ОБЪЕДИНЕННЫЙ
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ДУБНА

E5 - 10758

W.Timmermann

IDEALS
IN ALGEBRA OF UNBOUNDED OPERATORS

1977

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Submitted to Mathematische Nachrichten

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E5 - 10758

Идеалы в алгебрах неограниченных операторов

Дается общая конструкция идеалов в алгебрах неограниченных операторов на основе идеалов в $\mathfrak{B}(\mathcal{H})$. Исследуются алгебраические и топологические свойства таким образом полученных идеалов, порожденных хорошо известными симметрично-нормированными идеалами $\mathfrak{S}_{\Phi}(\mathcal{H})$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1977

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E5 - 10758

Ideals in Algebras of Unbounded Operators

A general procedure is given to get ideals in algebras of unbounded operators starting with ideals in $\mathfrak{B}(\mathcal{H})$. Algebraical and topological properties of ideals obtained in this manner from the well-known symmetrically-normed ideals $\mathfrak{S}_{\Phi}(\mathcal{H})$ are described.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1977

In this paper we continue the investigations on ideals in algebras of unbounded operators begun in /6/ and /7/ in a more systematic way. The first section deals with the description how one can get ideals in $\mathcal{L}^+(\mathfrak{D})$ starting with ideals in $\mathfrak{B}(\mathcal{H})$. We give the definition of two types of ideals: one contains only bounded operators, to the other belong also unbounded operators.

In section 2 some algebraic properties of ideals $\mathfrak{S}_{\Phi}(\mathfrak{D})$ derived from the well-known symmetrically normed ideals $\mathfrak{S}_{\Phi}$ are investigated.

Topologies in such ideals are introduced in the last section. There are also mentioned some results connected with topological properties of these ideals.

1. PRELIMINARIES AND BASIC DEFINITIONS

We use the following notions and notations (cf. /1/,/2/). For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathcal{H}$ we denote by $\mathcal{L}^+(\mathfrak{D})$ the $*$-algebra of all operators A (bounded or not) for which $A\mathfrak{D} \subset \mathfrak{D}$ and $A^*\mathfrak{D} \subset \mathfrak{D}$. The involution is given by $A \longrightarrow A^+ = A^* \restriction \mathfrak{D}$. $\mathcal{L}^+(\mathfrak{D})$ defines a natural topology t on the domain $\mathfrak{D}$ given by the directed system of seminorms $\phi \longrightarrow \|A\phi\|$ for all $A \in \mathcal{L}^+(\mathfrak{D})$. $\mathcal{L}^+(\mathfrak{D})$ is said to be <u>closed</u> if $\mathfrak{D}[t]$ is a complete space, or equivalently, if $\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(\bar{A})$. $\mathcal{L}^+(\mathfrak{D})$ is said to be <u>selfadjoint</u> if $\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(A^*)$. By $\mathfrak{F}(\mathfrak{D})$ we denote the set of finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$. $\mathfrak{F}(\mathfrak{D})$ is the two-sided minimal $*$-ideal of $\mathcal{L}^+(\mathfrak{D})$.

For a completely continuous operator $T \in \mathcal{B}(\mathcal{H})$ by $(s_n(T))$ we denote the sequence of s-numbers, $s_1(T) \geq s_2(T) \geq \ldots$ (each number repeated according to its multiplicity). $\Phi(.)$ stands for a symmetric norming function and $\mathfrak{S}_\Phi(\mathcal{H})$ or simple $\mathfrak{S}_\Phi$ denotes the corresponding symmetrically-normed ideal with the norm $\| \ \|_\Phi$ given by

$$\|T\|_\Phi = \Phi(s_1(T), s_2(T), \ldots)$$

The ideals $\mathfrak{S}_\infty(\mathcal{H})$, $\mathfrak{S}_p(\mathcal{H})$, $p \geq 1$, are special cases of such symmetrically-normed ideals. For details the reader may consult /1/. Now we give a general procedure to generate ideals in $\mathcal{L}^+(\mathcal{D})$.

Definition 1

Let $\mathfrak{J} = \mathfrak{J}(\mathcal{H})$ be a two-sided $*$-ideal in $\mathcal{B}(\mathcal{H})$. By $\mathfrak{J}(\mathcal{D})$ we denote the set

$$\mathfrak{J}(\mathcal{D}) = \{ T \in \mathcal{L}^+(\mathcal{D}): \ XTY \in \mathfrak{J}(\mathcal{H}) \text{ for all } X,Y \in \mathcal{L}^+(\mathcal{D}) \}$$

(Clearly, to be more exact, we had to write: XTY bounded on $\mathcal{D}$ and $\overline{XTY} \in \mathfrak{J}(\mathcal{H})$. For simplicity let us use the notation mentioned above)

Lemma 2

i) The set $\mathfrak{J}(\mathcal{D})$ given in Definition 1 is a two-sided $*$-ideal in $\mathcal{L}^+(\mathcal{D})$, called the corresponding ideal to $\mathfrak{J}(\mathcal{H})$; $\mathfrak{J}(\mathcal{D}) \subset \mathfrak{J}(\mathcal{H})$.

ii) If $\mathcal{L}^+(\mathcal{D})$ is selfadjoint, then from $XTY \in \mathfrak{J}(\mathcal{H})$ for all $X,Y \in \mathcal{L}^+(\mathcal{D})$ it follows $T \in \mathcal{L}^+(\mathcal{D})$.

Proof:

i) The linearity is clear, the $*$-property follows because $XTY \in \mathfrak{J}(\mathcal{H})$ implies $(XTY)^* = Y^*T^*X^* = Y^+T^+X^+$ (when restricted to $\mathcal{D}$) also in $\mathfrak{J}(\mathcal{H})$, but this means $T^+ \in \mathfrak{J}(\mathcal{D})$. Let $A,B \in \mathcal{L}^+(\mathcal{D})$ be arbitrary operators, then from $X(ATB)Y = UTV$, $U,V \in \mathcal{L}^+(\mathcal{D})$, it is seen that $ATB \in \mathfrak{J}(\mathcal{D})$. From $X = Y = I$ it follows $T \in \mathfrak{J}(\mathcal{H})$ if $T \in \mathfrak{J}(\mathcal{D})$.

ii) $XTY \in \mathfrak{J}(\mathcal{D})$ means especially XT and TY bounded for all $X,Y \in \mathcal{L}^+(\mathcal{D})$ and dense defined. Then Lemma 1.1 from /4/ gives $T^*\mathcal{H} \subset \mathcal{D}(Y^*)$ for all $Y \in \mathcal{L}^+(\mathcal{D})$, but this means $T^*\mathcal{H} \subset \mathcal{D}$ (as $\mathcal{L}^+(\mathcal{D})$ is selfadjoint). A slight generalization of /4/, Lemma 1.2 shows that $T\mathcal{H} \subset \mathcal{D}(\overline{X})$ for all $X \in \mathcal{L}^+(\mathcal{D})$, i.e. $T\mathcal{H} \subset \mathcal{D}$. So, T and T^* map $\mathcal{H}$ in $\mathcal{D}$, hence $T \in \mathcal{L}^+(\mathcal{D})$.

Q.E.D.

Remark 3

i) If $\mathcal{D} = \mathcal{H}$, then it is well-known /2/ that $\mathcal{L}^+(\mathcal{D}) = \mathcal{B}(\mathcal{H})$ and consequently $\mathfrak{J}(\mathcal{D}) = \mathfrak{J}(\mathcal{H})$. In general it is not so that $\mathfrak{J}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D}) \cap \mathfrak{J}(\mathcal{H})$ (to such a reading the notation $\mathfrak{J}(\mathcal{D})$ could lead). For example, if $\mathcal{L}^+(\mathcal{D})$ contains unbounded operators, then $I \in \mathcal{L}^+(\mathcal{D}) \cap \mathcal{B}(\mathcal{H})$ but, of course, $I \notin \mathcal{B}(\mathcal{D})$.

ii) As it can be seen from the proof of Lemma 2 it is not necessary for $\mathfrak{J}(\mathcal{H})$ to be a $*$-ideal to obtain an ideal $\mathfrak{J}(\mathcal{D})$ in the mentioned manner. It would be sufficient if $\mathfrak{J}(\mathcal{H})$ were a linear $*$-space. But this seems to be too general, because we like to atart with "well-known" substructures of $\mathcal{B}(\mathcal{H})$ (as $\mathfrak{S}_\Phi(\mathcal{H})$) to get substructures of $\mathcal{L}^+(\mathcal{D})$ which have also good properties.

The following lemma shows, roughly speaking, that in $\mathcal{L}^+(\mathcal{D})$ there are many distinct ideals if the operators are not "too unbounded".

Lemma 4

Let $\mathcal{L}^+(\mathcal{D})$ be such that there is an $N \in \mathcal{L}^+(\mathcal{D})$ with $N^{-1} \in \mathfrak{J}(\mathcal{H})$ for some ideal of $\mathcal{B}(\mathcal{H})$. Then $\mathfrak{J}(\mathcal{D}) = \mathfrak{F}(\mathcal{D})$ for all ideals $\mathfrak{F}$ with $\mathfrak{J}(\mathcal{H}) \subset \mathfrak{F}(\mathcal{H})$.

Proof:

It is enough to show $\mathfrak{J}(\mathcal{D}) = \mathcal{B}(\mathcal{D})$. But this follows from $XTY = XTYNN^{-1} = BN^{-1}$, where $T \in \mathcal{B}(\mathcal{D})$, i.e. $XTYN$ is bounded, consequently $BN^{-1} \in \mathfrak{J}(\mathcal{H})$. Hence $T \in \mathfrak{J}(\mathcal{D})$.

Q.E.D.

Next we define ideals which contain also unbounded operators (by ideals in what follows we always mean two-sided $*$-ideals).

Definition 5

Let $\mathfrak{J}(\mathcal{H})$, $\mathfrak{F}(\mathcal{H})$ be two ideals in $\mathcal{B}(\mathcal{H})$, $\mathfrak{J}(\mathcal{D})$, $\mathfrak{F}(\mathcal{D})$ the corresponding ideals in $\mathcal{L}^+(\mathcal{D})$. Put

$$M(\mathfrak{J}(\mathcal{D}), \mathfrak{F}(\mathcal{D})) =: M(\mathfrak{J}, \mathfrak{F}) = \{ A \in \mathcal{L}^+(\mathcal{D}): AT, A^+T \in \mathfrak{F}(\mathcal{D}) \text{ for all } T \in \mathfrak{J}(\mathcal{D}) \}.$$

Now we collect some simple properties of these sets, then we give an example.

Lemma 6

i) $\mathfrak{M}(\mathfrak{I}(\mathcal{D}), \mathfrak{J}(\mathcal{D}))$ is an ideal in $\mathcal{L}^+(\mathcal{D})$.

ii) If $\mathfrak{I}(\mathcal{H}) \subseteq \mathfrak{J}(\mathcal{H})$, so $\mathfrak{M}(\mathfrak{I}, \mathfrak{J}) = \mathcal{L}^+(\mathcal{D})$.

iii) If $\mathfrak{I}_1(\mathcal{H}) \subseteq \mathfrak{I}_2(\mathcal{H})$, so $\mathfrak{M}(\mathfrak{I}_2, \mathfrak{J}) \subseteq \mathfrak{M}(\mathfrak{I}_1, \mathfrak{J})$.

If $\mathfrak{J}_1(\mathcal{H}) \subseteq \mathfrak{J}_2(\mathcal{H})$, so $\mathfrak{M}(\mathfrak{I}, \mathfrak{J}_1) \subseteq \mathfrak{M}(\mathfrak{I}, \mathfrak{J}_2)$.

Proof:

ii) and iii) are trivial consequences of the definition.
i) The $*$-property and linearity are clear. Let $X \in \mathcal{L}^+(\mathcal{D})$, $A \in \mathfrak{M}(\mathfrak{I}, \mathfrak{J})$, $T \in \mathfrak{I}(\mathcal{D})$, then $(AX)T = AS$ with $S \in \mathfrak{I}(\mathcal{D})$, hence $AS \in \mathfrak{J}(\mathcal{D})$. $(AX)^+T = X^+A^+T = X^+R$ with $R \in \mathfrak{J}(\mathcal{D})$, hence $X^+R \in \mathfrak{J}(\mathcal{D})$. Therefore $AX \in \mathfrak{M}(\mathfrak{I}, \mathfrak{J})$ and analogously $XA \in \mathfrak{M}(\mathfrak{I}, \mathfrak{J})$.

Q.E.D.

We state an equivalent characterization of $\mathfrak{M}(\mathfrak{I}, \mathfrak{J})$. The simple proof is omitted.

Lemma 7

$$\mathfrak{M}(\mathfrak{I}, \mathfrak{J}) = \{A \in \mathcal{L}^+(\mathcal{D}): XAT,\ TAX \in \mathfrak{J}(\mathcal{D}) \text{ for all } X \in \mathcal{L}^+(\mathcal{D}) \text{ and all } T \in \mathfrak{I}(\mathcal{D})\}.$$

Next we give an example which shows that $\mathfrak{M}(\mathfrak{I}, \mathfrak{J})$ can contain also unbounded operators.

Example 8

Let $\mathcal{D} = \bigcap_n \mathcal{D}(R^n)$, $R = R\{(r_n),(\phi_n)\} \geq I$ a diagonal operator such that $\sup r_n = \infty$ but (r_n) does not increase "too fast"; for example, R^{-k} is for all k not nuclear. Regard $\mathfrak{M}(\mathfrak{S}_\infty(\mathcal{D}), \mathfrak{S}_1(\mathcal{D}))$. Let (r_{n_1}) be such a subsequence of (r_n) that $\sum r_{n_1}^{-1} < \infty$. The diagonal operator $A = A\{(a_n),(\phi_n)\}$ with

$$a_i = \begin{cases} r_{n_1} & \text{if } i = n_1 \\ 0 & \text{if } i \neq n_1 \end{cases}$$

belongs to $\mathfrak{M}(\mathfrak{S}_\infty(\mathcal{D}), \mathfrak{S}_1(\mathcal{D}))$. To see this, remark that $B = B\{(b_n),(\phi_n)\}$ with

$$b_i = \begin{cases} r_{n_1} & \text{if } i = n_1 \\ 0 & \text{if } i \neq n_1 \end{cases}$$

is nuclear and for all natural k: $R^k = BR^{k+1}A$. Let $C \in \mathfrak{S}_\infty(\mathcal{D})$, then $R^kAC = B(R^{k+1}AC)$ is nuclear for all k because $R^{k+1}AC$ is bounded

(even completely continuous). For an arbitrary $X \in \mathcal{L}^+(\mathcal{D})$ the closed graph theorem gives $\|XAC\phi\| \leq K\|R^mAC\phi\|$ for all $\phi \in \mathcal{D}$, suitable m and constant K. Because the operator on the right-hand side is bounded this estimation is valid for all $\phi \in \mathcal{H}$. The nuclearity of R^mAC then implies the nuclearity of XAC. Therefore, $X(AC)Y = XAD$, $D \in \mathfrak{S}_\infty(\mathcal{D})$ is nuclear for all $X,Y \in \mathcal{L}^+(\mathcal{D})$, $C \in \mathfrak{S}_\infty(\mathcal{D})$. This means $AC \in \mathfrak{S}_1(\mathcal{D})$, hence $A = A^+ \in \mathfrak{M}(\mathfrak{S}_\infty(\mathcal{D}), \mathfrak{S}_1(\mathcal{D}))$ and A is unbounded.

Q.E.D.

Remark 9

There is a lot of possibilities to introduce LK- or LK*-topologies in $\mathfrak{J}(\mathcal{D})$ and $\mathfrak{M}(\mathfrak{I}, \mathfrak{J})$. How to do this we demonstrate in the following section for $\mathfrak{I}(\mathcal{D})$ in the case where $\mathfrak{I}(\mathcal{D}) = \mathfrak{S}_\Phi(\mathcal{D})$. Some further topologies on $\mathfrak{S}_\Phi(\mathcal{D})$, topologies on $\mathfrak{M}(\mathfrak{I}, \mathfrak{J})$ and duality properties of these ideals will be investigated in a forthcoming paper.
We start with the consideration of algebraical properties of $\mathfrak{S}_\Phi(\mathcal{D})$.

2. THE IDEALS $\mathfrak{S}_\Phi(\mathcal{D})$ (ALGEBRAICAL PROPERTIES)

In this section we investigate algebraical properties of the ideals $\mathfrak{B}(\mathcal{D})$, $\mathfrak{S}_\Phi(\mathcal{D})$, corresponding to $\mathfrak{B}(\mathcal{H})$ and to the symmetrically-normed ideals $\mathfrak{S}_\Phi(\mathcal{H}) \subset \mathfrak{T}(\mathcal{H})$. As mentioned in /3/,/4/, /5/ the ideal $\mathfrak{S}_1(\mathcal{D})$ plays a key role in the investigation of physical observables-states-systems, other ideals are connected with the classification of domains of operator algebras.

The main point of this section is to give an equivalent characterization of $\mathfrak{S}_\Phi(\mathcal{D})$ which can be better handled than that given in Definition 1. For the remainder of this paper we suppose $\mathcal{L}^+(\mathcal{D})$ to be selfadjoint. Let us mention the following fact used in the sequel. If $T \in \mathfrak{S}_\Phi(\mathcal{H})$ and $T = (T_1-T_2) + i(T_3-T_4)$ is the decomposition of T such that $T_j \geq 0$, $(T_1-T_2) = (1/2)(T+T^*)$, $(T_3-T_4) = (1/2i)\cdot(T-T^*)$, so $\|T_j\|_\Phi \leq \|T\|_\Phi$, $1 \leq j \leq 4$ (use $\|S\|_\Phi = \|S^*\|_\Phi$). Moreover, $A \in \mathfrak{S}_\Phi(\mathcal{H})$, $\|B\psi\| \leq \|A\psi\|$ for all $\psi \in \mathcal{H}$ implies $B \in \mathfrak{S}_\Phi(\mathcal{H})$ and $\|B\|_\Phi \leq \|A\|_\Phi$. We begin with a proposition on $\mathfrak{B}(\mathcal{D})$ and $\mathfrak{S}_\infty(\mathcal{D})$ which we use permanent in the sequel. Then the main lemma will be proved which is the base of the equivalent characterization of $\mathfrak{S}_\Phi(\mathcal{D})$.

Proposition 10

i) If $0 \leq T = T^* \in \mathcal{B}(\mathfrak{D})$, then $T^\alpha \in \mathcal{B}(\mathfrak{D})$ for all $\alpha > 0$. Moreover, for $T \in \mathcal{L}^+(\mathfrak{D})$ any of the following equivalent conditions is equivalent to $T \in \mathcal{B}(\mathfrak{D})$:
   a) TA, T^*A bounded for all $A \in \mathcal{L}^+(\mathfrak{D})$
   b) AT, AT^* bounded for all $A \in \mathcal{L}^+(\mathfrak{D})$
   c) $T \in \mathcal{B}(\mathcal{H})$, $T\mathcal{H} \subset \mathfrak{D}$, $T^*\mathcal{H} \subset \mathfrak{D}$.

ii) If $0 \leq T = T^* \in \mathcal{S}_\infty(\mathfrak{D})$, then $T^\alpha \in \mathcal{S}_\infty(\mathfrak{D})$ for all $\alpha > 0$. Moreover, for $T \in \mathcal{L}^+(\mathfrak{D})$ any of the following equivalent conditions is equivalent to $T \in \mathcal{S}_\infty(\mathfrak{D})$:
   a) TA, T^*A completely continuous for all $A \in \mathcal{L}^+(\mathfrak{D})$
   b) AT, AT^* completely continuous for all $A \in \mathcal{L}^+(\mathfrak{D})$
   c) $T \in \mathcal{S}_\infty(\mathcal{H})$, $T\mathcal{H} \subset \mathfrak{D}$, $T^*\mathcal{H} \subset \mathfrak{D}$.

Proof:

i) From /4/ (Lemmata 1.1 and 1.2) $T \in \mathcal{B}(\mathfrak{D})$ implies $T\mathcal{H} \subset \mathfrak{D}$, $T^*\mathcal{H} \subset \mathfrak{D}$. The selfadjointness of $\mathcal{L}^+(\mathfrak{D})$ and the Kato-Heinz-inequality /8/ imply that from $T = T^* \geq 0$, $T\mathcal{H} \subset \mathfrak{D}$ one can conclude $T^\alpha \mathcal{H} \subset \mathfrak{D}$ for all $\alpha > 0$. But then /4/ gives us $T^\alpha \in \mathcal{B}(\mathfrak{D})$ for all $\alpha > 0$. This proves the first assertion. Once again referring to /4/ we have the following implications: $T \in \mathcal{B}(\mathfrak{D}) \longrightarrow$ a),b),c); a) $\longleftrightarrow$ b) (by the selfadjointness of $\mathcal{L}^+(\mathfrak{D})$); a) and b) $\longrightarrow$ c); c) $\longrightarrow$ a) and b). Hence it remains to see that c) $\longrightarrow T \in \mathcal{B}(\mathfrak{D})$. Because T and T^* map $\mathcal{H}$ in $\mathfrak{D}$ we may suppose $T = T^*$. Moreover, the decomposition $T = T_+ - T_-$ leads to operators T_+ and T_- which also map $\mathcal{H}$ in $\mathfrak{D}$. Hence our first assertion implies that $T_+^{1/2}$ and $T_-^{1/2}$ map $\mathcal{H}$ in $\mathfrak{D}$ and consequently $AT_+B = (AT_+^{1/2})(T_+^{1/2}B)$ and AT_-B are bounded operators for all $A, B \in \mathcal{L}^+(\mathfrak{D})$, that means ATB is bounded. Therefore $T \in \mathcal{B}(\mathfrak{D})$.

ii) Remark that $0 \leq T = T^* \in \mathcal{S}_\infty(\mathcal{H})$ implies $T^\alpha \in \mathcal{S}_\infty(\mathcal{H})$ for all $\alpha > 0$. All considerations are similar to those of i). To see the implication c) $\longrightarrow T \in \mathcal{S}_\infty(\mathfrak{D})$ suppose again $T \geq 0$, but then $ATB = (AT^{1/3})\, T^{1/3}\, (T^{1/3}B) = UT^{1/3}V$, where U, V are bounded by i) and $T^{1/3}$ is completely continuous. Hence $ATB \in \mathcal{S}_\infty(\mathcal{H})$ which means $T \in \mathcal{S}_\infty(\mathfrak{D})$.

Q.E.D.

Main Lemma 11

Let $A \in \mathcal{L}^+(\mathfrak{D})$, $0 \leq T = T^* \in \mathcal{S}_\infty(\mathfrak{D})$. For fixed $\phi, \psi \in \mathcal{R}(T) \oplus \mathcal{N}(T)$ the function $f(z) = \langle \phi , T^z A T^{1-z} \psi \rangle$ fulfils on the strip $S = \{ z = x+iy : 0 \leq x \leq 1 \}$ the assumptions of Hadamard's three line theorem /9/, namely

i) $f(z)$ is analytic in the interior of S
ii) $f(z)$ is bounded on S
iii) $f(z)$ is continuous on S.

Proof:

Let us premise some remarks. $T = \sum \lambda_n \langle \phi_n , \cdot \rangle \phi_n$ and $T^{iy} = \sum \lambda_n^{iy} \langle \phi_n , \cdot \rangle \phi_n$ imply that for all y: $\mathcal{N}(T) = \mathcal{N}(T^{iy})$, $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^{iy})}$, T^{iy} is a partially isometric operator mapping $\overline{\mathcal{R}(T)}$ isometrically onto itself. Moreover, $T^0 = \sum \langle \phi_n , \cdot \rangle \phi_n$ is the projection P on $\overline{\mathcal{R}(T)}$ (the closed subspace spanned by (ϕ_n)). Now we go on to the proof of the Lemma.

i) For any $0 < \varepsilon < 1$ the function $f(z) = \langle \phi , T^{z-\varepsilon} B T^{1-z} \psi \rangle$ with $B = T^\varepsilon A$ (which is bounded according to Proposition 10) is analytic in the strip $\{ z : \varepsilon < x < 1-\varepsilon \}$, hence $f(z)$ is analytic in the interior of the strip S.

ii) The boundedness of the function $f(z)$ can be seen as follows. $|f(z)| = |\langle \phi , T^{iy} T^x A T^{1-x} T^{-iy} \psi \rangle| \leq \|\phi\| \cdot \| T^x A T^{1-x} \| \cdot \|\psi\|$. We show $\sup \{ \| T^x A T^{1-x} \| : 0 \leq x \leq 1 \} < \infty$. $\sup \| T^x A T^{1-x} \| \leq \max \{ \sup_{0 \leq x \leq 1/2} \| T^x A T^{1-x} \|, \sup_{0 \leq x \leq 1/2} \| T^x A T^{1-x} \| \}$. But $\sup_{0 \leq x \leq 1/2} \| T^x A T^{1-x} \| \leq \sup_{0 \leq x \leq 1/2} \| T \|^x \sup_{0 \leq x \leq 1/2} \| A T^{1-x} \| \leq$ $= \sup \| T \|^x \sup \| A T^{1/4} T^{3/4-x} \| \leq \| T \|^{3/4} \| A T^{1/4} \| < \infty$.

The estimation of the other term follows analogously. Thus, ii) is proved.

iii) It is easy to see that

(1a) $T^{iy} \longrightarrow T^{iy_0}$ (for $y \longrightarrow y_0$)
(1b) $T^{1-x} \longrightarrow T$ (for $x \longrightarrow 0$)
(1c) $T^x \longrightarrow P$ (for $x \longrightarrow 0$)

in the sense of strongly convergence on $\mathcal{H}$. Clearly, according to i) we must show the continuity of $f(z)$ only on the boundary of the strip S, say for $x = 0$ (the case $x = 1$ is got by symmetry). We estimate $|f(z) - f(iy_0)|$ for fixed y_0, i.e.

$|\langle \phi , (T^{iy}T^xAT^{1-x}T^{-iy} - T^{iy_0}PATT^{-iy_0})\psi\rangle|$. Using the identity $T^{iy}T^xAT^{1-x}T^{-iy} - T^{iy_0}PATT^{-iy_0} = (T^{iy}T^xAT^{1-x}T^{-iy} - T^{iy_0}T^xAT^{1-x}T^{-iy}) + (T^{iy_0}T^xAT^{1-x}T^{-iy} - T^{iy_0}PATT^{-iy_0})$ one gets $|f(z) - f(iy_0)| \le$

(2) $\le |\langle \phi , (T^{iy_0}T^xAT^{1-x}T^{-iy} - T^{iy_0}PATT^{-iy_0})\psi\rangle| +$

(3) $+ |\langle \phi , (T^{iy}T^xAT^{1-x}T^{-iy} - T^{iy_0}T^xAT^{1-x}T^{-iy})\psi\rangle|$.

First we show that (3) $\longrightarrow 0$ as $y \longrightarrow y_0$.

(3) = $|\langle (T^{-iy} - T^{-iy_0})\phi , (T^xAT^{1-x})T^{-iy}\psi\rangle| \le \|(T^{-iy}-T^{-iy_0})\phi\| \, \|T^xAT^{1-x}\psi\|$. Thus ii) and(1a) give the desired result. Now we show that (2) $\longrightarrow 0$ as $y \longrightarrow y_0$, $x \longrightarrow 0$. Simple manipulations give (2) $\le |\langle T^{-iy_0}\phi , PAT(T^{-iy}-T^{-iy_0})\psi\rangle| + |\langle T^{-iy_0}\phi , (T^xAT^{1-x} - PAT)T^{-iy}\psi\rangle|$. The first term tends to zero because

$|\langle (PAT)^* T^{-iy_0}\phi , (T^{-iy}-T^{-iy_0})\psi\rangle| \le K \, \|(T^{-iy}-T^{-iy_0})\psi\|$.

The second term can be written as $|\langle T^{-iy_0}\phi , (T^xAT^{1-x} + T^xAT - T^xAT - PAT)T^{-iy}\psi\rangle| \le |\langle (T^xA)^* T^{-iy_0}\phi , (T^{1-x}-T)T^{-iy}\psi\rangle| + |\langle (P-T^x)T^{-iy_0}\phi , ATT^{-iy}\psi\rangle| \le \|(T^xA)^* T^{-iy_0}\phi\| \cdot \|(T^{1-x} - T) \cdot T^{-iy}\psi\| + \|(P-T^x)T^{-iy_0}\phi\| \cdot \|AT\| \cdot \|T^{-iy}\psi\|$. Using (1b),(1c) this expression goes to zero if we show that $\|(T^xA)^* T^{-iy_0}\phi\| \le L$ for all x, $0 \le x \le 1$. Now it will be used that $\phi \in \mathcal{R}(T) \oplus \mathcal{N}(T)$, i.e. $\phi = \phi_1 + \phi_2$, $\phi_1 \in \mathcal{R}(T)$, $\phi_2 \in \mathcal{N}(T)$. It is $\|A^* T^x T^{-iy_0}\phi\| = \|A^* T^x T^{-iy_0}\phi_1\| = \|A^* T^x T^{-iy_0} T\chi\|$ for some $\chi \in \mathcal{H}$ because $\phi_1 = T\chi$, so $\|A^* T^x T^{-iy_0}\phi\| \le \|A^* T\| \, \|T^x\| \, \|\chi\|$. Hence the continuity is proved. Q.E.D.

From this Lemma we deduce an important result. Before doing this, let us remark the following fact which can be proved by simple estimations. Let $\mathfrak{D}$ be an arbitrary dense manifold in $\mathcal{H}$, $\mathfrak{F}(\mathfrak{D}) \subset \mathcal{L}^+(\mathfrak{D})$, $\mathfrak{F} \subset \mathcal{B}(\mathcal{H})$ the corresponding sets of all finite dimensional operators contained in $\mathcal{L}^+(\mathfrak{D})$, $\mathcal{B}(\mathcal{H})$ resp.. If Φ is a symmetric norming function, then $\mathfrak{F}(\mathfrak{D})$ is $\|\ \|_\Phi$ -dense in $\mathfrak{F}$. Now we prove the equivalent characterization of $\mathcal{S}_\Phi(\mathfrak{D})$.

<u>Proposition 12</u>

Let Φ be a symmetric norming function, then

$\mathcal{S}_\Phi(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}): AT, AT^* \in \mathcal{S}_\Phi(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathfrak{D})\}$.

<u>Proof:</u>

The case where $\|\ \|_\Phi$ is equivalent to the operator norm, i.e. $\mathcal{S}_\Phi(\mathcal{H}) = \mathcal{S}_\infty(\mathcal{H})$ was regarded in Proposition 10. Thus, let $\|\ \|_\Phi$ be non-equivalent to the operator norm. The selfadjointness of $\mathcal{L}^+(\mathfrak{D})$ implies again $\{T \in \mathcal{L}^+(\mathfrak{D}): AT, AT^* \in \mathcal{S}_\Phi(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathfrak{D})\} = \{T \in \mathcal{L}^+(\mathfrak{D}): TA, T^*A \in \mathcal{S}_\Phi(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathfrak{D})\}$. Moreover, as in the proof of Proposition 10, $AT, AT^* \in \mathcal{S}_\Phi(\mathcal{H})$ implies that the decomposition $T=(T_1-T_2)+i(T_3-T_4)$, $T_j \ge 0$, $j=1,2,3,4$ leads to operators T_j with $AT_j \in \mathcal{S}_\Phi(\mathcal{H})$, too. Consequently, we can restrict ourselves to operators $T = T^* \ge 0$ and must show that $AT \in \mathcal{S}_\Phi(\mathcal{H})$ implies $ATB \in \mathcal{S}_\Phi(\mathcal{H})$ for all $A,B \in \mathcal{L}^+(\mathfrak{D})$. Let $\mathfrak{D}' = \mathcal{R}(T) \oplus \mathcal{N}(T)$, $F \in \mathfrak{F}(\mathfrak{D}')$ arbitrary, $F = \sum_{k=1}^{n} \mu_k \langle \varrho_k, \cdot \rangle \chi_k$, (ϱ_k),(χ_k) orthonormal systems in $\mathfrak{D}'$. Consider the function

$g(z) = \mathrm{Tr}\, T^z A T^{1-z} F$ on the strip $S = \{ z=x+iy: 0 \le x \le 1, -\infty < y < \infty \}$.

Because $\mathrm{Tr}\, T^z A T^{1-z} F = \sum_{k=1}^{n} \mu_k \langle \varrho_k , T^z A T^{1-z} \chi_k \rangle$, $g(z)$ is the linear combination of n functions satisfying on S the assumptions of Lemma 11, hence for $g(z)$ the three line theorem is avalable. Using $F = (F_1-F_2) + i(F_3-F_4)$ on the line $x = 0$ we have the estimation

$|\mathrm{Tr}\, T^{iy}ATT^{-iy}F| \le \sum_{j=1}^{4} |\mathrm{Tr}\, T^{iy}ATT^{-iy}F_j| = \sum \|T^{iy}ATT^{-iy}\|_\Phi \|F_j\|_{\Phi^*} = 4\|AT\|_\Phi \cdot \|F\|_{\Phi^*}$ (because $\|T^{iy}\| = \|T^{-iy}\| = 1$). Analogously, on the line $x = 1$: $|\mathrm{Tr}\, T^{iy}ATT^{-iy}F| \le 4\|TA\|_\Phi \|F\|_{\Phi^*}$. Here Φ^* is the symmetric norming function conjugate to Φ and the estimations follow from the corresponding properties of Φ and Φ^* (cf./1/). The three line theorem now gives:

$|\mathrm{Tr}\, T^{1/2}AT^{1/2}F| \le (4\|AT\|_\Phi \|F\|_{\Phi^*})^{1/2} (4\|TA\|_\Phi \|F\|_{\Phi^*})^{1/2} \le$

$\le 4\|F\|_{\Phi^*} (\|AT\|_\Phi + \|TA\|_\Phi)$. Thus,

$\sup_{F\in\mathfrak{F}(\mathfrak{D}')} \frac{|\operatorname{Tr} T^{1/2}AT^{1/2}F|}{\|F\|_{\Phi^*}} \leq 4(\|AT\|_\Phi + \|TA\|_\Phi) < \infty$. Together with the remark before Proposition 12 and /1/ (chap.III,Lemma 12.1) this estimation gives us $T^{1/2}AT^{1/2} \in \mathcal{S}_\Phi(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathfrak{D})$. Then $(T^{1/2}B)(AT^{1/2}) \in \mathcal{S}_\Phi(\mathcal{H})$ and consequently, $AT^{1/2}(T^{1/2}B) \in$ $\in \mathcal{S}_\Phi(\mathcal{H})$ for all $A,B \in \mathcal{L}^+(\mathfrak{D})$.

Q.E.D.

This characterization of $\mathcal{S}_\Phi(\mathfrak{D})$ will be useful for the investigation of topologies on $\mathcal{M}(\cdot,\cdot)$ and in duality-considerations. We remark that the above result can be obtained at least for $\mathcal{S}_1(\mathfrak{D})$ and $\mathcal{S}_2(\mathfrak{D})$ by direct computation without using interpolation methods. Furthermore, the proof of Proposition 12 gives us the possibility to derive elements of a functional calculus for the ideals $\mathcal{S}_p(\mathfrak{D})$ analogous to the case of $\mathcal{S}_p(\mathcal{H})$. We collect some results in the following Proposition.

<u>Proposition 13</u>

Let $1 \leq p < \infty$.

i) If $T \in \mathcal{S}_p(\mathfrak{D})$, so $T^n \in \mathcal{S}_{p/n}(\mathfrak{D})$ for all naturals n.

ii) If $0 \leq T = T^* \in \mathcal{S}_p(\mathfrak{D})$, so $T^\alpha \in \mathcal{S}_{p/\alpha}(\mathfrak{D})$ for all $\alpha > 0$.

iii) If $S \in \mathcal{S}_p(\mathfrak{D})$, $T \in \mathcal{S}_q(\mathfrak{D})$, so $ST \in \mathcal{S}_r(\mathfrak{D})$, $1/r = (1/p)+(1/q)$.

<u>Proof:</u>

i) Let $n \geq 2$, then it is for $AT^nB = ATT^{n-2}TB$, $AT, TB \in \mathcal{S}_p(\mathcal{H})$ and $T^{n-2} \in \mathcal{S}_{p/n-2}(\mathcal{H})$ in consequence of the well-known properties of the ideals $\mathcal{S}_p(\mathcal{H})$. Thus $ATB \in \mathcal{S}_q(\mathcal{H})$ with $1/q = (1/p)+(1/p)+(n-2/p) =$ $= n/p$. Therefore, i) is proved. iii) follows analogously.

ii) To prove this, we remark that $0 \leq T = T^* \in \mathcal{S}_p(\mathfrak{D})$ implies $T^{1/2}A^+AT^{1/2} \in \mathcal{S}_p(\mathcal{H})$ for all $A \in \mathcal{L}^+(\mathfrak{D})$ (see the end of the proof of Proposition 12). Consequently, $T^{1/2}A$ and $AT^{1/2} \in \mathcal{S}_{2p}(\mathcal{H})$, i.e. $T^{1/2} \in \mathcal{S}_{2p}(\mathfrak{D})$, and general: $T^{2^{-n}} \in \mathcal{S}_{2^n p}(\mathfrak{D})$. Hence, let $\alpha > 0$ be arbitrary, $\alpha = (2/2^n) + \beta$ for some natural n and $\beta > 0$. Then $AT^\alpha B = (AT^{2^{-n}})T^\beta(T^{2^{-n}}B)$ belongs to $\mathcal{S}_q(\mathcal{H})$ with $1/q = (\beta/p)$ $+ (2^{-n}/p) + (2^{-n}/p) = \alpha/p$, i.e. $T^\alpha \in \mathcal{S}_{p/\alpha}(\mathfrak{D})$.

Q.E.D.

We conclude the investigation of algebraical properties with a result which can be roughly expressed as follows: The orthonormal system occuring in the representation $T = \sum \lambda_n \langle \phi_n, \cdot \rangle \psi_n$ of an arbitrary operator $T \in \mathcal{S}_\infty(\mathfrak{D})$ are the same for all ideals $\mathcal{S}_\Phi(\mathfrak{D})$. Or in other words: if $T \in \mathcal{S}_\infty(\mathfrak{D})$, then only the decrease of the sequence (λ_n) decides whether $T \in \mathcal{S}_\Phi(\mathfrak{D})$ or not. More precisely:

<u>Lemma 14</u>

Let $T = \sum \lambda_n \langle \phi_n, \cdot \rangle \phi_n \in \mathcal{S}_\infty(\mathfrak{D})$, $T \geq 0$. Then there is a continuous function f with $f(x) > 0$ for $x > 0$ such that

$$f(T) = \sum f(\lambda_n) \langle \phi_n, \cdot \rangle \phi_n \in \mathcal{S}_1(\mathfrak{D}).$$

<u>Proof:</u>

It is easy to see that there is a continuous function g with the properties: $g(x) > 0$ for $x > 0$ and $\sum g(\lambda_n) < \infty$. Then f with $f(x) =$ $= xg(x)$ is the desired function. To see this we show $Af(T)$ nuclear for all $A \in \mathcal{L}^+(\mathfrak{D})$. It is $\sum \|Af(T)\phi_n\| \leq \sum g(\lambda_n) \|AT\phi_n\| \leq$ $\leq C \sum g(\lambda_n) < \infty$ since $AT \in \mathcal{S}_\infty(\mathfrak{D})$, i.e. AT bounded. Therefore, the bounded operator $Af(T)$ is nuclear.

Q.E.D.

3. TOPOLOGIES ON $\mathcal{S}_\Phi(\mathfrak{D})$

In this section we introduce some topologies on $\mathcal{S}_\Phi(\mathfrak{D})$. They are more or less suggested already by the ideal structure. Let us remark that there are many possibilities for defining a topology on these ideals and the choice of the topology depends on the problem we are dealing with. In a forthcoming paper where we will consider the ideals $\mathcal{M}(\ ,\ )$ and questions concerning duality some other topologies will be useful.

<u>Definition 15</u>

On $\mathcal{B}(\mathfrak{D})$, $\mathcal{S}_\Phi(\mathfrak{D})$ the following topologies (given by generating systems of seminorms) are introduced:

τ_Φ^r : $\mathcal{S}_\Phi(\mathfrak{D}) \ni T \longrightarrow \|T\|^r_{A,\Phi} = \|TA\|_\Phi$ for all $A \in \mathcal{L}^+(\mathfrak{D})$

τ_Φ^l : $\mathcal{S}_\Phi(\mathfrak{D}) \ni T \longrightarrow \|T\|^l_{A,\Phi} = \|AT\|_\Phi$ for all $A \in \mathcal{L}^+(\mathfrak{D})$

τ_Φ : $\mathcal{S}_\Phi(\mathfrak{D}) \ni T \longrightarrow \max(\|AT\|_\Phi, \|TA\|_\Phi)$ for all $A \in \mathcal{L}^+(\mathfrak{D})$

τ^Φ : $\mathcal{S}_\Phi(\mathfrak{D}) \ni T \longrightarrow \|T\|_{A,B,\Phi} = \|ATB\|_\Phi$ for all $A,B \in \mathcal{L}^+(\mathfrak{D})$.

On $\mathfrak{B}(\mathfrak{D})$ the same seminorms as on $\mathcal{J}_\infty(\mathfrak{D})$ are used.

The following Lemma summarizes some simple properties.

Lemma 16

i) All systems of seminorms are directed; $\tau^r_\Phi \prec \tau_\Phi \prec \tau^\Phi$, $\tau^l_\Phi \prec \tau_\Phi \prec \tau^\Phi$.

ii) The ideals equipped with these topologies become locally convex algebras with separately continuous multiplication. The involution is continuous with respect to τ_Φ and τ^Φ.

iii) If the topology t is given by the system of norms $\{\| \ \|_\alpha = \| \ \|_{A_\alpha}, \alpha \in \mathfrak{A}\}$, then any of the topologies defined above can be given in which there occur only operators A_α, $\alpha \in \mathfrak{A}$. Especially, if t is metrizable, so also any of these topologies.

Proof:

First of all let us remark the following fact: If $R, S \in \mathfrak{B}(\mathcal{H})$, $S \in \mathcal{J}_\Phi(\mathcal{H})$ and $\|R\phi\| \leq \|S\phi\|$ for all $\phi \in \mathcal{H}$, then $R \in \mathcal{J}_\Phi(\mathcal{H})$ and $\|R\|_\Phi \leq \|S\|_\Phi$.

i) Given $A, B \in \mathcal{L}^+(\mathfrak{D})$. As the system defining t is directed, there is a $C \in \mathcal{L}^+(\mathfrak{D})$ with $\|A\phi\| \leq \|C\phi\|$, $\|B\phi\| \leq \|C\phi\|$ for all $\phi \in \mathfrak{D}$, hence for $T \in \mathcal{J}_\Phi(\mathfrak{D})$: $\|AT\phi\| \leq \|CT\phi\|$, $\|BT\phi\| \leq \|CT\phi\|$. By the remark above: $\|AT\|_\Phi \leq \|CT\|_\Phi$, $\|BT\|_\Phi \leq \|CT\|_\Phi$. Using the fact that $\|R\|_\Phi = \|R^*\|_\Phi$ it is easy to derive that the systems of seminorms defining the other topologies are also directed.
$\tau^r_\Phi \prec \tau_\Phi$, $\tau^l_\Phi \prec \tau_\Phi$ is trivial. Let $A \in \mathcal{L}^+(\mathfrak{D})$, then $\max(\|AT\|_\Phi, \|TA\|_\Phi) \leq \|AT\|_\Phi + \|TA\|_\Phi = \|T\|_{A,I,\Phi} + \|T\|_{I,A,\Phi}$. Hence $\tau_\Phi \prec \tau^\Phi$.

iii) follows from i).

ii) Only the assertions about multiplication and the involution must be varified. Let $S, T \in \mathcal{J}_\Phi(\mathfrak{D})$, $A \in \mathcal{L}^+(\mathfrak{D})$, then $\|STA\|_\Phi \leq \|S\| \, \|TA\|_\Phi$, $\|STA\|_\Phi \leq \|S\|_\Phi \|TA\|$ and similarly, the assertions for the other topologies follow. For the involution it is $\max(\|AT^+\|_\Phi, \|T^+A\|_\Phi) = \max(\|TA^+\|_\Phi, \|A^+T\|_\Phi)$ and analogously for τ^Φ.

Q.E.D.

Lemma 17

$\mathcal{J}_\Phi(\mathfrak{D})[\tau^\Phi]$ and $\mathcal{J}_\Phi(\mathfrak{D})[\tau_\Phi]$ are complete locally convex spaces.

Proof:

Let (T_α) be a generalized sequence in $\mathcal{J}_\Phi(\mathfrak{D})$ which is a τ^Φ-Cauchy sequence (in the case of the topology τ_Φ all considerations are similar). Because $T \longrightarrow T^+$ is τ^Φ-continuous, $T = T^+$ may be assumed. Thus, for $A, B \in \mathcal{L}^+(\mathfrak{D})$, $(AT_\alpha B)$ and (T_α) are $\| \ \|_\Phi$-Cauchy sequences and since $\mathcal{J}_\Phi(\mathcal{H})$ is $\| \ \|_\Phi$-complete: $AT_\alpha B \longrightarrow S$, $T_\alpha \longrightarrow T$; $S, T \in \mathcal{J}_\Phi(\mathcal{H})$. It must be shown that $ATB = S$. For this it is sufficient to see that $AT_\alpha B$ converges on $\mathfrak{D}$ weakly to ATB (since then $AT_\alpha B$ converges on $\mathcal{H}$ weakly to ATB and we can apply /1/, chap.III,Theorem 5.1). But the weak convergence follows from $|\langle (AT_\alpha B - ATB)\phi, \psi\rangle| = |\langle (T_\alpha - T)B\phi, A^+\psi\rangle|$. Hence $ATB = S$ and consequently $T \in \mathcal{J}_\Phi(\mathfrak{D})$. The only gap in the proof is to show that ATB makes sense on $\mathfrak{D}$. For this we show $T\mathcal{H} \subset \mathfrak{D}$. Let $\phi \in \mathcal{H}$ be arbitrary, then $T_\alpha\phi = \psi_\alpha$ converges to $T\phi$. Moreover, for any $A \in \mathcal{L}^+(\mathfrak{D})$, $AT_\alpha\phi$ is a Cauchy sequence in $\mathcal{H}$ (because AT_α is a Cauchy sequence with respect to $\| \ \|_\Phi$). A closable and $T_\alpha\phi = \psi_\alpha \in \mathfrak{D}(\bar{A})$ imply $AT_\alpha\phi \longrightarrow \bar{A}T\phi$ and $T\phi \in \mathfrak{D}(\bar{A})$. Hence, $T\mathcal{H} \subset \mathfrak{D}(\bar{A})$ for all $A \in \mathcal{L}^+(\mathfrak{D})$. Since $\mathcal{L}^+(\mathfrak{D})$ was assumed to be selfadjoint the assertion follows.

Q.E.D.

The lemmas below give some examples how these topologies can be applied to get results analogous to the case $\mathcal{J}_\Phi(\mathcal{H})$ (cf. /1/, chap.III).

Lemma 18

i) For each $T \in \mathcal{J}_\Phi(\mathfrak{D})$, $A, B \in \mathcal{L}^+(\mathfrak{D})$

$$(4) \quad \min\{\|T-F\|^r_{A,\Phi}\} = \Phi(s_{n+1}(TA), s_{n+2}(TA), \ldots)$$

$$(5) \quad \min\{\|T-F\|^l_{A,\Phi}\} = \Phi(s_{n+1}(AT), s_{n+2}(AT), \ldots)$$

$$(6) \quad \min\{\|T-F\|_{A,B,\Phi}\} = \Phi(s_{n+1}(ATB), s_{n+2}(ATB), \ldots).$$

The minimum in (4)-(6) is taken over all $F \in \mathfrak{F}_n(\mathfrak{D}) = \{G \in \mathfrak{F}(\mathfrak{D}) : \dim G \leq n\}$.

ii) If Φ is mono-norming, then $\mathfrak{F}(\mathfrak{D})$ is dense in $\mathcal{J}_\Phi(\mathfrak{D})$ with respect to τ^r_Φ, τ^l_Φ, τ_Φ, and τ^Φ.

Proof:

The properties of Φ and $\tau_\Phi \prec \tau^\Phi$ give the implication i) $\longrightarrow$ ii). To prove i) we restrict ourselves to (4) since the other statements are established in the same way. Remark that TA is completely continuous on $\mathcal{H}$, so $TA = \sum \lambda_j \langle \phi_j , \cdot \rangle \psi_j$ with orthonormal systems (ϕ_j), (ψ_j) in $\mathcal{H}$. (4) follows from the analogous result for $\mathcal{S}_\Phi(\mathcal{H})$ used for TA. The only point where we must be careful is the restriction $F \in \mathcal{F}_n(\mathcal{D})$. Thus, we will show that there is an $F \in \mathcal{F}_n(\mathcal{D})$ such that $FA = \sum_{j=1}^{n} \lambda_j \langle \phi_j , \cdot \rangle \psi_j$, then all is clear.

Set $F = \sum_{j=1}^{n} \lambda_j \langle \rho_j , \cdot \rangle \psi_j$ and determine ρ_j in the necessary way. The desired operator F is obtained for $\rho_j = \lambda_j^{-1} \; T^+ \psi_j$ which is easy to verify using the equation

$$FA = \sum_{j=1}^{n} \lambda_j \langle A^+ \rho_j , \cdot \rangle \psi_j \; .$$

Q.E.D.

As in case $\mathcal{S}_\Phi(\mathcal{H})$ one could introduce the τ_Φ^r-, ..., τ^Φ-closures of $\mathcal{F}(\mathcal{D})$ in $\mathcal{S}_\Phi(\mathcal{D})$ and would get corresponding one- or two-sided ideals in $\mathcal{L}^+(\mathcal{D})$. Moreover one could prove some results about separability of the so-obtained ideals. For brevity we indicate such a result for mono-norming Φ.

<u>Lemma 19</u>

Let Φ be mono-norming. If $\mathcal{D}[t]$ is separable, then $\mathcal{S}_\Phi(\mathcal{D})$ is separable when equipped with any of the topologies τ_Φ^r, τ_Φ^l, τ_Φ, τ^Φ.

<u>Proof:</u>

Because of Lemma 15 i) it is sufficient to consider the topology τ^Φ. Let $\mathcal{N}$ be an arbitrary countable t-dense subset of $\mathcal{D}$ and put

$$\mathcal{M} = \{ F = \sum_{\text{finite}} \langle \phi_j , \cdot \rangle \psi_j \; ; \; \phi_j , \psi_j \in \mathcal{N} \} \subset \mathcal{F}(\mathcal{D}).$$

We show that $\mathcal{M}$ is τ^Φ-dense in $\mathcal{F}(\mathcal{D})$ and so as a consequence of Lemma 18 ii) the Lemma is proved.

Let $G = \sum_{j=1}^{n} \langle \rho_j , \cdot \rangle \chi_j \in \mathcal{F}(\mathcal{D})$, $A, B \in \mathcal{L}^+(\mathcal{D})$, $\varepsilon > 0$ be arbitrarily given. If there is an $F \in \mathcal{M} : \| G - F \|_{A,B,\Phi} < \varepsilon$ then we arrive at the desired result because the system of seminorms for τ^Φ is directed.

For $F = \sum_{j=1}^{n} \langle \phi_j , \cdot \rangle \psi_j$ the estimation $\| G - F \|_{A,B,\Phi} =$

$= \| \sum_{j=1}^{n} (\langle B^+ \rho_j , \cdot \rangle A \chi_j - \langle B^+ \phi_j , \cdot \rangle A \psi_j) \|_\Phi \leq \sum \| \langle B^+ (\rho_j - \phi_j) , \cdot \rangle A \chi_j$

$+ \langle B^+ \phi_j , \cdot \rangle (A \chi_j - A \psi_j) \|_\Phi \leq \sum \{ \| B^+ (\rho_j - \phi_j) \| \cdot \| A \chi_j \| +$

$+ \| B^+ \phi_j \| \, \| A (\chi_j - \psi_j) \| \}$ shows that the ϕ_j, ψ_j can be chosen so that $\| G - F \|_{A,B,\Phi} < \varepsilon$.

Q.E.D.

We conclude with some criteria (corresponding to those for $\mathcal{S}_\Phi(\mathcal{H})$) for $T \in \mathcal{S}_\Phi(\mathcal{D})$.

<u>Lemma 20</u>

Let Φ be such that $\mathcal{S}_\Phi(\mathcal{H}) \neq \mathcal{L}_\infty(\mathcal{H})$.

i) If $(T_m) \subset \mathcal{S}_\Phi(\mathcal{D})$ converges on $\mathcal{D}$ weakly to $T \in \mathcal{L}^+(\mathcal{D})$ and $\sup_m \| T_m \|_{A,B,\Phi} = \sup \| A T_m B \|_\Phi < \infty$ for all $A, B \in \mathcal{L}^+(\mathcal{D})$, then $T \in \mathcal{S}_\Phi(\mathcal{D})$.

ii) If $\mathcal{L}^+(\mathcal{D})$ contains a monotonically increasing sequence (P_n) of finite dimensional orthoprojections converging t-strongly to I, i.e. $\| A (P_n - I) \phi \| \longrightarrow 0$ for all $\phi \in \mathcal{D}$, $A \in \mathcal{L}^+(\mathcal{D})$ and $(P_n T P_n)$ is τ^Φ-bounded for some $T \in \mathcal{L}^+(\mathcal{D})$, then $T \in \mathcal{S}_\Phi(\mathcal{D})$

iii) Let (P_n) be as in ii), $S, T \in \mathcal{L}^+(\mathcal{D})$. If $(S P_n T P_n)$ is τ^Φ-bounded, then $ST \in \mathcal{S}_\Phi(\mathcal{D})$.

<u>Proof:</u>

We will only prove i) since the other assertions follow by similar considerations. In /1/ (Theorem 5.1, chap. III) it is shown that $(S_m) \subset \mathcal{S}_\Phi(\mathcal{H})$, $S_m \longrightarrow S$ (weakly on $\mathcal{H}$) and $\sup \| S_m \|_\Phi < \infty$ implies $S \in \mathcal{S}_\Phi(\mathcal{H})$. Applying this result to the sequence $A T_m B$ it remains to show that $A T_m B$ converges on $\mathcal{H}$ weakly to ATB. It is $\lim \langle A T_m B \phi , \psi \rangle = \lim \langle T_m B \phi , A^+ \psi \rangle = \langle ATB \phi , \psi \rangle$ for all $\phi, \psi \in \mathcal{D}$. Hence $A T_m B \longrightarrow ATB$ weakly on $\mathcal{D}$. Because $(A T_m B)$ is bounded with respect to the **operator norm this sequence is also weakly** convergent on $\mathcal{H}$ to the bounded operator ATB.

Q.E.D.

Remark that the existence of a sequence (P_n) such as mentioned in ii) and iii) of the above Lemma is guaranteed for example in the important case where $\mathfrak{D} = \bigcap_n \mathfrak{D}(R^n)$, $R = R^*$.

Acknowledgement

The author is grateful to Prof.G.Lassner for stimulating discussions and Prof.A.Uhlmann for his interest in this work.

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Received by Publishing Department on June 15, 1977.