space with scalar product $\langle .,. \rangle$ and $\mathcal{H}$ its completion. By $\mathcal{L}^+(\mathcal{D})$ we denote the $*$-algebra of all operators A with $A\mathcal{D} \subset \mathcal{D}$, $\mathcal{D} \subset \mathcal{D}(A^*)$, $A^*\mathcal{D} \subset \mathcal{D}$ and with the involution $A \longrightarrow A^+ = A^*|_{\mathcal{D}}$. An Op$*$-algebra $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ with identity I. Every Op$*$-algebra $\mathcal{A}(\mathcal{D})$ defines a topology $t_{\mathcal{A}}$ on $\mathcal{D}$ given by the system of seminorms:

$$\varphi \longrightarrow \|\varphi\|_A = \|A\varphi\| \quad \text{for all } A \in \mathcal{A}(\mathcal{D}).$$

An Op*-algebra is said to be closed, if $\mathcal{D}[t_{\mathcal{A}}]$ is complete.

An Op*-algebra $\mathcal{A}(\mathcal{D})$ can be equipped with different topologies (/2/, /3/). We will use:

the uniform topology $\tau_{\mathcal{D}}$, defined by the seminorms:

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi\in\mathcal{M}} |\langle\varphi, A\psi\rangle|,$$

where $\mathcal{M}$ runs over all subsets $\mathcal{M}\subset\mathcal{D}$, for which $\|A\|_{\mathcal{M}}<\infty$ for all $A\in\mathcal{A}(\mathcal{D})$;

the strong topology τ_s, defined by the seminorms:

$$A \longrightarrow \|A\|_{\varphi} = \|A\varphi\| \quad \text{for all } \varphi\in\mathcal{D}.$$

A realization of $\mathcal{P}(x_1,\dots,x_n)$ is an algebraical *-isomorphism onto an appropriate Op*-algebra $\mathcal{A}(\mathcal{D}) = \mathcal{P}(A_1,\dots,A_n)$ given by $x_i \longrightarrow A_i = A_i^+ \in \mathcal{L}^+(\mathcal{D})$, $1\le i\le n$. For brevity we also call the algebra $\mathcal{P}(A_1,\dots,A_n)$ realization of $\mathcal{P}(x_1,\dots,x_n)$.

The realization is said to be closed if $\mathcal{P}(A_1,\dots,A_n)$ is a closed Op*-algebra.

In considering the polynomial algebra $\mathcal{P}(x_1,\dots,x_n)$ we use the favourable notations of /5/. Hence, let J be the set N^n (we shall assume $0\in N$) with the following operations:

$$\mathfrak{i} \pm \mathfrak{j} = (i_1,\dots,i_n) \pm (j_1,\dots,j_n) = (i_1\pm j_1,\dots,i_n\pm j_n)$$

Further, let $\pi(\mathfrak{i}) = i$ be a bijective map from J onto N with:

1. $\sum_{s=0}^{n} i_s < \sum_{s=0}^{n} j_s \implies \pi(\mathfrak{i}) < \pi(\mathfrak{j})$

2. $\sum_{s=0}^{n} i_s = \sum_{s=0}^{n} j_s$ and $i_s < j_s$ for the smallest s with $i_s \ne j_s \implies \pi(\mathfrak{i}) < \pi(\mathfrak{j})$.

With the help of this numeration map π one defines an order and a semiorder in J by:

(1) $\mathfrak{j} \le \mathfrak{i} \iff \pi(\mathfrak{j}) \le \pi(\mathfrak{i})$

(2) $\mathfrak{j} \prec 2\mathfrak{i} \iff \mathfrak{j} = \mathfrak{r} + \mathfrak{s}$, $0 \le \pi(\mathfrak{r}), \pi(\mathfrak{s}) \le \pi(\mathfrak{i})$
$(\pi(\mathfrak{r}),\pi(\mathfrak{s})) \ne (\pi(\mathfrak{i}),\pi(\mathfrak{i}))$

We write: $x^{\mathfrak{i}} = x_1^{i_1}\dots x_n^{i_n}$ and denote by $d(x^{\mathfrak{i}}) = i_1+\dots+i_n$ the degree of $x^{\mathfrak{i}}$.

Then $\mathcal{P}(x_1,\dots,x_n)$ is the linear span of the algebraical basis $\{x^{\mathfrak{i}}\}_{\mathfrak{i}\in J}$, i.e.

$$\mathcal{P}(x_1,\dots,x_n) = \left\{ p(x_1,\dots,x_n) = \sum_{\mathfrak{i}\in J} \alpha_{\mathfrak{i}} x^{\mathfrak{i}},\ \alpha_{\mathfrak{i}} = \alpha_{i_1\dots i_n} \in C \right\}$$

equipped with the multiplication

$$\Big(\sum_{\mathfrak{i}\in J} \alpha_{\mathfrak{i}} x^{\mathfrak{i}}\Big)\Big(\sum_{\mathfrak{j}\in J} \beta_{\mathfrak{j}} x^{\mathfrak{j}}\Big) = \sum_{\mathfrak{i},\mathfrak{j}\in J} \alpha_{\mathfrak{i}}\beta_{\mathfrak{j}}\, x^{\mathfrak{i}+\mathfrak{j}}$$

and the involution

$$\Big(\sum_{\mathfrak{i}\in J} \alpha_{\mathfrak{i}} x^{\mathfrak{i}}\Big)^+ = \sum_{\mathfrak{i}\in J} \overline{\alpha_{\mathfrak{i}}}\, x^{\mathfrak{i}}.$$

Then

$$\mathcal{P}(A_1,\dots,A_n) = \left\{ p(A_1,\dots,A_n) = \sum_{\mathfrak{i}\in J} \alpha_{\mathfrak{i}} A^{\mathfrak{i}},\ \alpha_{\mathfrak{i}} = \alpha_{i_1\dots i_n} \in C \right\}$$

(where $A^{\mathfrak{i}} = A_1^{i_1}\dots A_n^{i_n}$ and the operations are defined as above).

The order, semiorder, resp. defined by (1), (2) can be transformed to $\{x^{\mathfrak{i}}\}$, $\{A^{\mathfrak{i}}\}$. For example: $x^{\mathfrak{j}} \le x^{\mathfrak{i}} \iff \mathfrak{j} \le \mathfrak{i}$ and $x^{\mathfrak{j}} \prec x^{2\mathfrak{i}} \iff \mathfrak{j} \prec 2\mathfrak{i}$ and so on, (thus π can be interpreted as a map which preserves the degree, and within the set of elements with the same degree π preserves also the lexicographic order)

For an algebra $\mathcal{R}$ with a countable algebraical basis $\{b_i\}$ the strongest locally convex topology τ_{st} is given by one of the following systems of seminorms:

$$X = \sum_i \beta_i b_i \longrightarrow \|X\|_{(\gamma_i)} = \sum_i \gamma_i |\beta_i|$$

or

$$X = \sum_i \beta_i b_i \longrightarrow \|X\|'_{(\gamma_i)} = \Big(\sum_i \gamma_i^2 |\beta_i|^2\Big)^{1/2},$$

where (γ_i) runs over all sequences of nonnegative numbers. It is clear that we can restrict ourselves to sequences (γ_i) with $1\le\gamma_0\le\gamma_1\le\gamma_2\le\dots$, γ_i naturals. For the countable case it is easy to see, that the systems $\|.\|_{(\gamma_i)}$, $\|.\|'_{(\gamma_i)}$ are equivalent, but we remark that for the uncountable case it is not so. (/1/)

In our case, i.e. for $\mathcal{P}(x_1,\dots,x_n)$, $\mathcal{P}(A_1,\dots,A_n)$, resp. this topology τ_{st} is defined by

$$p(x_1,\dots,x_n) \longrightarrow \|p(x_1,\dots,x_n)\|_{(\gamma_{\mathfrak{i}})} = \sum_{\mathfrak{i}\in J} \gamma_{\mathfrak{i}}\alpha_{\mathfrak{i}}$$

or

$$p(x_1,\dots,x_n) \longrightarrow \|p(x_1,\dots,x_n)\|'_{(\gamma_i)} = \Big(\sum_{i\in J} \gamma_i^2 |\alpha_i|^2\Big)^{1/2},$$

where $(\gamma_i) = (\gamma_{i_1\dots i_n})$ is an arbitrary sequence of naturals.

2. Now we can formulate our results.

Theorem 1

Let $\mathcal{A}(\mathcal{D}) = \mathcal{P}(A_1,\dots,A_n)$ be a realization of the algebra $\mathcal{P}(x_1,\dots,x_n)$ on $\mathcal{D}$.

If for any given sequence $(\gamma_i)_{i\in J}$ of nonnegative numbers there is a sequence $(\varphi_i)_{i\in J}\in\mathcal{D}$ with

(i) $|\langle\varphi_i, A^i\varphi_i\rangle| > \gamma_i + 1 + \sum_{r,s<i} |\langle\varphi_r, A^i\varphi_s\rangle|$

(ii) $|\langle\varphi_s, A^i\varphi_s\rangle| < \frac{1}{2^{\rho(s)}}$ for $i<s$

(iii) $\langle\varphi_i, A^r\varphi_s\rangle = 0$ for $r<i$, $s<i$

then on $\mathcal{P}(A_1,\dots,A_n)$ the uniform topology $\tau_\mathcal{D}$ coincides with the strongest locally convex topology τ_{st}: $\tau_\mathcal{D}=\tau_{st}$.
Here $\rho(s)$ is fixed for all s and such that

$$\sum_s \frac{1}{2^{\rho(s)}} < 1.$$

Proof:

1. $\tau_\mathcal{D} \prec \tau_{st}$ is trivial.
2. To see $\tau_{st}\prec\tau_\mathcal{D}$ we show that for any given sequence (γ_i) there is a $t_{\mathcal{P}(A_1,\dots,A_n)}$-bounded set $\mathcal{M}\subset\mathcal{D}$ with

$$\|p(A_1,\dots,A_n)\|_{(\gamma_i)} \le \sup_{\varphi,\psi\in\mathcal{M}} |\langle\varphi, p(A_1,\dots,A_n)\psi\rangle|$$

Let $\|\cdot\|_{(\gamma_i)}$ be a given seminorm. Then by the assumptions of our theorem there is a sequence $(\varphi_i)\in\mathcal{D}$ with the properties (i) - (iii). Put

$$\mathcal{M} = \Big\{\hat\varphi_s = \sum_{j<s} \varepsilon_j\varphi_j, \quad \varepsilon_j\in C, \quad |\varepsilon_j|=1, \quad s\in J\Big\}.$$

This set is $t_{\mathcal{P}(A_1,\dots,A_n)}$-bounded (because of (ii), (iii)):

$$\sup_{\hat\varphi_s\in\mathcal{M}} \|A^i\hat\varphi_s\|^2 = \sup_{\hat\varphi_s\in\mathcal{M}} |\langle\hat\varphi_s, A^{2i}\hat\varphi_s\rangle|$$
$$\le \Big|\Big\langle \sum_j \varepsilon_j\varphi_j, A^{2i}\sum_r \varepsilon_r\varphi_r\Big\rangle\Big|$$

$$\le \Big|\Big\langle\sum_{j\le 2i}\varepsilon_j\varphi_j, A^{2i}\sum_{r\le 2i}\varepsilon_r\varphi_r\Big\rangle\Big| + \Big|\Big\langle\sum_{j>2i}\varphi_j, A^{2i}\sum_{r>2i}\varphi_r\Big\rangle\Big|$$
$$\le \Big|\Big\langle\sum_{j\le 2i}\varepsilon_j\varphi_j, A^{2i}\sum_{r\le 2i}\varepsilon_r\varphi_r\Big\rangle\Big| + \sum_{j>2i}\frac{1}{2^{\rho(j)}} < \infty$$

For a given polynomial $p(A_1,\dots,A_n) = \sum_{i<r}\alpha_i A^i$ choose:

$$\hat\psi_{p(A_1,\dots,A_n)} = \sum_{j<r}\varepsilon_j\varphi_j, \qquad \hat\varphi_{p(A_1,\dots,A_n)} = \sum_{j<r}\varphi_j$$

with complex numbers ε_j such that

$$\varepsilon_j\alpha_j\langle\varphi_j, A^j\varphi_j\rangle = |\alpha_j|\,|\langle\varphi_j, A^j\varphi_j\rangle|$$

With the help of the assumptions (i) - (iii) we estimate:

$$\|p(A_1,\dots,A_n)\|_\mathcal{M} = \sup_{\varphi,\psi\in\mathcal{M}} |\langle\hat\varphi, p(A_1,\dots,A_n)\hat\psi\rangle|$$
$$\ge |\langle\hat\varphi_{p(A_1,\dots,A_n)}, p(A_1,\dots,A_n)\hat\psi_{p(A_1,\dots,A_n)}\rangle|$$
$$= \Big|\Big\langle\sum_{s<r}\varphi_s, \sum_{i<r}\alpha_iA^i\sum_{j<r}\varepsilon_j\varphi_j\Big\rangle\Big| = \Big|\sum_{i<r}\alpha_i\sum_{j,s<r}\varepsilon_j\langle\varphi_s, A^i\varphi_j\rangle\Big|$$
$$\ge \Big|\sum_{i<r}\alpha_i\varepsilon_i\langle\varphi_i, A^i\varphi_i\rangle\Big| - \sum_{i<r}|\alpha_i|\sum_{s,j<i}|\langle\varphi_s, A^i\varphi_j\rangle|$$
$$-\sum_{i<r}|\alpha_i|\sum_{j>i}|\langle\varphi_j, A^i\varphi_j\rangle| - 2\sum_{i<r}|\alpha_i|\sum_{j>s>i}|\langle\varphi_s, A^i\varphi_j\rangle|$$
$$-2\sum_{i<r}|\alpha_i|\sum_{j<i}|\langle\varphi_i, A^i\varphi_j\rangle| - 2\sum_{i<r}|\alpha_i|\sum_{j>i}|\langle\varphi_i, A^i\varphi_j\rangle|$$
$$\ge \sum_{i<r}|\alpha_i|\Big\{\gamma_i + 1 + \sum_{s,j<i}|\langle\varphi_s, A^i\varphi_j\rangle|\Big\}$$
$$-\sum_{i<r}|\alpha_i|\sum_{s,j<i}|\langle\varphi_s, A^i\varphi_j\rangle| - \sum_{i<r}|\alpha_i|\sum_{j>i}|\langle\varphi_j, A^i\varphi_j\rangle|$$
$$\ge \sum_{i<r}|\alpha_i|\,\gamma_i = \|p(A_1,\dots,A_n)\|_{(\gamma_i)}$$

Q.E.D.

Now we consider the strong topology τ_s and formulate the following theorem.

Theorem 2

Let $\mathcal{A}(\mathcal{D}) = \mathcal{P}(A_1,\dots,A_n)$ be a realization of the algebra $\mathcal{P}(x_1,\dots,x_n)$ on $\mathcal{D}$.
If for a given sequence $(\delta_i)_{i\in J}$ of positive numbers there is a $\varphi\in\mathcal{D}$ with

(i) $\|\varphi\|^2 \geq \delta_0$

(ii) $\langle\varphi, A^{2i}\varphi\rangle \geq \delta_i\left[\max\limits_{j\prec 2i}\{1, |\langle\varphi, A^{j}\varphi\rangle|\}\right]^{\pi(i)+1}$

then on the algebra $\mathcal{P}(A_1,\dots,A_n)$ the strong topology τ_s coincides with the strongest locally convex topology τ_{st}: $\tau_s = \tau_{st}$.

In this theorem and its proof for τ_{st} the seminorms $\|\cdot\|'_{(\gamma_i)}$ are used.

Proof:

1. $\tau_s \prec \tau_{st}$ is trivial.

2. $\tau_{st} \prec \tau_s$:

We will show: For a given sequence $(\gamma_i)_{i\in J}$ of nonnegative numbers there is a $\varphi\in\mathcal{D}$ with

$$\|p(A_1,\dots,A_n)\|'^2_{(\gamma_i)} \leq \|p(A_1,\dots,A_n)\|^2_{\varphi}$$

for all $p(A_1,\dots,A_n)\in\mathcal{P}(A_1,\dots,A_n)$.
This is the same as

$$\sum_i \gamma_i^2\,|\alpha_i|^2 \leq \Big\langle \sum_i \alpha_i A^i\varphi, \sum_j \alpha_j A^j\varphi\Big\rangle = \sum_{i,j}\overline{\alpha_i}\,\alpha_j\langle\varphi, A^{i+j}\varphi\rangle$$

or

$$\sum_{i,j}\left\{\langle\varphi, A^{i+j}\varphi\rangle - \gamma_i^2\,\delta_j^i\right\}\overline{\alpha_i}\,\alpha_j \geq 0$$

This means that the infinite matrix M must be positive definite.
In the main diagonal of this matrix M we have expressions of the form

$$\langle\varphi, A^{2\pi^{-1}(i)}\varphi\rangle - \gamma^2_{\pi^{-1}(i)} = \langle\varphi, A^{2i}\varphi\rangle - \gamma_i^2$$

and the whole matrix M has the following form:

$$M = \begin{pmatrix} \langle\varphi,\varphi\rangle - \gamma^2_{\pi^{-1}(0)} & \langle\varphi, A^{\pi^{-1}(1)}\varphi\rangle & \cdots \\ \langle\varphi, A^{\pi^{-1}(1)}\varphi\rangle & \langle\varphi, A^{2\pi^{-1}(1)}\varphi\rangle - \gamma^2_{\pi^{-1}(1)} & \cdots \\ \langle\varphi, A^{\pi^{-1}(2)}\varphi\rangle & \langle\varphi, A^{\pi^{-1}(1)+\pi^{-1}(2)}\varphi\rangle & \cdots \\ \langle\varphi, A^{\pi^{-1}(3)}\varphi\rangle & \langle\varphi, A^{\pi^{-1}(1)+\pi^{-1}(3)}\varphi\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Now we still show that under the assumptions of our Theorem 2 every finite dimensional principle minor $\|M_i\|$ is a positive one. By $\|M_i\| = \|(m_{kl})\|$, $0\leq k,l\leq i$, we denote this principle minor where $m_{ii} = \langle\varphi, A^{2i}\varphi\rangle - \gamma_i^2$.
By decomposition we get

$$\|M_i\| = (\langle\varphi, A^{2i}\varphi\rangle - \gamma_i^2)\,\|M_{i-1}\| + R_i$$

In R_i there are products with factors of the form

$$(\langle\varphi, A^{\cdots}\varphi\rangle - \gamma^2_{\cdots}) \quad\text{and}\quad \langle\varphi, A^{\cdots}\varphi\rangle$$

Let z_i be the number of terms (summands) which one gets multiplying all these products. Then we can estimate:

$$|R_i| \leq z_i \prod_{j<i}\gamma_j^2\left[\max_{j\prec 2i}\{1, |\langle\varphi, A^j\varphi\rangle|\}\right]^{i+1}$$

Let now

$$\delta_i = 3\,z_i\prod_{j\leq i}\gamma_j^2$$

and φ the element corresponding to this sequence according to the assumptions of our theorem.
By induction we show: $\|M_i\| \geq 1$

$$\|M_0\| = \langle\varphi,\varphi\rangle - \gamma_0^2 \geq 1$$

is trivial.
Let $\|M_n\| \geq 1$ for $0\leq n\leq i-1$.
Then the above decomposition of $\|M_i\|$ and estimation of $|R_i|$ lead to

$$\|M_i\| \geq (\langle\varphi, A^{2i}\varphi\rangle - \gamma_i^2)\,\|M_{i-1}\| - |R_i|$$

$$\geq \langle\varphi, A^{2i}\varphi\rangle - \gamma_i^2 - z_i \prod_{j<i} \gamma_j^2 \Big[\max_{j \prec 2i}\{1, |\langle\varphi, A^{j}\varphi\rangle|\}\Big]^{i+1}$$

$$\geq \langle\varphi, A^{2i}\varphi\rangle + 1 - 3z_i \prod_{j\leq i} \gamma_j^2 \Big[\max_{j \prec 2i}\{1, |\langle\varphi, A^{j}\varphi\rangle|\}\Big]^{i+1}$$

$$\geq \langle\varphi, A^{2i}\varphi\rangle + 1 - \delta_i \Big[\max_{j \prec 2i}\{1, |\langle\varphi, A^{j}\varphi\rangle|\}\Big]^{i+1}$$

$$\geq 1$$

Q.E.D.

Remark

Analogously to /4/ there may be given examples which show that Theorem 2 is valid for closed and also for non-closed realizations.

3. In the following we will demonstrate under which conditions on the operators A_i and on the domain $\mathcal{D}$ the assumptions of our Theorem 1 and Theorem 2 are fulfilled. For simplicity we restrict ourselves to the case n=2.
We start with the following lemma. For brevity we omit the proof.

Lemma 3

Let $A_1, \ldots, A_n, B \in \mathcal{L}^+(\mathcal{D})$, $\sup_{\varphi\in\mathcal{U}} |\langle\varphi, B\varphi\rangle| = \infty$ where $\mathcal{U} = \{\varphi\in\mathcal{D}: |\langle\varphi, A_i\varphi\rangle| \leq 1, 1\leq i\leq n\}$. Furthermore let $\mathcal{H}_0$ be a subspace of $\mathcal{D}$ of finite dimension.
Then for $\mathcal{U}_0 = \mathcal{U} \cap (\mathcal{H}_0^\perp \cap \mathcal{D})$ also

$$\sup_{\varphi\in\mathcal{U}_0} |\langle\varphi, B\varphi\rangle| = \infty .$$

Now we can prove the following proposition.

Proposition 4

Let $A_1, A_2 \in \mathcal{L}^+(\mathcal{D})$, $[A_1, A_2] = 0$ on $\mathcal{D}$. If for all $i \in J = N\times N$ $\sup_{\varphi\in\mathcal{U}_i} |\langle\varphi, A^i\varphi\rangle| = \infty$ where $\mathcal{U}_i = \{\varphi\in\mathcal{D}: |\langle\varphi, A^j\varphi\rangle| \leq 1$ for all $j < i\}$, then the assumptions (i) - (iii) of Theorem 1 are fulfilled, i.e. on $\mathcal{P}(A_1, A_2)$ the uniform topology $\tau_{\mathcal{D}}$ coincides with the strongest locally convex topology τ_{st}.

Proof:

We show by induction the existence of a sequence $(\varphi_j) \in \mathcal{D}$ with

(i) $|\langle\varphi_j, A^j\varphi_j\rangle| > \gamma_j + 1 + \sum_{r,s<j} |\langle\varphi_r, A^j\varphi_s\rangle|$

(ii) $|\langle\varphi_s, A^j\varphi_s\rangle| < \frac{1}{2^{\varrho(s)}}$ for $j < s$

(iii) $\langle\varphi_j, A^r\varphi_s\rangle = 0$ for $r < j$, $s < j$

Let $\varphi_0 \in \mathcal{D}$ with $|\langle\varphi_0, \varphi_0\rangle| \geq \gamma_0 + 1$. If φ_j, $j < r$ are chosen then put

$$\mathcal{H}_r = \mathcal{L}\{A^s\varphi_j,\ s \leq r,\ j < r\} \quad (\mathcal{L} \text{ means "linear span"})$$

and take $\varphi_r \in \mathcal{D}_r = \mathcal{D} \cap \mathcal{H}_r^\perp$ so that

$$|\langle\varphi_r, A^r\varphi_r\rangle| > \gamma_r + 1 + \sum_{i,j<r} |\langle\varphi_i, A^r\varphi_j\rangle|$$

and

$$|\langle\varphi_r, A^s\varphi_r\rangle| < \frac{1}{2^{\varrho(r)}} \qquad \text{for } s < r$$

This is possible because of our assumptions and Lemma 3 (applied to $\mathcal{U}_r \cap \mathcal{D}_r$).
It is easy to see that (iii) also holds by the construction above.

Q.E.D.

Remark 5

From the proof of Proposition 4 it can be seen that the commutativity of A_1, A_2 is not used (as in Theorem 1, too!). Hence, if $\mathcal{A}$ is an arbitrary Op*-algebra with algebraic basis $\{B_i\}$ and $\sup_{\varphi\in\mathcal{U}_k} |\langle\varphi, B_k\varphi\rangle| = \infty$ with $\mathcal{U}_k = \{\varphi\in\mathcal{D}: |\langle\varphi, B_i\varphi\rangle| \leq 1, i<k\}$ then $\tau_{\mathcal{D}} = \tau_{st}$ on $\mathcal{A}$.

The analogous proposition for the strong topology reads as follows:

Proposition 6

Let $A_1, A_2 \in \mathcal{L}^+(\mathcal{D})$, $[A_1, A_2] = 0$ on $\mathcal{D}$, $\mathcal{D}[t_{\mathcal{P}(A_1,A_2)}]$ complete. If for all $i \in J = N\times N$ $\sup_{\varphi\in\mathcal{U}_i} |\langle\varphi, A^{2i}\varphi\rangle| = \infty$ where $\mathcal{U}_i = \{\varphi\in\mathcal{D}: |\langle\varphi, A^j\varphi\rangle| \leq 1$ for all $j \prec 2i\}$ then the assumptions (i) and (ii) of Theorem 2 are fullfilled, i.e. $\tau_{st} = \tau_s$ on $\mathcal{P}(A_1, A_2)$.

The proof is similar to that of Proposition 4. The element φ is constructed as $\varphi = \sum_{j} \varphi_j$, $\varphi_j \in \mathcal{M}_j$. To be sure that $\varphi \in \mathcal{D}$ the assumption "$\mathcal{D}[t_{\mathcal{P}}]$ complete" is used.

The following proposition gives more transparent conditions which guarantee that $\tau_{\mathcal{D}} = \tau_s = \tau_{st}$ on $\mathcal{P}(A_1, A_2)$.

To formulate this result we use the following (may be somewhat "non-standard") definition.

Definition 7

Let $A_1, A_2 \in \mathcal{L}^+(\mathcal{D})$, $[A_1, A_2] = 0$ on $\mathcal{D}$. By the common spectrum $\sigma(A_1, A_2)$ in the strong sense we mean the following set:

$$\sigma(A_1, A_2) = \{ (\lambda_1, \lambda_2) \in R^2 : \exists \ (\varphi_i) \in \mathcal{D},\ \|\varphi_i\| = 1,$$
$$\|(A^{r} - \lambda^{r})\varphi_i\| \longrightarrow 0 \text{ for } i \longrightarrow \infty,$$
$$\text{for all } r \in N \times N \}.$$

$(r = (r_1, r_2),\ \lambda^{r} = \lambda_1^{r_1}\lambda_2^{r_2})$.

Remark, that the sequence (φ_i) depends only on λ but not on r !

Proposition 8

Let $A_1, A_2 \in \mathcal{L}^+(\mathcal{D})$, $[A_1, A_2] = 0$ on $\mathcal{D}$. Further suppose that $\sigma(A_1, A_2)$ is such that for any α, $0 < \alpha < 1$ there is a β, $\alpha < \beta < 1$ and between the curves $\lambda_2 = \lambda_1^{\alpha}$ and $\lambda_2 = \lambda_1^{\beta}$ lies an unbounded subset of $\sigma(A_1, A_2)$. Then

(i) on $\mathcal{P}(A_1, A_2)$ the uniform topology $\tau_{\mathcal{D}}$ coincides with the strongest locally convex topology τ_{st}.

(ii) If moreover $\mathcal{D}[t_{\mathcal{P}(A_1, A_2)}]$ is complete, then on $\mathcal{P}(A_1, A_2)$ the strong topology τ_s coincides with the strongest locally convex topology τ_{st}.

Roughly speaking this Proposition 8 says that $\tau_{\mathcal{D}} = \tau_s = \tau_{st}$ if $\sigma(A_1, A_2)$ is rich enough.

In the proof we use the following lemma for which the proof is omitted.

Lemma 9

(i) For any $i = (i_1, i_2) \in J$ there are $0 < \varepsilon_1 < 1$, $\vartheta > 0$ with

$$\lim_{\lambda_1 \to \infty} \lambda_1^{i_1 + \rho i_2 - \vartheta} = \infty$$

$$\lim_{\lambda_1 \to \infty} \lambda_1^{j_1 + \rho j_2 - \vartheta} = 0 \quad \text{for all } j < i \text{ and all fixed } \rho \in (\varepsilon_1, 1).$$

(ii) For any $i = (i_1, i_2) \in J$ there are $0 < \varepsilon_1 < 1$, $\vartheta > 0$ with

$$\lim_{\lambda_1 \to \infty} \lambda_1^{2i_1 + 2\rho i_2 - 2\vartheta} = \infty$$

$$\lim_{\lambda_1 \to \infty} \lambda_1^{j_1 + \rho j_2 - 2\vartheta} = 0 \quad \text{for all } j \propto 2i \text{ and all fixed } \rho \in (\varepsilon_1, 1).$$

Proof of Proposition 8:

(i) Let $(\lambda_1, \lambda_2) = (\lambda_1, \lambda_1^{\rho}) \in \sigma(A_1, A_2)$, $\lambda_1 > 0$, $\rho > 0$.

Then there is a sequence $(\varphi_i(\lambda)) = (\varphi_i)$ with $\|\varphi_i\| = 1$ and

$$\|(A_1^{r_1}A_2^{r_2} - \lambda_1^{r_1}\lambda_2^{r_2})\varphi_i\| \longrightarrow 0 \quad \text{for } i \longrightarrow \infty.$$

Therefore

$$|\langle \varphi_i, (A_1^{r_1}A_2^{r_2} - \lambda_1^{r_1}\lambda_2^{r_2})\varphi_i\rangle| \leq \varepsilon \quad \text{for } i \geq i_0(\lambda, \varepsilon).$$

Hence

$$\lambda_1^{r_1 + \rho r_2} - \varepsilon < \langle \varphi_i, A^{r}\varphi_i\rangle < \lambda_1^{r_1 + \rho r_2} + \varepsilon$$

or if we divide by λ_1^{ϑ}, $\vartheta > 0$:

$$(3) \quad \lambda_1^{r_1 + \rho r_2 - \vartheta} - \frac{\varepsilon}{\lambda_1^{\vartheta}} < \left\langle \frac{\varphi_i}{\lambda_1^{\vartheta/2}}, A^{r}\frac{\varphi_i}{\lambda_1^{\vartheta/2}} \right\rangle < \lambda_1^{r_1 + \rho r_2 - \vartheta} + \frac{\varepsilon}{\lambda_1^{\vartheta}}.$$

Analogously

$$(4) \quad \lambda_1^{s_1 + \rho s_2 - \vartheta} - \frac{\varepsilon}{\lambda_1^{\vartheta}} < \left\langle \frac{\varphi_i}{\lambda_1^{\vartheta/2}}, A^{s}\frac{\varphi_i}{\lambda_1^{\vartheta/2}} \right\rangle < \lambda_1^{s_1 + \rho s_2 - \vartheta} + \frac{\varepsilon}{\lambda_1^{\vartheta}}.$$

and $s < r$

Using Lemma 9 we can find $\rho > 0$, $\vartheta > 0$ such that in (3) the left-hand side and the right-hand side go to infinity while in (4) the corresponding expressions go to zero if $\lambda_1 \longrightarrow \infty$. Moreover the assumptions of Proposition 8 and Lemma 9 mean that

ϱ and ϑ can be chosen such that $(\lambda_1, \lambda_1^{\varrho})$ belongs to an unbounded subset of $\mathfrak{S}(A_1, A_2)$ which lies in the strip

$$\lambda_1^{\alpha} \le \lambda_1^{\varrho} \le \lambda_1^{\beta}, \quad 0 < \alpha < \beta < 1, \quad 0 < \lambda_1 < \infty .$$

Therefore, for any fixed $\varepsilon > 0$ (for example let $\varepsilon = 1/2$) we can choose a sequence (ψ_k) such that

$$|\langle \psi_k, A^{\tau} \psi_k \rangle| \ge k$$

$$0 \le |\langle \psi_k, A^{s} \psi_k \rangle| \le 1/2 \qquad \text{for all} \quad s < \tau$$

i.e. $(\psi_k) \in \mathcal{M}_{+} = \{ \varphi \in \mathcal{D} : |\langle \varphi, A^{s} \varphi \rangle| \le 1 \quad \text{for all } s < \tau \}$

and $\sup\limits_k |\langle \psi_k, A^{\tau} \psi_k \rangle| = \infty$

consequently: $\sup\limits_{\psi \in \mathcal{M}_+} |\langle \psi, A^{\tau} \psi \rangle| = \infty .$

(The vectors ψ_k are appropriate $\lambda_1^{-\vartheta/2} \varphi_i(\lambda_1)$ for $\lambda_1 \longrightarrow \infty$, $(\lambda_1, \lambda_1^{\varrho}) \in \mathfrak{S}(A_1, A_2)$.

Thus we have proved that the assumption of Proposition 4 is fulfilled, i.e., $\tau_{\mathcal{D}} = \tau_{st}$.
The proof of (ii) is quite analogous and uses Lemma 9 (ii).

Q.E.D.

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