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ON AN IDEAL IN ALGEBRAS OF UNBOUNDED OPERATORS

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О некотором идеале в алгебрах неограниченных операторов

Рассматривается замыкание множества конечномерных операторов из $\mathcal{L}^+(\mathfrak{D})$ относительно разных топологий. Полученные идеалы обладают многими свойствами анологичными идеалу вполне непрерывных операторов в гильбертовом пространстве. Например, при некоторых подходящих предположениях все непрерывные функционалы нормальны, неприводимые представления эквивалентны тождественному представлению и т.д.

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On an Ideal in Algebras of Unbounded Operators

The closure of the set of finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$ with respect to different topologies is considered. The obtained ideals have many properties similar to those of the ideal of completely continuous operators on Hilbert space. For example, under some appropriate assumptions all continuous functionals are normal, irreducible representations are equivalent to the identical representation and so on.

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In this paper we continue the considerations of /12/ and refer to this paper for general remarks on the subject. We concentrate our attention on the closure of the ideal of finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$ with respect to different topologies. The ideals obtained in this way reflect many properties of the ideal of completely continuous operators in Hilbert space. For example, the dual space can be identified with a certain ideal of trace class operators, irreducible representations are (under some natural restrictions) equivalent to the identical representation and so on.

1. PRELIMINARIES

To make the paper independent of /12/ we recall some definitions (see also /3/,/6/). For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathfrak{H}$ by $\mathcal{L}^+(\mathfrak{D})$ we denote the $*$-algebra of all operators A (bounded or not) with $A\mathfrak{D} \subset \mathfrak{D}$, $A^*\mathfrak{D} \subset \mathfrak{D}$. The involution is given by $A \longrightarrow A^+ = A^* \restriction \mathfrak{D}$. A $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$ is called Op*-algebra. $\mathcal{L}^+(\mathfrak{D})$ defines a natural topology t on $\mathfrak{D}$ given by the directed system of seminorms

$$\mathfrak{D} \ni \phi \longrightarrow \|\phi\|_A = \|A\phi\| \quad , A \in \mathcal{L}^+(\mathfrak{D}).$$

$\mathcal{L}^+(\mathfrak{D})$ is said to be closed if $\mathfrak{D}[t]$ is complete or equivalently, $\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(\bar{A})$. If $\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(A^*)$, $\mathcal{L}^+(\mathfrak{D})$ is called selfadjoint.

In $\mathcal{L}^+(\mathcal{D})$ we introduce the uniform topology $\tau_{\mathcal{D}}$ given by the system of seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\phi,\psi\in\mathcal{M}} |\langle\phi, A\psi\rangle|$$

and the quasiuniform topology $\tau^{\mathcal{D}}$ given by the system of seminorms

$$A \longrightarrow \|A\|^{\mathcal{M}}_{B} = \sup_{\phi\in\mathcal{M}} \|BA\phi\|,\ B \in \mathcal{L}^+(\mathcal{D}) \text{ arbitrary.}$$

In both the cases $\mathcal{M}$ runs over all t-bounded subsets of $\mathcal{D}$, i.e. $\sup_{\phi\in\mathcal{M}} \|A\phi\| < \infty$ for all $A \in \mathcal{L}^+(\mathcal{D})$. Remark that the seminorms defining $\tau_{\mathcal{D}}$ can be extended to all operators of $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}(\mathcal{D}[t])$, the continuous linear operators of $\mathcal{D}[t]$ into itself, the seminorms defining $\tau^{\mathcal{D}}$ can be extended to $\mathcal{L}(\mathcal{D}[t])$. The following definition is for simplicity given for selfadjoint $\mathcal{L}^+(\mathcal{D})$. A linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is said to be normal if it has the representation

$$\omega(A) = \text{Tr } AT = \text{Tr } TA \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}),$$

where T belongs to the two-sided $*$-ideal

$$\mathcal{S}_1(\mathcal{D}) = \{T \in \mathcal{L}^+(\mathcal{D}):\ TA,\ T^*A \text{ nuclear for all } A \in \mathcal{L}^+(\mathcal{D})\} =$$
$$= \{T \in \mathcal{L}^+(\mathcal{D}):\ AT,\ AT^* \text{ nuclear for all } A \in \mathcal{L}^+(\mathcal{D})\}$$

(cf./6/,/13/).

By $\mathcal{F}(\mathcal{D})$ we denote the minimal two-sided $*$-ideal of all finite dimensional operators of $\mathcal{L}^+(\mathcal{D})$. $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ (closure in $\mathcal{L}^+(\mathcal{D})$) is a two-sided $*$-ideal in $\mathcal{L}^+(\mathcal{D})$ (may be no proper one) /12/.

We remark that because of /12/ ($\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}} = \overline{\text{Com}(t,t)}^{\tau_{\mathcal{D}}} = \overline{\text{Com}(t,\|\cdot\|)}^{\tau_{\mathcal{D}}}$) all results concerning $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ are also valid for the just mentioned sets. Com(t,t) (Com(t,$\|\cdot\|$) resp.) denotes the set compact maps from $\mathcal{D}[t]$ in $\mathcal{D}[t]$ ($\mathcal{D}[\|\cdot\|]$ resp.) which belong to $\mathcal{L}^+(\mathcal{D})$.

2. A RESULT ON DUALITY

In this section two propositions are given which generalize and systematize some results of /6/ and /11/. For the proof we use the same idea as in /6/, Theorem 2.

Proposition 1

Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint and $\mathcal{D}[t]$ a metrizable space. Then any normal linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$-continuous, i.e. for any functional ω with $\omega(A) = \text{Tr } AT$, $T \in \mathcal{S}_1(\mathcal{D})$ there is a t-bounded set $\mathcal{M}$ such that

(1) $\quad |\omega(A)| \leq \|A\|_{\mathcal{M}}$ for all $A \in \mathcal{L}^+(\mathcal{D})$.

Proof:

Let $T \in \mathcal{S}_1(\mathcal{D})$, T = UH the polar decomposition, then $H = (T^*T)^{1/2} \in \mathcal{S}_1(\mathcal{D})$.
Let $H\phi_i = \lambda_i\phi_i$, $\lambda_i \neq 0$, (ϕ_i) the orthonormal system of corresponding eigenvectors. The system $\{\|\ \|_n,\ n \in \mathbb{N}\}$ of seminorms defining the topology t can be given by a family of operators

(2) $\quad \mathfrak{A} = \{A_n \in \mathcal{L}^+(\mathcal{D}): \|\phi\|_n = \|A_n\phi\|,\ I \leq A_n = A_n^+ \leq A_{n+1} = A_{n+1}^+ \leq \ldots$
so that $A_n \in \mathfrak{A}$ implies $A_n^2 = A_{m(n)} \in \mathfrak{A}\}$.

Take into account that U, U^* are bounded and thus A_nH, U^*A_nT nuclear, one gets the absolute convergence of the following series:

(3a) $\quad \sum_i \langle U^*A_nT\phi_i, \phi_i\rangle = \sum \lambda_i \langle U^*A_nU\phi_i, \phi_i\rangle < \infty$ for all n

(3b) $\quad \sum_i \langle A_nH\phi_i, \phi_i\rangle = \sum \lambda_i \langle A_n\phi_i, \phi_i\rangle < \infty$ for all n.

Similarly as in /6/ one shows that there is a sequence (α_i), $\alpha_i \geq 1$

(4a) $\quad \sum \lambda_i\alpha_i < \infty$

(4b) $\quad \sup_i (1/\alpha_i)\langle U^*A_nU\phi_i, \phi_i\rangle < \infty$ for all n

(4c) $\quad \sup (1/\alpha_i)\langle A_n\phi_i, \phi_i\rangle < \infty$ for all n.

Set $\mathcal{M} = \{(1/\alpha_i^{1/2})U\phi_i,\ (1/\alpha_i^{1/2})\phi_i,\ i=1,2,\ldots\}$

Remark that T, $T^* \in \mathcal{S}_1(\mathcal{D})$ implies that ϕ_i, $U\phi_i \in \mathcal{D}$, i.e. $\mathcal{M} \subset \mathcal{D}$.
Let us note that $T \in \mathcal{S}_1(\mathcal{D})$ does not imply $U \in \mathcal{L}^+(\mathcal{D})$, where T = UH.
$\mathcal{M}$ is a t-bounded set since

$\sup_i \|A_n(1/\alpha_i^{1/2})U\phi_i\|^2 = \sup (1/\alpha_i)\langle A_n^2U\phi_i, U\phi_i\rangle = \sup(1/\alpha_i)\cdot$
$\cdot\langle U^*A_{m(n)}U\phi_i, \phi_i\rangle < \infty$ ((2) and (4b)). In the same way it is

$\sup_i \|A_n(1/\alpha_i^{1/2})\phi_i\|^2 < \infty$ ((2) and (4c)). Moreover, if $A \in \mathcal{L}^+(\mathcal{D})$

arbitrary, the closed graph theorem gives $\|A\phi\| \le \sum_{\text{finite}} \|A_{n_i}\phi\|$, $\phi \in \mathcal{D}$
Hence, the t-boundedness of $\mathcal{M}$ is established. The $\tau_{\mathcal{D}}$-continuity of ω is seen from the estimation

$$|\omega(A)| = |\sum \langle AUH\phi_i, \phi_i\rangle| \le \sum \lambda_i |\langle AU\phi_i, \phi_i\rangle| =$$

$$= \sum (\lambda_i \alpha_i)\,|(1/\alpha_i)\langle AU\phi_i, \phi_i\rangle| \le (\sum \lambda_i\alpha_i)\,\|A\|_{\mathcal{M}} .$$

Q.E.D.

The next proposition deals with the question when a $\tau_{\mathcal{D}}$-continuous functional is a normal one.

Proposition 2

Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint and such that any normal functional is $\tau_{\mathcal{D}}$-continuous. Then any $\tau_{\mathcal{D}}$-continuous linear functional ω on $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ is normal on $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$, i.e.

(5) $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}[\tau_{\mathcal{D}}]' \cong \mathcal{J}_1(\mathcal{D})$ (algebraic isomorphism).

Proof:

The $\tau_{\mathcal{D}}$-continuity of the involution $A \longrightarrow A^+$ allows us to restrict ourselves to real functionals. Moreover since $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ is an Op*-algebra, the positive cone is normal with respect to $\tau_{\mathcal{D}}$ /9/ and consequently it is enough to regard only positive $\tau_{\mathcal{D}}$-continuous functionals /8/. Thus, let ω be such a functional with

$|\omega(A)| \le \|A\|_{\mathcal{M}}$ for suitable t-bounded $\mathcal{M}$ and all $A \in \overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$.

For $\langle\phi, .\rangle\psi \in \mathcal{F}(\mathcal{D})$, $\phi,\psi \in \mathcal{D}$, as a consequence of

$$|\omega(\langle\phi, .\rangle\psi)| \le \sup_{\rho,\chi\in\mathcal{M}} |\langle\phi,\chi\rangle||\langle\psi,\rho\rangle| \le K\|\phi\|\cdot\|\psi\|$$

the bilinear form $\omega(\langle\phi, .\rangle\psi)$ is $\|\cdot\|$-continuous on $\mathcal{D}$ and can be extended to such a one on $\mathcal{H}$. This implies

$\omega(\langle\phi, .\rangle\psi) = \langle\phi, T\psi\rangle$, $T \in \mathcal{B}(\mathcal{H})$, $\phi,\psi \in \mathcal{H}$.

The linearity of ω gives moreover

(6) $\omega(F) = \mathrm{Tr}\, FT$ for all $F \in \mathcal{F}(\mathcal{D})$.

T has the following properties: $T \ge 0$ since ω is positive. Let $\phi \in \mathcal{D}$, $\psi \in \mathcal{H}$ then

$$|\omega(\langle A\phi, .\rangle\psi)| = |\langle A\phi, T\psi\rangle| \le \sup_{\rho,\chi\in\mathcal{M}} |\langle A\phi,\rho\rangle\langle\psi,\chi\rangle| \le L(A,\mathcal{M})\,\|\phi\|$$

for arbitrary $A \in \mathcal{L}^+(\mathcal{D})$ means that $T\psi \in \mathcal{D}(A^*)$ for all $A \in \mathcal{L}^+(\mathcal{D})$. By the selfadjointness of $\mathcal{L}^+(\mathcal{D})$ this gives $T\mathcal{H} \subset \mathcal{D}$, hence TA and AT are bounded for all A. The proof that even $T \in \mathcal{J}_1(\mathcal{D})$ was given by Uhlmann /16/ (see also /10/) and uses essentially the positivity of ω. The assumptions of the Proposition say that in (5) there stand $\tau_{\mathcal{D}}$-continuous functionals on both sides, thus (5) can be extended to $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$ and one gets the desired result.

Q.E.D.

Corollary 3 (Theorem 3,/6/)

Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint and $\mathcal{D}$ such that there is an operator $N \in \mathcal{L}^+(\mathcal{D})$ with N^{-1} is nuclear. Then the $\tau_{\mathcal{D}}$-continuous (positive) functionals on $\mathcal{L}^+(\mathcal{D})$ coincide with the normal ones.

Remark 4

Proposition 2 reflects the result valid for the ideal $\mathcal{C}(\mathcal{H})$ of completely continuous operators on $\mathcal{H}$ which can be expressed as

$$\mathcal{C}(\mathcal{H})[\|\ \|]' \cong \mathcal{J}_1(\mathcal{H})[\|\ \|_1]$$

where $\|\cdot\|$ is the operator norm, $\|\ \|_1$ the trace norm, $\mathcal{J}_1(\mathcal{H})$ the ideal of trace class operators. Here " $\cong$ " means algebraic and topological isomorphism /1/.

3. FURTHER PROPERTIES OF $\overline{\mathcal{F}(\mathcal{D})}^{\tau_{\mathcal{D}}}$

For completeness let us recall the following definition.

Definition 5

A topological algebra $\mathcal{B}[\tau]$ is said to be an annihilator algebra if $\mathcal{R}(\mathcal{J}) = \{A \in \mathcal{B} : \mathcal{J}A = 0\} \neq \{0\}$ for any closed proper left ideal $\mathcal{J}$ and $\mathcal{L}(\mathcal{J}) = \{A \in \mathcal{B} : A\mathcal{J} = 0\} \neq \{0\}$ for any closed proper right ideal $\mathcal{J}$ and $\mathcal{R}(\mathcal{B}) = \mathcal{L}(\mathcal{B}) = \{0\}$.

In what follows we need two lemmata.

Lemma 6

If $\mathcal{J} \subset \mathcal{L}^+(\mathcal{D})$ is a $\tau_{\mathcal{D}}$-closed left ideal so $\mathcal{J}^+ = \{A^+ : A \in \mathcal{J}\}$ is a $\tau_{\mathcal{D}}$-closed right ideal.

This is a consequence of the $\tau_{\mathfrak{D}}$-continuity of the involution.

Lemma 7 (/2/, Theorem 3.5, adapted)

Let $\mathcal{A}(\mathfrak{D})$ be an Op*-algebra which contains $\mathcal{F}(\mathfrak{D})$. Then $\mathcal{R}(\mathfrak{J}) = \{0\}$ if and only if $\mathcal{F}(\mathfrak{D}) \subset \mathfrak{J}$ for any left ideal $\mathfrak{J}$ in $\mathcal{A}(\mathfrak{D})$.

Since $\mathcal{F}(\mathfrak{D})^+ = \mathcal{F}(\mathfrak{D})$ these two Lemmas give $\mathcal{L}(\mathfrak{J}) = \{0\}$ if and only if $\mathcal{F}(\mathfrak{D}) \subset \mathfrak{J}$ for any right ideal $\mathfrak{J}$ in $\mathcal{A}(\mathfrak{D})$ ($\mathcal{A}(\mathfrak{D})$ as in the Lemma above). Now it is easy to prove:

Proposition 8

$\overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}} [\tau_{\mathfrak{D}}]$ is an annihilator algebra.

Proof:

Setting $\mathcal{B} = \overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}}$, Lemma 7 immediately gives $\mathcal{R}(\mathcal{B}) = \mathcal{L}(\mathcal{B}) = \{0\}$. Let $\mathfrak{J}$ be a $\tau_{\mathfrak{D}}$-closed proper left ideal, then $\mathcal{F}(\mathfrak{D}) \not\subset \mathfrak{J}$ and consequently the same Lemma shows $\mathcal{R}(\mathfrak{J}) \neq \{0\}$. Analogously for a proper closed right ideal $\mathfrak{J}$: $\mathcal{L}(\mathfrak{J}) \neq \{0\}$.

Q.E.D.

In /15/ Uhlmann among other things proves the following interesting result: Two maximal Op*-algebras $\mathcal{L}^+(\mathfrak{D})$ and $\mathcal{L}^+(\mathfrak{D}')$ are $*$-isomorphic if and only if there is a unitary operator U such that $U\mathfrak{D} = \mathfrak{D}'$.

Now we give a slight generalization ; simultaneously this result corresponds to /1/, Theorem 4.1.8 for the case $\mathcal{L}(\mathcal{H})$. For simplicity let us assume that $\mathfrak{D}$ and $\mathfrak{D}'$ are contained in the same Hilbert space .

Proposition 9

Let π be a $*$-isomorphism between $\mathcal{F}(\mathfrak{D})$ and $\mathcal{F}(\mathfrak{D}')$.

i) π can be extended to an algebraic and topological isomorphism π_1 between $\mathcal{L}^+(\mathfrak{D})[\tau_{\mathfrak{D}}]$ and $\mathcal{L}^+(\mathfrak{D}')[\tau_{\mathfrak{D}'}]$.

ii) π_1 is given by a unitary operator U with $U\mathfrak{D} = \mathfrak{D}'$ such that

$$\pi_1(A) = UAU^{-1} \quad \text{for all } A \in \mathcal{L}^+(\mathfrak{D}).$$

iii) A unitary operator V gives π_1 (resp. π) if and only if $V = \lambda U$ with $|\lambda| = 1$.

Proof:

The proof of Uhlmann's result does not use the $*$-isomophism of $\mathcal{L}^+(\mathfrak{D})$ and $\mathcal{L}^+(\mathfrak{D}')$, rather it is enough that $\mathcal{F}(\mathfrak{D})$ and $\mathcal{F}(\mathfrak{D}')$ are $*$-isomorphic (by the way, this follows from the $*$-isomorphy of the maximal Op*-algebras). Therefore a unitary U exists with $U\mathfrak{D} = \mathfrak{D}'$ and $\pi(A) = UAU^{-1}$ for all $A \in \mathcal{F}(\mathfrak{D})$. Clearly $\pi_1(A) = UAU^{-1}$ for all $A \in \mathcal{L}^+(\mathfrak{D})$ extends this isomorphism in the desired way, hence ii) holds.

Now let $\mathfrak{M}'$ be a t'-bounded subset of $\mathfrak{D}'$. Then $U^{-1}\mathfrak{M}' = \mathfrak{M}$ is a t-bounded subset of $\mathfrak{D}$ since $\sup_{\psi \in \mathfrak{M}} \|A\psi\| = \sup_{\phi \in \mathfrak{M}'} \|AU^{-1}\phi\| = \sup \|UAU^{-1}\phi\| = \sup \|A'\phi\| < \infty$. Further

$$\|\pi_1(A)\|_{\mathfrak{M}'} = \sup_{\phi,\psi \in \mathfrak{M}'} |\langle \phi, \pi_1(A)\psi \rangle| = \sup |\langle U^{-1}\phi , AU^{-1}\psi \rangle| =$$

$$= \sup_{\varphi,\chi \in \mathfrak{M}} |\langle \varphi, A\chi \rangle| = \|A\|_{\mathfrak{M}} \text{ , where } \mathfrak{M} = U^{-1}\mathfrak{M}' .$$

Thus i) holds.

One direction of iii) is trivial. Let $\pi_1(A) = UAU^{-1} = VAV^{-1}$, consequently $V^{-1}UA = AV^{-1}U$, i.e., $WA = AW$ for all $A \in \mathcal{L}^+(\mathfrak{D})$ and the unitary operator W. But any operator commuting with $\mathcal{L}^+(\mathfrak{D})$ must be a multiple of the identity, thus $W = \lambda I$ and the unitarity implies $|\lambda| = 1$.

Q.E.D.

Remark 10

Of course, in the above Proposition the topology $\tau_{\mathfrak{D}}$ is by no means distinguished. The same result is valid for the weak topology $\sigma_{\mathfrak{D}}$, the strong topology $\sigma^{\mathfrak{D}}$, the quasi-uniform topology $\tau^{\mathfrak{D}}$ or other suitable topologies as defined for example in /4/, /5/.

An important property of the ideal of completely continuous operators on Hilbert space reads as follows:" any irreducible representation (distinct from the null-representation) is equivalent to the identical representation". We prove the corresponding result.

Proposition 11

Let $\mathfrak{D}$ be such a domain that any $\tau_{\mathfrak{D}}$-continuous linear positive functional ω on $\overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}} =: \mathcal{B}$ is normal. Then: any weakly continuous irreducible GNS-representation π_ω of the ideal $\overline{\mathcal{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}}$ (distinct from the null-representation) is equivalent to the identical representation.

Proof:

The GNS-representation π_ω is characterized by: the state ω, the $*$-algebra of operators $\pi_\omega(\mathcal{B})$ defined on the linear manifold $\mathcal{D}_\omega$ endowed with the scalar product $<,>_\omega$ and the cyclic vector ϕ_ω which represents the state ω as a vector state by

(7) $\quad \omega(A) = <\phi_\omega, \pi_\omega(A)\phi_\omega>_\omega \;, \|\phi_\omega\|_\omega = 1$.

Weak continuity of π_ω means especially $|<\phi_\omega, \pi_\omega(A)\phi_\omega>_\omega| \leq \|A\|_{\mathcal{M}}$ for a suitable t-bounded set $\mathcal{M} \subset \mathcal{D}[t]$. This together with (7) leads to the $\tau_{\mathcal{D}}$-continuity of ω . By assumption, this implies the normality of ω . Further, irreducibility of π_ω results in that ω is a pure state /7/. This and the normality of ω give that ω is a vector state /11/: $\omega = <\phi, \cdot\,\phi>$, $\|\phi\| = 1$. Hence $\omega(A) = <\phi, A\phi> = <\phi_\omega, \pi_\omega(A)\phi_\omega>_\omega$. This equation gives us the desired result that π_ω and the identical representation are equivalent because ϕ and ϕ_ω are not only cyclic but even generating vectors, i.e. $\mathcal{D} = \{A\phi : A \in \mathcal{B}\}$, $\mathcal{D}_\omega = \{\pi_\omega(A)\phi_\omega : A \in \mathcal{B}\}$, and the representation is determined by the functional ω up to equivalence /1/.

Q.E.D.

Remarks 12

i) By "irreducible" we mean here that the weak commutant is trivial (cf./7/,/14/,/17/).

ii) In the bounded case the assumption "weakly continuous GNS-representation" is unnecessary because $\mathcal{L}(\mathcal{H})$ is a C^*-algebra, so that any $*$-representation is continuous. Moreover any irreducible $*$-representation is a GNS-representation since in the bounded case one has (irreducible) $\longleftrightarrow$ (any $\phi \in \mathcal{D}_\pi$, $\phi \neq 0$ is cyclic) $\longrightarrow$ (π is GNS-representation).

iii) The assumption that any $\tau_{\mathcal{D}}$-continuous state is normal is used in the proof only to get the implication (ω pure) $\longrightarrow$ (ω is a vector state) , since we know only this result /11/.

We conclude the paper with a proposition about centralizers. For genera (topological) algebras the concept of centralizers was rather extensively studied by Johnson /2/.

Definition 13

Let $\mathcal{R}$ be an algebra. We denote by

i) $C_1(\mathcal{R})$ the set of all left-centralizers T on $\mathcal{R}$,i.e. T is a linear map from $\mathcal{R}$ in $\mathcal{R}$ such that $T(x)y = T(xy)$ for all $x,y \in \mathcal{R}$.

ii) $C_r(\mathcal{R})$ the set of all right-centralizers T on $\mathcal{R}$,i.e. T is a linear map from $\mathcal{R}$ in $\mathcal{R}$ such that $xT(y) = T(xy)$ for all $x,y \in \mathcal{R}$

The proposition below generalizes a result of Johnson (/2/,Theorem 18, for $\mathcal{L}(E)$, E - Banach space) to the algebra $\mathcal{L}(\mathcal{D}[t])$. In /12/, Lemma 8 we proved that $\mathcal{F}(\mathcal{D})$ is $\tau^{\mathcal{D}}$-dense in $\mathcal{F}$,the set of all finite dimensional operators of $\mathcal{L}(\mathcal{D}[t])$. Hence $\mathcal{A} := \overline{\mathcal{F}(\mathcal{D})}^{\tau^{\mathcal{D}}} = \overline{\mathcal{F}}^{\tau^{\mathcal{D}}}$ where $\overline{}^{\tau^{\mathcal{D}}}$ means the closure in $\mathcal{L}(\mathcal{D}[t])$. $\mathcal{A}$ is a two-sided ideal in $\mathcal{L}(\mathcal{D}[t])$ (may be no proper one).

Proposition 14

Let $\mathcal{D}[t]$ be an (F)-space, then there exists an algebraic isomorphism π from $\mathcal{L}(\mathcal{D}[t])$ onto $C_1(\mathcal{A})$ given by

(8) $\quad \pi(B) = T_B \;,\; T_B(C) = BC \;,\; B \in \mathcal{L}(\mathcal{D}[t]),\ C \in \mathcal{A}$

Proof:

It is easy to see that (8) defines an isomorphism "in". To show that this is also an isomorphism "onto" we use an idea from /2/. Let $T \in C_1(\mathcal{A})$. We show the existence of an operator $B \in \mathcal{L}(\mathcal{D}[t])$ such that $T = T_B$. Let $\phi_1, \phi_2 \in \mathcal{D}$, $C_1, C_2 \in \mathcal{A}$, $C_1\phi_1 = C_2\phi_2$, then

(9) $\quad T(C_1)\phi_1 = T(C_2)\phi_2$

To see this suppose $\phi_1 \neq 0$, let $\omega \in \mathcal{D}[t]'$ with $\omega(\phi_1) = 1$. By P,Q resp., we denote the following one dimensional operators ($\in \mathcal{A}$): $P\psi = \omega(\psi)\phi_1$, $Q\psi = \omega(\psi)\phi_2$ for all $\psi \in \mathcal{D}$. Then $P\phi_1 = \phi_1$, $Q\phi_1 = \phi_2$, $C_1P = C_2Q$ and consequently $T(C_1)\phi_1 = T(C_1)P\phi_1 = T(C_1P)\phi_1 = T(C_2Q)\phi_1 = T(C_2)\phi_2$, i.e. (9) holds. Therefore

(10) $\quad B: \quad B\psi = T(C)\phi \;,\; C\phi = \psi \;,\; C \in \mathcal{A} \;,\; \phi \in \mathcal{D}, \psi \in \mathcal{D}$

is a correctly defined operator which maps $\mathcal{D}$ into $\mathcal{D}$. Remark that (10) means $BC\phi = T(C)\phi$,i.e. $T = T_B$ (cf.(8)). It remains to show that B maps $\mathcal{D}[t]$ into itself continuously. The assumption on $\mathcal{D}[t]$ allows us to write:

(11) $\quad \mathcal{D} = \bigcap_n \mathcal{D}(\overline{A}_n) \;,\; A_n = A_n^+ \in \mathcal{L}^+(\mathcal{D}),\ \|\phi\| \leq \|A_i\phi\| \leq \|A_j\phi\| \;, i < j$

Let $\{\phi_n\}$ be an arbitrary infinite orthonormal system (with respect to the scalar product $\langle , \rangle$ in $\mathcal{H}$) in $\mathcal{D}$. From the countability of the system of norms $\{\| \ \|_n\}$ defining the topology t one easily derives the existence of a sequence (β_n), $\beta_n > 0$ such that the sequence $(\beta_n \phi_n)$ is t-convergent to zero. Suppose $B \notin \mathcal{L}(\mathcal{D}[t])$, then there is a sequence $(\psi_n) \subset \mathcal{D}$ such that $\psi_n \longrightarrow 0$ (with respect to t), but $(B\psi_n)$ does not converge to zero. If necessary we choose a subsequence $\psi_{a_k} = \chi_k$ fulfilling

$\| A_n \chi_n \| \leq \alpha_n \beta_n$ for all n , $\sum \alpha_n < \infty$, $(B\chi_n)$ is not t-convergent to zero. Consider the following series:

$$C = \sum (1/\beta_n) \langle \phi_n , \cdot \rangle \chi_n .$$

1. For each $\phi \in \mathcal{D}$: $\sum (1/\beta_n) \langle \phi_n , \phi \rangle \chi_n$ is t-convergent, so C defines an operator mapping $\mathcal{D}$ into $\mathcal{D}$ (as $\mathcal{D}[t]$ is complete). This follows from the estimation

$$\| A_j \sum_{n=1}^{N} (1/\beta_n) \langle \phi_n, \phi \rangle \chi_n \| \leq \|\phi\| \sum_{n=1}^{N} (1/\beta_n) \| A_j \chi_n \| \leq$$

$$\leq \sum_{n=1}^{j} (1/\beta_n) \| A_j \chi_n \| + \sum_{n>j} (1/\beta_n) \| A_n \chi_n \| \leq K < \infty .$$

2. The operators $C_N = \sum_{n=1}^{N} (1/\beta_n) \langle \phi_n , \cdot \rangle \chi_n \in \mathcal{A}$ converge to C with respect to the topology $\tau^{\mathcal{D}}$. To see this, let $D \in \mathcal{L}^+(\mathcal{D})$, $\mathcal{M} \subset \mathcal{D}[t]$ bounded, then

$$\| C - C_N \|_D^{\mathcal{M}} \leq \sup_{\phi \in \mathcal{M}} \sum_{n>N} (1/\beta_n) |\langle \phi_n , \phi \rangle| \cdot \| D \chi_n \| \leq \sup_{\phi \in \mathcal{M}} \| \phi \| \cdot$$

$$\cdot \sum_{n>N} (1/\beta_n) \| A_r \chi_n \| \leq L \sum_{n>N} (1/\beta_n) \| A_n \chi_n \| \leq L \sum_{n>N} \alpha_n \ ;$$ the

last term goes to zero as $N \longrightarrow \infty$. (for the estimation the closed graph theorem was used).

3. C is continuous from $\mathcal{D}[t]$ into $\mathcal{D}[t]$. Let $(\varphi_n) \subset \mathcal{D}$, $\varphi_n \longrightarrow 0$ with respect to t, then $C\varphi_n \longrightarrow 0$ since

$$\| A_k C \varphi_j \| \leq \sum_n (1/\beta_n) |\langle \phi_n , \varphi_j \rangle| \, \| A_k \chi_n \| \leq M \sum (1/\beta_n) \| \varphi_j \| \, \| A_n \chi_n \| \leq$$

$\leq \| \varphi_j \| \cdot M \sum \alpha_n$ and this term tends to zero as $j \longrightarrow \infty$.

The properties 1.-3. show that $C \in \mathcal{A}$. Moreover $C(\beta_n \phi_n) = \chi_n$ for all n and from (10) it follows

$$B\chi_n = T(C)(\beta_n \phi_n).$$

But this is a contradiction because $(\beta_n \phi_n)$ is t-convergent to zero, $T(C) \in \mathcal{A}$ hence $T(C)(\beta_n \phi_n)$ is also convergent to zero while $(B\chi_n)$ does not. So we have proved that $B \in \mathcal{L}(\mathcal{D}[t])$. This and (10) complete the proof.

Q.E.D.

<u>Remark 15</u>

Because many (F)-spaces used in analysis are of the form indicated in Proposition 14, this result holds for a sufficiently large class of locally convex spaces. For general (F)-spaces let us remark the following. Let $E[\tau]$ be an (F)-space and τ given by the system of seminorms $\{p_n;\ n=1,2,\dots\}$. In $\mathcal{L}(E[\tau])$ regard the topology τ^E of uniform convergence on the τ -bounded subsets $\mathcal{M} \subset E[\tau]$ (in our situation this is exactly $\tau^{\mathcal{D}}$). By $\mathcal{A}$ denote the τ^E -closure (in $\mathcal{L}(E[\tau])$) of the set of finite dimensional operators in $\mathcal{L}(E[\tau])$. Then Proposition 14 can be proved under the following assumptions: There is an infinite biorthogonal sequence $\{\phi_n, f_n\}$,

i.e. $(\phi_n) \subset E, (f_n) \subset E[\tau]'$, $f_n(\phi_m) = 0 \longleftrightarrow m \neq n$.

Further, (ϕ_n) is τ -bounded, (f_n) is equicontinuous, i.e. $|f_n(\phi)| \leq p_j(\phi)$ for all n, $\phi \in E$, for a suitable j.

We do not suppose (ϕ_n) to be a basis of $E[\tau]$. The only crucial point may be the equicontinuity of (f_n). In the proof one would set

$$C = \sum (1/f_n(\phi_n)) \chi_n \otimes f_n$$

where $(\chi_n \otimes f_n)(\psi) = f_n(\psi) \chi_n$. (χ_n) an appropriate subsequence of (ψ_n).

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