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ON THE FRACTAL MEANING OF THE ANOMALOUS DIFFUSION

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The anomalous diffusion, or Levy diffusion [1] is a stochastic process for which the root-mean-square displacement $\langle x^2 \rangle$ of the observable particle during the observation time t and the distribution function $P(x,t)$ of the corresponding process are represented in the form [1, p.58]:

$$\langle x^2 \rangle \approx D t_0 (t / t_0)^{2H}, \quad P(x,t) \approx (2\pi \langle x^2 \rangle)^{1/2} \cdot \exp\left(-\frac{x^2}{2\langle x^2 \rangle} \right). \tag{1}$$

Here D is a diffusion coefficient; t_0, a characteristic time («time scale»); H, the Hurst index. The value $H = 1/2$ corresponds to Fickian («normal») diffusion. All the cases $H \neq 1/2$ refer to the Levy diffusion: $H > 1/2$ — i.e., the «enhanced» diffusion; $H < 1/2$ — the diffusion with the «geometric constraints» [2].

The Fickian diffusion with a constant D value is usually considered as «smooth continuous process», because it is described by the linear differential equation having the well-known analytic solution. There is no analogous differential representation to describe the anomalous diffusion, that's why the Levy diffusion is considered to be the «discontinuous process» connected with the fractal properties of the object under consideration [1, pp.41—76]. In that case either the discrete walks models [1] and their continuous integro-differential representation — «continuous-time random walks» (CTRW) — are used to describe the discontinuous process or the effective diffusion coefficients $D = D(r,t)$ with model dependence from coordinates or time are introduced. Thus, in order to obtained the well-known Richardson dependence — time evolution of relative distance r between particles typical for turbulent diffusion $\langle r^2 \rangle \sim t^3$ — it is possible to introduce $D(r,t) \sim r^{4/3}$ or $D(r,t) \sim t^2$ as well as to consider CTPW-model postulating «hard» relation between r and t in the expression $\psi(r,t)$ for the distribution function of the length jump r in the time interval from t till $t + dt$: $\psi(r,t) \sim r^{-\mu} \delta(r - at^{\nu})$, where a, μ and ν are parameters [2]. It should be also indicated that the experimental analysis of the well-developed turbulence on different space scales testifies [3,4] to its adequate use for modelling the Bachelier–Smoluchowski–Chapman–Kolmogorov (BSCK) general kinetic equation [1, pp.45—47], which can be reduced to nonlinear diffusion equation (more exactly — to the Fokker–Planck equation with the coefficients dependent on the pulsation velocities and scales).

We will show below that the Hurst index can be received on the basis of BSCK equation, which is often called Smoluchowski equation and is used in the analysis of the Markov stochastic processes (see, for example, [5]) if we introduce the fractional transformations of the dynamic variables and time which appear in this equation and reflect the physical meaning of the space and time fractalness of the natural objects and their evolution [6,7]. Let us consider the simplest case of the one-dimensional motion on the whole infinite axis $(-\infty \le x \le \infty)$ and following [1] analyse the BSCK equation for characteristic function $\phi(k,t)$ determined as Fourier transform of the probability density function $P(x,t)$:

$$\phi(k,t)=\int_{-\infty}^{\infty}\exp(ikx)\cdot P(x,t)\,dx;\quad P(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-ikx)\cdot\phi(k,t)\,dk. \tag{2}$$

Note that according to normalization of the function $P(x,t)$ we have: $\phi(0,t)=1$ for all values of t. The corresponding BSCK equation for the function $\phi(k,t)$ is represented in the form:

$$\phi(k,t)=\phi(k,t-\tau)\phi(k,\tau). \tag{3}$$

We obtain the so-called «Levy-approximation» [1, p.46] by confining to the analysis of the central symmetrical solution:

$$\phi_l(k,t)=\exp\left(-Dtx_0^{\mu-2}|k|^{\mu}\right), \tag{4}$$

where μ is a constant; x_0, a characteristic length (the «length scale»); D, the effective diffusion coefficient. The introduced parameters should be changed within the limits $0<\mu\le 2$ and $D\ge 0$, so that the inverse Fourier transform gives the probability density function. Then we obtain the following equation for the Levy-approximation of the density function:

$$P_l(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-ikx)\cdot\phi_l(k,t)\,dk. \tag{5}$$

Dependence $P_l(x,t)$ can be represented in the analytical form only when the values of the parameter μ are *2/2, 1* and *2*. In particular, when $\mu=2$, the case of the Fickian diffusion is:

$$P_l(x,t)=(4\pi Dt)^{-1/2}\exp\left(-\frac{x^2}{4Dt}\right) \tag{6}$$

(we consider $P_L(x,0)=\delta(x)$), and when $\mu=1$ (the Cauchy case), we have:

$$P_L(x,t)=\frac{1}{\pi}\cdot\frac{Dtx_0}{(x/x_0)^2+(Dt)^2}. \tag{7}$$

The important feature of the Levy-distributions: with the exception of the case when $\mu=2$, all the functions $P_L(x,t)$ have no finite moments of n-th power if $n\ge\mu$. In other words for these distributions only the first moment is nonzero if $1<\mu<2$. Such conclusion can be made from the asymptotic dependence (when $x\to\infty$) [1, p.46]:

$$\lim P_{L(x,t)}\to\mu Dt\Gamma(\mu)\sin\frac{\mu\pi}{2}\cdot\frac{x_0^{\mu-2}}{\pi|x|^{\mu+1}}. \tag{8}$$

Our aim is to find the asymptotic (when $t\to\infty$) dependence for root-mean-square displacement $<x^2(t)>$ of the stochastically moving particle during the observation time t when realizing the Levy-distribution for the physically defined dynamic variables determining these motions. As it was mentioned above the latter means that we are interested in the diffusional motion connected with the fractal character of the objects under consideration. Here we measure the displacements $x(t)$ of the particle in the usual configuration space, when the physical environment for this particle is the fractal object. It means that the space and time variables describing the evolution of the particle in the fractal environment have another meaning in comparison with a coordinate and time used by the «observer» to describe the visible displacement of the particle. Let us suppose that the above BSCK equation and the expression for the probability density function $P_L(x,t)$ characterize the displacement of the particle in the physical (fractal) environment being analysed. In connection with the fact that we shall distinguish between the coordinates (X and x) and time (T and t) related to the «displacement along the fractal» and to the «laboratory system». It means that the above expressions for the probability density function should be written as $P_L(x,t)$. Then we may write the equation for the moment of the 2-nd power in the «laboratory system»:

$$\langle x^2\rangle=\int_{-\infty}^{\infty}x^2P(X,T)\,dx=\frac{2}{\pi}\int_{-\infty}^{\infty}x^2dx\int_0^{\infty}dk\cdot\exp(-DTx_0^{\mu-2}k^{\mu})\cos(kX). \tag{9}$$

We suppose that the physical characteristics X and T of the process are connected with the fractional dependences which are measured by values x and t, and it reflects the internal «fractal geometry» with its own «fractal time» of the physical system under consideration:

$$X=x_0\left(\frac{|x|}{x_0}\right)^b,\; T=t_0\left(\frac{|t|}{t_0}\right)^a, \tag{10}$$

where a and b are parameters $(a>0; 3>b>2/(\mu+1))$. The substitution of (10) into (9) allows one to find the asymptotic dependence:

$$\langle x^2\rangle\approx\frac{x_0^2(Dt_0/x_0^2)^{(3-b)/\mu b}}{\mu}\cdot\left(\frac{t}{t_0}\right)^{a(3-b)/\mu b}. \tag{11}$$

Thus, we receive for the Hurst index:

$$2H = \frac{a(3-b)}{\mu b}. \tag{12}$$

From the expression (12), when $a = b = 1$ (the coincidence of the «physical» and the «laboratory» scales), it follows the equation (4.48) of the work [1] with $H > 1/2$ when $\mu < 2$. We should also mention that the Fickian case $2H = 1$ isn't always connected with the smooth Fickian diffusion corresponding to the parameters: $\mu = 2$, $a = b = 1$. Probably the observation of the fractal structure of diffusional fronts in a number of experiments indicates the realization of this particular possibility.

The obtained expressions (11), (12) actually reveal the «microscopic» meaning of the calculated moments determined by the dynamic parameters: the parameter μ characterizing the probabilities of the process transition and a and b parameters which reflect the fractalness of the trajectory (see below). Thus, we reveal the physical meaning of a number of phenomenological parameters of stationary nonlinear dissipational processes introduced in [6,7] to describe difference moments («structural functions») $\Phi_p(\tau)$ of the p-th power:

$$\Phi_p(\tau) = \langle |x(t) - x(t+\tau)|^p \rangle$$

when the value of the «time shift» τ is relatively small. We should note that the postulate adopted in [6,7] about the information significance only of the irregularities of the analysed evolution characterized by the self-similarity at different space and time scales turns out to be equivalent to the notion of evolution as the time fractal. Now we will show that the analysis of the scale transformations (10) leads to the same conclusion.

First of all, the introduction of the two characteristic times — the universal «laboratory» (external) time t and the specific «individual» (internal) time T of the system — allows one to realize the Prigogine idea [9] going back to Aristoteles about the two types of time. According to Prigogine (see also [10]) the time of the first type, or «time-parameter», has the dimension equal to unity. This is the usual «geometrized» time in the Minkovsky space-time continuum in the interval $(-\infty, +\infty)$. The time of the second type, or «time-age», has the beginning and the end, and in accordance with the above statements it may have the dimensions differing from unity. The most important thing here is that, in principle, each system may have its «time-age» with different dimensions. Thus, the possibility appears to build in the future the universal theory of evolution in which the particular classes of the evolutionary systems will differ from each other, mainly through the dimensions of their «time-age».

The scale transformation of time is usually connected with «motion along the fractal» when a number of states in the space in which the fractal object is «inserted» turns out to be excluded. Such an approach is important for the description of the evolution, because it allows one to introduce [11] the systems with «partial memory» and ability to forget in addition to the systems with the «full memory» and «without memory» at all.

From mathematical point of view the account of the above-mentioned restrictions (with the «counting» of the allowed states) in model using the Kantor set results in the so-called «fractional integral» and «fractional derivation» [11, 12]. In particular, a part of such allowed states $\alpha (0 < \alpha \le 1)$ coinciding with fractal dimensions the Kantor set is introduced to analyse this problem in the simplest case of one-dimensional diffusion motion $(-\infty \le x \le \infty)$. The corresponding equation using the fractional derivatives for the probability density function is written in the form:

$$\frac{\partial^\alpha P_\alpha(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 P_\alpha(x,t)}{\partial x^2}, \tag{13}$$

where D_α is the diffusion effective coefficient on the fractal. The procedure of fractional differentiation is determined by the introduction of the Laplace transform for the function $P_\alpha(x,t)$ with generalization of the appropriate equation for the operational image $F(x,p)$ of this function when $\alpha \ne 1$:

$$F_{xx} - \frac{p^\alpha}{D_\alpha t_0^{1-\alpha}} F + \frac{1}{D_\alpha} P_\alpha(x,0) = 0; \; F(x,p) = \int_0^\infty \exp(-pt) P_\alpha(x,t) dt, \tag{14}$$

(the case $\alpha = 1$ corresponds to the «ordinary» time differentiation). The solution of Eq. (14) with the initial condition $P_\alpha(x,0) = \delta(x)$ is of the form:

$$F(x,p) = \frac{t_0^{(1-\alpha)/2}}{(4D_\alpha p^\alpha)^{1/2}} \exp\left[-\frac{p^{\alpha/2}|x|}{D_\alpha^{1/2} t_0^{(1-\alpha)/2}} \right] \tag{15}$$

and we receive for the moments $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \frac{2}{\Gamma(\alpha + \alpha n/2)} (D_\alpha t_0) \left(\frac{t}{t_0} \right)^{2\alpha - 1}. \tag{16}$$

We find the following by comparing (16) with (11) when $b = 1$ (in the absence of scale transformation of coordinates) and $\mu = 2$ (when the exponents of a power of the products Dt_0 and $D_\alpha t_0$ are equal):

$$\alpha = \alpha + \frac{2(\alpha - 1)}{n}. \tag{17}$$

The equation (17) shows the interrelation of the index of the time scale transform with a part $\alpha (0 < \alpha \le 1)$ states allowed for the motion (belonging to the fractal) which has been introduced into (10). When the laboratory time of the system and the individual one $(\alpha = 1)$ coincide, $\alpha = 1$ follows from (17). It means that all the states of the system are allowed.

In this case we should note that unlike the present study, in [6,7] time dynamics of the non-linear dissipative system is represented as «real» (but not «mathematical») time fractal, when only points of the real «markers-irregularities» — splashes, jumps, breaks of a derivative of a dynamic variable being analysed — are significant (from the point of view of the reception of the information about the system) on the time axis. The latter condition gives to the real fractals considered in [6,7] a kind of «polychromism» (according to the types of irregularities). Moreover, it is so, because the information of «various colours» — about splashes and jumps — is received from the analyses of different dependences, power spectrum and difference moments accordingly.

Particular physical meaning can also be given to the index b [see (10)] of the scale transformation of coordinates. According to [13] such type of fractional transformation of coordinates can be achieved in the analysis of «inter-particle» interaction in the evolution of complex nonlinear system. Here we deal with the general enough fundamental result going back to A. Einstein. It has been demonstrated in [13] that the idea of «canonical transformations» of dynamical variables which further goes to the description of «non-interacting (or weakly interacting) quasi-particles» in the space of fractional coordinates turns out to be effective both in physics and in the theory of biological evolution. In this case the components of the appropriate metric tensor, which can be revealed on the basis of the comparison of the model theoretical expressions with the experimental data, should be considered as the phenomenological parameters of the process being studied.

It is this particular meaning that we put in the introduced parameter b of scale transformation of coordinates. We think that the analysis of experimental dependences of the anomaluous diffusion in various complex systems with the determination of both root-mean-square displacement of the moving particles and the excess indexes and other cumulants allows one to find a and b parameters of the «fractional dynamics» introduced by us in the studied physical systems and thus determine the extent of adequacy of the proposed parametrization of the dynamics being investigated.

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