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SIMPLE PROPERTIES OF SOME IDEALS OF COMPACT OPERATORS IN ALGEBRAS OF UNBOUNDED OPERATORS

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Простые свойства некоторых идеалов компактных операторов в алгебрах неограниченных операторов

Определены разные идеалы компактных и вполне непрерывных операторов в алгебрах неограниченных операторов. Обсуждаются простые свойства, как например, плотность конечномерных операторов. Указана связь с проблемой аппроксимации.

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Simple Properties of Some Ideals of Compact Operators in Algebras of Unbounded Operators

In the paper there are defined several ideals of compact and completely continuous operators in algebras of unbounded operators. Simple properties as the density of the finite dimensional operators are discussed. The connection with the approximation problem is indicated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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In recent years non-normable topological $*$-algebras and their representations as $*$-algebras of unbounded operators in Hilbert space have been more and more extensively studied. In effect there are two main lines of investigations. One deals with the systematic development of general aspects as, for example, representation theory of topological algebras, topologization of operator algebras, reduction theory, general structure properties and so on (see, for example, /2/,/12/, /14/,/17/,/18/,/27/,/31/,/32/,/44/). The second line concerns, for example, the study of special classes of such algebras, several questions connected with the structure of algebras of unbounded operators, their domains, their continuous functionals and so on (see /1/-/3/,/5/-/8/,/11/,/13/,/15/,/16/,/19/-/22/,/24/,/33/-/39/,/42/,/43/).

Let us remark that we have not mentioned most of the papers which deal with the physical relevance of these algebras or with the test function algebra.

To speak about ideal theory we note that in the case of bounded operators there is the already classical theory of ideals in Hilbert space of Schatten and von Neumann /30/ and more general of Gohberg and Krein /10/. And there is the well worked out theory of ideals of operators over Banach spaces of Pietsch and coworkers (see /25/ and the references quoted there). Against this, in the non-normable case there are only elements of an ideal theory, for example for LMC*-algebras (see /9/,/23/,/26/,/31/,/33/ and the references there).

In this paper we describe simple properties of some ideals of algebras of unbounded operators. These ideals are composed by compact resp. completely continuous operators with respect to different topologies on the domain where they are regarded. A forthcoming paper /40/ deals with the investigation of a special ideal which reflects many well-known properties of the ideal of completely continuous operators in Hilbert space. In /41/ we will give a systematic treatment of a class of ideals derived from the S_p- (or more general S_Φ -) ideals in Hilbert space. Some of these ideals will be used already in this paper.

1. PRELIMINARIES

For a dense linear manifold $\mathfrak{D}$ in a separable Hilbert space $\mathcal{H}$ by $\mathcal{L}^+(\mathfrak{D})$ /14/ we denote the $*$-algebra of all operators A, bounded or not, for which $A\mathfrak{D} \subset \mathfrak{D}$, $A^*\mathfrak{D} \subset \mathfrak{D}$. The involution is given by $A \longrightarrow A^+ = A^* \restriction \mathfrak{D}$. $\mathcal{L}^+(\mathfrak{D})$ defines a natural topology t on the domain $\mathfrak{D}$ given by the (directed) set of seminorms

$$\mathfrak{D} \ni \phi \longrightarrow \|\phi\|_A = \|A\phi\|, \quad A \in \mathcal{L}^+(\mathfrak{D}).$$

Let us remark that $\mathfrak{D}[t]$ is the projective limit of Hilbert spaces. $\mathcal{L}^+(\mathfrak{D})$ is said to be <u>closed</u> (<u>selfadjoint</u>, resp.) if

$$\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(\bar{A}) \qquad \Big(\mathfrak{D} = \bigcap_{A \in \mathcal{L}^+(\mathfrak{D})} \mathfrak{D}(A^*) \text{ , resp.}\Big)$$

By $\mathfrak{F}(\mathfrak{D})$ we denote the set of all finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$. This is the minimal two-sided $*$-ideal in $\mathcal{L}^+(\mathfrak{D})$. Among the possible locally convex topologies in $\mathcal{L}^+(\mathfrak{D})$ we mention the following two used in the sequel:
the <u>uniform topology</u> $\tau_{\mathfrak{D}}$ given by the seminorms

$$A \longrightarrow \|A\|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} |\langle \phi, A\psi \rangle|$$

the <u>quasi-uniform topology</u> $\tau^{\mathfrak{D}}$ given by the seminorms

$$A \longrightarrow \|A\|_B^{\mathcal{M}} = \sup_{\phi \in \mathcal{M}} \|BA\phi\| \quad , B \in \mathcal{L}^+(\mathfrak{D}) \text{ arbitrary.}$$

In both cases $\mathcal{M}$ runs over all subsets of $\mathfrak{D}$ with $\sup_{\phi \in \mathcal{M}} \|A\phi\| < \infty$ for all $A \in \mathcal{L}^+(\mathfrak{D})$, the <u>t-bounded</u> subsets of $\mathfrak{D}$. In what follows we make use without comment of the following remark. The seminorms $\|.\|_{\mathcal{M}}$ are well-defined (i.e. $< \infty$) also for all bounded operators on $\mathcal{H}$ and for all operators of $\mathcal{L}(\mathfrak{D}[t])$, the continuous operators from $\mathfrak{D}[t]$ into itself.
$\| \ \|_B^{\mathcal{M}}$ is well defined for all operators of $\mathcal{L}(\mathfrak{D}[t])$.

Furthermore, we need the following sets of operators /41/ (and they are used for simplicity only in the context of selfadjoint $\mathcal{L}^+(\mathfrak{D})$)

$$\mathcal{B}(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}) : TA, T^*A \text{ are bounded for all } A \in \mathcal{L}^+(\mathfrak{D})\}$$

$$\mathcal{S}_\infty(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}) : TA, T^*A \text{ are completely continuous } \forall A \in \mathcal{L}^+(\mathfrak{D})\}$$

Clearly, to be more exact, these definitions must be unterstood in the sense that, for example in $\mathcal{S}_\infty(\mathfrak{D})$, TA is bounded on $\mathfrak{D}$ and $\overline{TA}$ is completely continuous on $\mathcal{H}$. Without proof we mention the following properties and equivalent characterizations (cf. /41/)

i) $\mathcal{B}(\mathfrak{D})$ and $\mathcal{S}_\infty(\mathfrak{D})$ are two-sided $*$-ideals in $\mathcal{L}^+(\mathfrak{D})$.

ii) $\mathcal{B}(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}) : AT, AT^* \text{ are bounded } \forall A \in \mathcal{L}^+(\mathfrak{D})\} =$
$= \{T \in \mathcal{B}(\mathcal{H}) : T\mathcal{H} \subset \mathfrak{D}, T^*\mathcal{H} \subset \mathfrak{D}\}$

iii) $\mathcal{S}_\infty(\mathfrak{D}) = \{T \in \mathcal{L}^+(\mathfrak{D}) : AT, AT^* \text{ are completely continuous } \forall A \in \mathcal{L}^+(\mathfrak{D})\} =$
$= \{T \in \mathcal{S}_\infty(\mathcal{H}) : T\mathcal{H} \subset \mathfrak{D}, T^*\mathcal{H} \subset \mathfrak{D}\}$

2. DEFINITIONS AND RESULTS

Since we are interested only in such algebras $\mathcal{L}^+(\mathfrak{D})$ which contain also unbounded operators, the topology t is not a norm-topology. Therefore, we must distinguish between compact and completely continuous operators on $\mathfrak{D}[t]$. This is reflected in the following

<u>Definition 1</u>

An operator $A \in \mathcal{L}^+(\mathfrak{D})$ is contained in

i) <u>Com(t,t)</u> if and only if there is a t-neighbourhood $\mathcal{U}$ of zero such that $A\mathcal{U}$ is t-relatively compact

ii) <u>Com(t,‖ ‖)</u> if and only if there is a t-neighbourhood $\mathcal{U}$ of zero such that $A\mathcal{U}$ is ‖ ‖-relatively compact

iii) <u>Vol(t,t)</u> if and only if $A\mathcal{M}$ is t-relatively compact for all t-bounded sets $\mathcal{M} \subset \mathfrak{D}$

iv) <u>Vol(t,‖ ‖)</u> if and only if $A\mathcal{M}$ is ‖ ‖-relatively compact for all t-bounded sets $\mathcal{M} \subset \mathfrak{D}$.

We summarize some simple properties of these sets of operators.

Lemma 2

i) $\mathcal{F}(\mathcal{D}) \subset \mathrm{Com}(t,t) \subset \mathrm{Com}(t,\|\ \|) \subset \mathrm{Vol}(t,\|\ \|)$
$\mathrm{Com}(t,t) \subset \mathrm{Vol}(t,t) \subset \mathrm{Vol}(t,\|\ \|)$

ii) All sets are algebras, moreover

iii) $\mathrm{Com}(t,t)$ is a two-sided ideal in $\mathcal{L}^+(\mathcal{D})$,
$\mathrm{Com}(t,\|\ \|)$ is a right ideal in $\mathcal{L}^+(\mathcal{D})$.

iv) $\mathrm{Vol}(t,t)$ is a two-sided ideal in $\mathcal{L}^+(\mathcal{D})$,
$\mathrm{Vol}(t,\|\ \|)$ is a right ideal in $\mathcal{L}^+(\mathcal{D})$.

Proof:

i) is trivial. Concerning ii) remark that linearity of these spaces is obvious, the algebra-property follows from iii) and iv).
iii) Let $A \in \mathrm{Com}(t,t)$, $B \in \mathcal{L}^+(\mathcal{D})$. Then B is t-t-continuous and maps any t-relatively compact set in such a one, this proves $BA \in \mathrm{Com}(t,t)$. On the other hand, let $\mathcal{U}$ be a t-neighbourhood of zero such that $A\,\mathcal{U}$ is t-relatively compact. There is a t-neighbourhood of zero, say $\mathcal{V}$, such that $B\mathcal{V} \subset \mathcal{U}$, i.e. $AB\mathcal{V} \subset A\mathcal{U}$ is t-relatively compact, hence $AB \in \mathrm{Com}(t,t)$. The other assertions follow in a similar way. Q.E.D.

The following conclusion gives a simple example for the case $\mathrm{Vol}(t,\|\ \|) = \mathrm{Com}(t,\|\ \|)$.

Conclusion 3

Let $\mathcal{D}$ be such a domain that $\mathcal{L}^+(\mathcal{D})$ contains an operator N with $N^{-1} \in \mathrm{Com}(t,\|\ \|)$. Then $\mathcal{L}^+(\mathcal{D}) = \mathrm{Com}(t,\|\ \|) = \mathrm{Vol}(t,\|\ \|)$.

Proposition 4

i) $\mathrm{Com}(t,t) \subset \mathcal{S}_\infty(\mathcal{D})$

ii) If t is metrizable, then $\mathrm{Com}(t,t) = \mathcal{S}_\infty(\mathcal{D})$ and consequently $\mathrm{Com}(t,t)$ is a $*$-ideal.

Proof:

i) $A \in \mathrm{Com}(t,t)$ implies A^+A and $AA^+ \in \mathrm{Com}(t,t)$ (ideal property). Let $\mathcal{U}$ be a t-neighbourhood of zero given by

$$\mathcal{U} = \{\varphi \in \mathcal{D} :\ \|B\varphi\| \le 1\ ,\ B = B^+ \ge I,\ B \in \mathcal{L}^+(\mathcal{D})\}$$

such that $A^+A\mathcal{U}$ is t-relatively compact. Using, for example /4/, chap.II § 4.1 one sees that A^+A can be extended to a compact map from $\mathcal{D}(\bar{B})$ to $\mathcal{D}[t]$ (use the same symbol for the extension). Especially, A^+A is continuous from $\mathcal{D}(\bar{B})$ to $\mathcal{D}[t]$, i.e.

(1) $\|A^+A\varphi\|_C = \|CA^+A\varphi\| \le K(C)\,\|B\varphi\|$ for all $\varphi \in \mathcal{D}$, $C \in \mathcal{L}^+(\mathcal{D})$

Put $C = B$, then $\|BA^+A\varphi\| \le K\,\|B\varphi\|$ which means that the symmetric operator A^+A is continuous with respect to the norm $\|\ \|_B$ stronger than the Hilbert space norm $\|\ \|$. By Theorem 2.1 of /14/ A^+A is $\|\ \|$-bounded, i.e. A and A^+ are $\|\ \|$-bounded.
Put $C = BDD^+$, $D \in \mathcal{L}^+(\mathcal{D})$, then $\|BDD^+A^+A\varphi\| \le K(C)\,\|B\varphi\|$. Substitute $\varphi = D\psi$, $\psi \in \mathcal{D}$,

(2) $\|BD(D^+A^+AD)\psi\| \le K(C)\,\|BD\psi\|$

Furthermore, $\|\varphi\| \le \|B\varphi\|$, (1) and (2) imply

(3) $\|D^+A^+AD\psi\| \le \|BD^+A^+AD\psi\| \le L\,\|BD\psi\|$

Hence the symmetric operator $(D^+A^+)(AD)$ is continuous with respect to the norm $\|\ \| + \|\ \|_{BD} \ge \|\ \|$. Consequently, AD is $\|\ \|$-bounded for all $D \in \mathcal{L}^+(\mathcal{D})$. Repeating all these considerations for AA^+ instead of A^+A we get the boundedness of A^+D for all $D \in \mathcal{L}^+(\mathcal{D})$. Thus we arrived at $A \in \mathcal{B}(\mathcal{D})$. Putting the results together we have A^+ maps $\mathcal{H}$ continuously in $\mathcal{D}[t]$, A maps $\mathcal{D}[t]$ compactly in $\mathcal{D}[t]$, i.e. AA^+ maps $\mathcal{H}$ compactly in $\mathcal{D}[t]$. Especially, AA^+ is a completely continuous map from $\mathcal{H}[\|\ \|]$ in $\mathcal{H}[\|\ \|]$. From the polar decomposition it is seen that both, A and A^+ are completely continuous maps from $\mathcal{H}[\|\ \|]$ in $\mathcal{H}[\|\ \|]$. This means $A \in \mathcal{S}_\infty(\mathcal{D})$ (see section 1).
ii) It is only to show that for metrizable t the inclusion $\mathcal{S}_\infty(\mathcal{D}) \subset \mathrm{Com}(t,t)$ holds. $A \in \mathcal{S}_\infty(\mathcal{D})$ implies AB and A^+B completely continuous on $\mathcal{H}$ for all $B \in \mathcal{L}^+(\mathcal{D})$. Let t be given by the set of seminorms $\varphi \longrightarrow \|B_n\varphi\|$, $n = 1,2,\ldots$, $B_n \in \mathcal{L}^+(\mathcal{D})$.
We show that $A\,\mathcal{U}$ is t-relatively compact, where $\mathcal{U} = \{\varphi \in \mathcal{D} : \|\varphi\| \le 1\}$. For this purpose it is sufficient to prove that any sequence $(\varphi_n) \subset A\mathcal{U}$ contains a t-convergent subsequence. Put $\varphi_n = A\psi_n$, $\psi_n \in \mathcal{U}$ and remember that B_jA are completely continuous on $\mathcal{H}$ for all $j = 1,2,\ldots$. The well-known diagonal procedure leads to a subsequence (ψ_{n_k}) such that $(B_jA\psi_{n_k})$ is a $\|\ \|$-convergent sequence for all j, i.e. $(B_j\varphi_{n_k})$ is $\|\ \|$-convergent for all j, i.e. (φ_{n_k}) is t-convergent. Hence $A \in \mathrm{Com}(t,t)$.

Q.E.D.

The subsequent propositions deal with the question whether the sets Vol(·,·) are closed in $\mathcal{L}^+(\mathfrak{D})$ with respect to $\tau_\mathfrak{D}$.

Proposition 5

Let t be metrizable, then Vol(t,t) is $\tau_\mathfrak{D}$-closed in $\mathcal{L}^+(\mathfrak{D})$.

Proof:

Let $\{B_n\}$ be a system of operators contained in $\mathcal{L}^+(\mathfrak{D})$ such that $\{\|\ \|_{B_n} = \|\ \|_n,\ n = 1,2,\dots\}$ defines the topology t. Let $A \in \overline{Vol(t,t)}^{\tau_\mathfrak{D}}$, $\mathfrak{M}$ an arbitrary t-bounded set. As in the proof of the last proposition we show: for any sequence $(\psi_n) \subset \mathfrak{M}$ there is a subsequence $(\phi_i = \psi_{n_i})$ such that $(A\phi_i)$ is t-convergent, i.e. $(B_nA\phi_i)$ is $\|\ \|$-convergent for all n. Regard the estimations

$$\|BA(\phi_i-\phi_j)\|^2 \le |\langle B(A-A_\delta)\phi_i, BA(\phi_i-\phi_j)\rangle| +$$
$$+ |\langle BA_\delta(\phi_i-\phi_j), BA(\phi_i-\phi_j)\rangle| + |\langle B(A-A_\delta)\phi_j, BA(\phi_i-\phi_j)\rangle|$$

Since $(\phi_i) \subset \mathfrak{M}$ and for t-bounded $\mathfrak{M}$, $A, B \in \mathcal{L}^+(\mathfrak{D})$ also the set $\mathfrak{N} = B^+BA(\mathfrak{M}-\mathfrak{M}) \cup \mathfrak{M}$ is t-bounded, the estimation can be continued as follows:

$$\|BA(\phi_i-\phi_j)\|^2 \le |\langle (A-A_\delta)\phi_i, B^+BA(\phi_i-\phi_j)\rangle| +$$
$$+ |\langle A_\delta(\phi_i-\phi_j), B^+BA(\phi_i-\phi_j)\rangle| + |\langle (A-A_\delta)\phi_j, B^+BA(\phi_i-\phi_j)\rangle| \le$$
$$= 2\sup_{\phi,\psi\in\mathfrak{N}} |\langle (A-A_\delta)\phi, \psi\rangle| + \|A_\delta(\phi_i-\phi_j)\| \cdot \sup_{\phi\in\mathfrak{N}} \|\psi\|$$

Now we choose $A_\delta \in Vol(t,t)$ so that $\|A-A_\delta\|_\mathfrak{N} < \varepsilon/3$, in dependence of A_δ we choose (ϕ_i) so that $(A_\delta\phi_i)$ is norm-convergent and so the last term in the estimation above can be made $< \varepsilon/3$ for sufficiently large i,j. Thus, $(BA\phi_i)$ is norm-convergent. Now we perform this procedure for the operators $B = B_n$, n = 1,2,... with the following t-bounded sets:

for B_1 choose A_{δ_1}, $(\phi_i^{(1)} = \psi_{n_i}^{(1)}) \subset (\psi_n)$ so that $(B_1A\phi_i^{(1)})$ is norm-convergent, and general,

for B_n choose A_{δ_n}, $(\phi_i^{(n)} = \phi_{n_i}^{(n-1)}) \subset (\phi_n^{(n-1)})$ so that $(B_nA\phi_i^{(n)})$ is norm-convergent. The diagonal sequence $(\phi_i = \phi_i^{(i)})$ is a subsequence of (ψ_n) and $(B_nA\phi_i)$ is norm-convergent for all n.

Q.E.D.

The proof shows that the metrizability of t is only used to prove that $A\mathfrak{M}$ is relatively compact in $\mathfrak{D}[t]$. Therefore

Corollary 6

Vol(t,‖ ‖) is $\tau_\mathfrak{D}$-closed in $\mathcal{L}^+(\mathfrak{D})$ and consequently $\overline{\mathfrak{F}(\mathfrak{D})}^{\tau_\mathfrak{D}} \subset Vol(t,\|\ \|)$.

Now we prove a result about the density of the finite dimensional operators in Com(t,‖ ‖) with respect to $\tau_\mathfrak{D}$.

Proposition 7

The finite dimensional operators $\mathfrak{F}(\mathfrak{D}) \subset \mathcal{L}^+(\mathfrak{D})$ are $\tau_\mathfrak{D}$-dense in Com(t,‖ ‖), hence

$$\overline{\mathfrak{F}(\mathfrak{D})}^{\tau_\mathfrak{D}} = \overline{Com(t,t)}^{\tau_\mathfrak{D}} = \overline{Com(t,\|\ \|)}^{\tau_\mathfrak{D}}.$$

Proof:

Let $A \in Com(t,\|\ \|)$ and $\mathfrak{U}$ a t-neighbourhood of zero so that $A(\mathfrak{U})$ is a ‖ ‖-relatively compact set. There exists an $\mathfrak{V} = \{\phi \in \mathfrak{D}: \|B\phi\| \le 1\} \subset \mathfrak{U}$, $B \in \mathcal{L}^+(\mathfrak{D})$, and without loss of generality we may assume that $\langle\ ,\ \rangle_B : \langle\phi,\psi\rangle_B = \langle B\phi, B\psi\rangle$ is a scalar product. Then $A(\mathfrak{V})$ is a ‖ ‖-relatively compact set and as in Proposition 4i) A extends to a compact operator from the Hilbert space $\mathfrak{D}(\overline{B})[\langle,\rangle_B]$ in the Hilbert space $\mathcal{H}$. For simplicity denote $\langle,\rangle_B$ by $\langle,\rangle'$ and the extension of A also by A. Therefore A has the form $A = \sum_n \lambda_n \langle\phi_n,\cdot\rangle'\psi_n$ with (λ_n) tending to zero, $\{\phi_n\}$ is a $\langle,\rangle'$-orthonormal system; $\{\psi_n\}$, an orthonormal system in $\mathcal{H}$. Now choose $(\phi_n^{(k)}), (\psi_n^{(k)}) \subset \mathfrak{D}$ so that

$$\|\phi_n - \phi_n^{(k)}\|' \longrightarrow 0, \quad \|\psi_n - \psi_n^{(k)}\| \longrightarrow 0 \quad \text{for } k \longrightarrow \infty$$

for all n. Put

$\hat{\mathfrak{F}}(\mathfrak{D}) = \{A^M_{(r_n),(s_n)} = \sum_{n=1}^{M} \lambda_n \langle\phi_n^{(r_n)},\cdot\rangle'\psi_n^{(s_n)}$; M, an arbitrary natural, $(r_n),(s_n)$ arbitrary monotonously increasing sequences of naturals $\}$. $\hat{\mathfrak{F}}(\mathfrak{D}) \subset \mathfrak{F}(\mathfrak{D})$. Let $\mathfrak{M}$ t-bounded, $\varepsilon > 0$ be arbitrary, $\mathfrak{U}_{\mathfrak{M},\varepsilon} = \{C \in \mathcal{L}^+(\mathfrak{D}): \|A - C\|_\mathfrak{M} < \varepsilon\}$

We show that there is a $C \in \hat{\mathfrak{F}}$ with $C \in \mathfrak{U}_{\mathfrak{M},\varepsilon}$.

For $A_k = \sum_{n=1}^{k} \lambda_n \langle \phi_n \cdot \rangle' \psi_n$

$$\| A-A_k \|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} \left| \left\langle \sum_{n=k+1}^{\infty} \lambda_n \langle \phi_n, \phi \rangle' \psi_n, \psi \right\rangle \right| \leq$$

$$\leq \sup \sum |\lambda_n| \, |\langle \phi_n, \phi \rangle'| \, |\langle \psi_n, \psi \rangle| \leq \sup_{n>k} |\lambda_n| \sup_{\phi,\psi \in \mathcal{M}} \left(\sum_n |\langle \phi_n, \phi \rangle'|^2 \right)^{1/2} \cdot$$

$$\cdot \left(\sum_n |\langle \psi_n, \psi \rangle|^2 \right)^{1/2} \leq \sup_{n>k} |\lambda_n| \sup_{\phi \in \mathcal{M}} \| B\phi \| \cdot \sup_{\psi \in \mathcal{M}} \| \psi \| \longrightarrow 0 \quad \text{as } k \to \infty$$

Choose k so large that $\| A - A_k \|_{\mathcal{M}} < \varepsilon/2$. Now we show that there is an $A^k_{(r_n),(s_n)} \in \hat{\mathcal{F}}$ with $\| A_k - A^k_{(r_n),(s_n)} \|_{\mathcal{M}} < \varepsilon/2$. It is

$$\| A-A^k_{(r_n),(s_n)} \|_{\mathcal{M}} = \sup_{\phi,\psi \in \mathcal{M}} \left| \left\langle \sum_{n=1}^{k} \lambda_n \left[\langle \phi_n, \phi \rangle' \psi_n - \langle \phi_n^{(r_n)}, \phi \rangle' \psi_n^{(s_n)} \right], \psi \right\rangle \right| \leq$$

$$\leq \sup_{\phi,\psi \in \mathcal{M}} \sum |\lambda_n| \, | \langle \psi_n, \psi \rangle \langle \phi_n, \phi \rangle' - \langle \psi_n^{(s_n)}, \psi \rangle \langle \phi, \phi_n^{(r_n)} \rangle' |$$

If we put $\lambda = \max \{ |\lambda_n| , 1 \leq n \leq k \}$ and use $\langle \phi, \phi_n \rangle' \langle \psi_n, \psi \rangle -$
$- \langle \psi_n^{(s_n)}, \psi \rangle \langle \phi, \phi_n^{(r_n)} \rangle' = \langle \phi, \phi_n - \phi_n^{(r_n)} \rangle' \langle \psi_n, \psi \rangle - \langle \phi, \phi_n^{(r_n)} \rangle' \langle \psi_n^{(s_n)} - \psi_n, \psi \rangle$
then

$$\| A - A^k_{(r_n),(s_n)} \|_{\mathcal{M}} \leq \lambda \sum \sup_{\phi,\psi \in \mathcal{M}} \{ \|\phi\|' \|\phi_n - \phi_n^{(r_n)}\|' \, \|\psi_n\| \, \|\psi\| +$$

$$+ \|\phi\|' \|\phi_n^{(r_n)}\|' \, \|\psi_n^{(s_n)} - \psi_n\| \cdot \|\psi\| \} \leq \lambda \cdot L \sum \{ \|\psi_n\| \cdot \|\phi_n - \phi_n^{(r_n)}\|' +$$

$$+ \|\phi_n^{(r_n)}\|' \, \|\psi_n^{(s_n)} - \psi_n\| \} \quad \text{with } L = \sup_{\phi,\psi \in \mathcal{M}} \|\phi\|' \|\psi\| < \infty .$$

Take r_n, $1 \leq n \leq k$ so that $\| \phi_n - \phi_n^{(r_n)} \|' < \delta/2k$, then take s_n so that $\| \psi_n - \psi_n^{(s_n)} \| < \delta/(2k \| \phi_n^{(r_n)} \|')$. In doing so,

$\| A_k - A^k_{(r_n),(s_n)} \|_{\mathcal{M}} < \lambda \cdot \delta \cdot L$ and this gives for $\delta = \varepsilon/2L\lambda$

the desired result $A^k_{(r_n),(s_n)} \in \mathcal{U}_{\mathcal{M},\varepsilon}$.

Q.E.D.

In conclusion we discuss the connection with the approximation problem. Let $\mathfrak{D}[t]$ be complete and $\mathcal{F} = \{ F \in \mathcal{L}(\mathfrak{D}[t]) : \dim F < \infty \}$. If $\mathfrak{D} \neq \mathcal{H}$ so in any case $\mathcal{F}(\mathfrak{D}) \subsetneq \mathcal{F}$.

As it was mentioned in section 1 $\mathfrak{D}[t]$ is the projective limit of Hilbert spaces and therefore a locally convex space with approximation property. This means that each operator A which is a compact map from $\mathfrak{D}[t]$ into itself can be approximated (in the topology of uniform convergence on the t-bounded sets) by operators of $\mathcal{F}$. This topology, restricted to $\mathcal{L}^+(\mathfrak{D})$, coincides with $\tau^{\mathfrak{D}}$ (and is stronger than $\tau_{\mathfrak{D}}$). Using the remark after the introduction of these topologies in section 1 we can say that the operators of Com(t,t) can be approximated with respect to $\tau^{\mathfrak{D}}$ (and consequently with respect to $\tau_{\mathfrak{D}}$, too) by operators of $\mathcal{F}$. A priori it does not follow that these operators can be approximated by operators of $\mathcal{F}(\mathfrak{D})$. To get this result we prove

<u>Lemma 8</u>

$\mathcal{F}(\mathfrak{D})$ is $\tau^{\mathfrak{D}}$-dense in $\mathcal{F}$ and consequently $\tau_{\mathfrak{D}}$-dense, too.

<u>Proof:</u>

First of all we show that the one-dimensional operators F have the form

(4) $\qquad F\phi = \omega(\phi)\psi = \langle \bar{C}\chi, \bar{C}\phi \rangle \psi$

with $\psi \in \mathfrak{D}$, ω a t-continuous functional, $\chi \in \mathcal{H}$, $C \in \mathcal{L}^+(\mathfrak{D})$. The first equation in (4) is clear from the t-continuity of F. Moreover this leads to

$$|\omega(\phi)| \leq \|C\phi\| \quad \text{for all } \phi \in \mathfrak{D} \text{ and some } C \in \mathcal{L}^+(\mathfrak{D}).$$

And again we may assume that $\langle C\phi, C\psi \rangle$ is a scalar product, so that ω extends to a continuous linear functional on the Hilbert space $\mathfrak{D}(\bar{C})[\langle,\rangle_{\bar{C}}]$. Therefore ω has the representation

$$\omega(\phi) = \langle \bar{C}\chi, \bar{C}\phi \rangle \quad \text{for all } \phi \in \mathfrak{D}(\bar{C}) \text{ and fixed } \chi \in \mathfrak{D}(\bar{C}).$$

This proves (4). Let $\| \; \|^{\mathcal{M}}_{\mathfrak{D}}$ be one of the seminorms defining the topology $\tau^{\mathfrak{D}}$. As an ansatz we put $F' = \langle \chi_1, \cdot \rangle_C \psi$, $\chi_1 \in \mathfrak{D}$. $F' \in \mathcal{L}^+(\mathfrak{D})$ because $F'\mathfrak{D} \subset \mathfrak{D}$ and $F'^* = \langle \psi, \cdot \rangle C^+C\chi_1$ also maps $\mathfrak{D}$ into $\mathfrak{D}$. Then

$$\| F - F' \|^{\mathcal{M}}_{\mathfrak{D}} = \sup_{\rho \in \mathcal{M}} \| \langle \chi_1, \rho \rangle_C D\psi - \langle \chi, \rho \rangle_{\bar{C}} \cdot D\psi \| \leq$$

$= \|D\psi\| \cdot \sup_{\rho \in \mathcal{M}} | \langle \bar{C}\chi_1 - \bar{C}\chi, \rho \rangle | \leq L \, \| \bar{C}(\chi_1 - \chi) \|$ and as $\mathfrak{D}$ is dense in $\mathfrak{D}(\bar{C})[\langle,\rangle_{\bar{C}}]$ the right-hand side can be made arbitrary small.

Q.E.D.

Corollary 9

$\mathfrak{F}(\mathfrak{D})$ is $\tau^{\mathfrak{D}}$-dense in Com(t,t); $\overline{\mathfrak{F}(\mathfrak{D})}^{\tau^{\mathfrak{D}}} = \overline{Com(t,t)}^{\tau^{\mathfrak{D}}}$

We conclude with the following remarks.

Remarks 10

i) Because the multiplication in $\mathcal{L}^{+}(\mathfrak{D})$ is separately continuous with respect to $\tau_{\mathfrak{D}}$ and $\tau^{\mathfrak{D}}$, the $\tau_{\mathfrak{D}}$-, $\tau^{\mathfrak{D}}$-closure, resp., of any ideal in $\mathcal{L}^{+}(\mathfrak{D})$ is again an ideal (may be no proper one). Hence,

$$\overline{\mathfrak{F}(\mathfrak{D})}^{\tau^{\mathfrak{D}}} = \overline{Com(t,t)}^{\tau^{\mathfrak{D}}} \text{ is a two-sided ideal in } \mathcal{L}^{+}(\mathfrak{D}).$$

ii) As the involution $A \longrightarrow A^{+}$ is $\tau_{\mathfrak{D}}$-continuous the $\tau_{\mathfrak{D}}$-closure of a $*$-ideal in $\mathcal{L}^{+}(\mathfrak{D})$ is again a $*$-ideal. Hence

$\overline{\mathfrak{F}(\mathfrak{D})}^{\tau_{\mathfrak{D}}} = \overline{Com(t,t)}^{\tau_{\mathfrak{D}}} = \overline{Com(t,**)}^{\tau_{\mathfrak{D}}}$ is a twosided $*$-ideal in $\mathcal{L}^{+}(\mathfrak{D})$.

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