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EXPANSIONS OF FERMI AND SYMMETRIZED FERMI INTEGRALS AND APPLICATIONS IN NUCLEAR PHYSICS*

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1 Introduction

The Fermi function (F-function):

$$f_F(r) = \frac{1}{1+e^{(r-c)/a}} \tag{1.1}$$

is widely used in nuclear physics. It has been extensively used [1]-[4], originally by the Stanford group, to represent the charge density $\rho_F(r) = \rho_0 f_F(r)$ of nuclei for a wide range of mass numbers. Then, beginning with [5] it was often used in the so-called high-energy approximation in calculating the charge form factors of nuclei. Furthermore, the "form factor" of the conventional Woods-Saxon potential [6], which is a fair first approximation to the self-consistent single-particle potential, is an F-function. Among other applications of the F-function, we mention its use in connection with the strong absorption models [7]-[11].

Another function which is closely related to $f_F(r)$ and which we also study in this paper is the symmetrized Fermi function (SF-function) (see,e.g., [12],[13]):

$$f_{SF}(r) = \frac{1}{1+e^{(r-c)/a}} + \frac{1}{1+e^{-(r+c)/a}} - 1. \tag{1.2}$$

The function $f_{SF}(r)$ has the property $f_{SF}(-r) = f_{SF}(r)$ and may also be written in the following forms:

$$f_{SF}(r) = \frac{1}{1+e^{(r-c)/a}} - \frac{1}{1+e^{(r+c)/a}}, \tag{1.3}$$

$$f_{SF}(r) = \frac{\sinh(c/a)}{\cosh(r/a)+\cosh(c/a)}, \tag{1.4}$$

$$f_{SF}(r) = \frac{1}{2}[\tanh(\frac{c+r}{2a}) + \tanh(\frac{c-r}{2a})]. \tag{1.5}$$

It is evident since $f_{SF}(r)$ is an even function that it can be expanded in even powers of r and has a zero slope at the origin $f'_{SF}(0) = 0$. Furthermore, it has certain analytic advantages. For light nuclei with $c/a > 1$, it resembles a Gaussian function while for heavier nuclei it goes over to the Fermi distribution. Thus, it might be said that it is quite appropriate to be considered as a "universal" nuclear density. In practice, however, at least for medium and heavy nuclei, it leads to results very similar to those of the usual Fermi distribution. We may also recall that the so called "cosh" [14] and the SF-potentials [15] are appropriate to represent cluster model potentials [14]. We finally note that very recently D.Sprung and J.Matorell [16] studied as well the symmetrized Fermi function and its transforms and also emphasized in their independent study pertinent analytic advantages.

In a recent publication [17] the "expansion of the Fermi distribution" was derived in terms of derivatives of the δ-function in an alternative way to the traditional one:

$$\frac{1}{1+e^{(r-c)/a}} = \theta(c-r) - \sum_{k=0}^{\infty} \delta^{(2k+1)}(r-c)a^{2k+2}A_{2k+1} \tag{1.6}$$

with the coefficients $A_n = A_{2k+1}$ expressed through the Bernoulli numbers. In the above expansion both sides should be understood under the integral sign, with a well-behaved function $q(r)$. These integrals were discussed in [2] and called "the Fermi type integrals". In those cases when eq.(1.6) has meaning, the corresponding integrals are corrected by the exponentially small terms of the order $\exp(-c/a)$. They have been omitted in [17] as well as in other studies (e.g., [18], [19]), where only the first terms of (1.6) have been derived. In the following Sects. the exact formulae and estimations for omitted terms will be given and some examples where their contribution can be important will be considered.

The purpose of the present paper is to extend these results in three directions. Firstly, in Sec.2 we extend the approach of [17] to the case of the SF-function, and we pay attention to the conditions of validity for expansions similar to (1.6). Secondly, in Sec.3, we allow for more general integration limits, namely from $R_i < c$ to $R_f > c$, including in the expansion the exponential terms in a convenient form. The same procedure is applied to the SF-function, and the results for both distributions are obtained in a unified way. Thirdly, in Sec.4 an alternative treatment is carried out on the basis of Fourier transforms and the properties of the hypergeometric functions. The results are obtained in a general form for the F- and SF-integrals with arbitrary limits, and in particular cases the expressions for the correction terms are given in "closed form" (i.e.,in terms of known functions). In the final section, specific cases are considered and numerical calculations are performed.

2 On an expansion of the symmetrized Fermi function

In this section we derive a general expansion of an integral containing the SF-function. Using for $f_{SF}(r)$ the form of (1.3) we write:

$$I_{SF} = \int_0^\infty f_{SF}(r)q(r)dr = I_F - \mathcal{J}^{(+)}, \tag{2.1}$$

where the "standard Fermi integral" considered previously in [17] is

$$I_F = \int_0^\infty \frac{q(r)}{1+e^{(r-c)/a}}dr. \tag{2.2}$$

As to the second term in (2.1), we introduce the designation $\mathcal{J}^{(\pm)}$, useful for calculations, with the replacement $r = az - c$:

$$\mathcal{J}^{(\pm)} = \int_0^\infty \frac{q(\pm r)}{1+e^{(r+c)/a}}dr = a\int_{c/a}^\infty \frac{q(\pm(az-c))}{1+e^z}dz. \tag{2.3}$$

In the following we shall simplify the method of [17] to make it more transparent and suitable for further considerations. To this aim let us transfom (2.2) by changing the variable $r = az + c$ to obtain:

$$I_F = a\int_0^\infty \frac{q(c+az)}{1+e^z}dz + a\int_0^{c/a}\frac{q(c-az)}{1+e^{-z}}dz. \tag{2.4}$$

Substituting into the second integral the $(1+\exp(-z))^{-1}$ by means of the identity $(1+\exp(-z))^{-1} = 1-(1+\exp z)^{-1}$ and then using the relation

$$\int_0^{c/a}\frac{q(c-az)}{1+e^z}dz = \int_0^\infty \frac{q(c-az)}{1+e^z}dz - \int_{c/a}^\infty \frac{q(c-az)}{1+e^z}dz, \tag{2.5}$$

one can write:

$$I_F = I_s + I_{as} + \mathcal{J}^{(-)}, \tag{2.6}$$

$$I_{SF} = I_s + I_{as} + \mathcal{J}, \tag{2.7}$$

where

$$I_s = a\int_0^{c/a} q(c-az)dz = \int_0^\infty \Theta(c-r)q(r)dr, \tag{2.8}$$

$$I_{as} = a\int_0^\infty \frac{q(c+az)-q(c-az)}{1+e^z}dz, \tag{2.9}$$

$$\mathcal{J} = \mathcal{J}^{(-)} - \mathcal{J}^{(+)}, \tag{2.10}$$

and $\Theta(x)$ is the unit step function:

$$\Theta(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 1. \end{cases}$$

The representation for the F- and SF-integrals (2.6) and (2.7) is rather instructive. Indeed, the first term I_s contains the very simple sharp cutoff function in an integrand. The second term I_{as} includes an "antisymmetric" function $g(z) = q(c+az) - q(c-az)$. The property $g(z) = -g(-z)$ enables one to simplify considerably its evaluation. Finally, the integrals $\mathcal{J}^{(\pm)}$ and $\mathcal{J}$ are usually exponentially small since merely the integration from a large number ($z = c/a \gg 1$) to ∞, where only the tail of the integrand function $(1+e^z)^{-1} \simeq e^{-z} \ll 1$ contributes to them, is involved.

Now, when calculating the I_{as}-integral we assume that $q(c\pm az)$ can be expanded in the series

$$q(c\pm az) = q(c) + \sum_{n=1}^\infty (\pm 1)^n a^n \frac{q^{(n)}(c)}{n!}z^n. \tag{2.11}$$

Inserting (2.11) into (2.9) and then changing the order of integration and summation (which is assumed to be valid) we get:

$$I_{as} = a\sum_{n=1}^\infty D_n a^n q^{(n)}(c), \tag{2.12}$$

where the decomposition coefficients D_n are related for the odd n-values to the Bernoulli numbers (see, e.g., [20], p.53 and [21]):

$$D_n = \frac{1-(-1)^n}{n!}\int_0^\infty \frac{z^n}{1+e^z}dz = \begin{cases} 0 & \text{for} \quad \text{even } n \\ \frac{2}{n!}\frac{\pi^{n+1}}{n+1}(2^n-1)|B_{n+1}| & \text{for} \quad \text{odd } n. \end{cases} \tag{2.13}$$

Thus, for example, one can obtain, the first coefficients:

$$D_1 = \frac{\pi^2}{6}, \quad D_3 = \frac{7}{4}\frac{\pi^4}{90}, \quad D_5 = \frac{31}{16}\frac{\pi^6}{945}. \tag{2.14}$$

Further, accepting the relation

$$q^{(n)}(c) = (-1)^n \int_0^\infty \delta^{(n)}(r-c)q(r)dr, \qquad (n = 1,2,3....), \tag{2.15}$$

as valid for some class of functions $q(r)$ (see, e.g., [22]) one can write the final result for I_{as}:

$$I_{as} = -a \sum_{n=1,odd}^{\infty} a^n D_n \int_0^{\infty} \delta^{(n)}(r-c) q(r) dr. \tag{2.16}$$

Thus, we obtain the integrals I_F and I_{SF} expanded in powers of the diffuseness parameter a:

$$I_{SF(F)} = \int_0^{\infty} f_{SF(F)}(r) q(r) dr = \int_0^{\infty} \Theta(c-r) q(r) dr - \sum_{n=1,odd}^{\infty} D_n a^n \int_0^{\infty} \delta^{(n)}(r-c) q(r) dr + \mathcal{J}\left(\mathcal{J}^{(-)}\right). \tag{2.17}$$

To this approximation when one can ignore the last terms in (2.17), the expansions for the SF- and F-functions coincide with each other, and therefore one can write:

$$f_{SF}(r) = \frac{\sinh(c/a)}{\cosh(r/a) + \cosh(c/a)} = \theta(c-r) - \sum_{n=1,odd}^{\infty} a^{n+1} D_n \delta^{(n)}(r-c). \tag{2.18}$$

The explicit form as a series with terms proportional to the odd derivatives of the δ-function may be useful for practical calculations. However, in all the cases one needs to keep in mind the conditions of its validity, viz., (i) existence of the expansion (2.11), (ii) possibility of the transition from (2.11) to (2.12), (iii) determination of the class of functions, on which the generalized δ-function and its derivatives act. As to the disregardness of integrals $\mathcal{J}\left(\mathcal{J}^{(-)}\right)$ their calculation is a separate task. For sufficiently smooth functions $q(r)$ they are thought to be of the order $e^{-(c/a)}$. Indeed, when evaluating the integrals $\mathcal{J}^{(\pm)}$ it is often convenient to use the following presentation:

$$\int_0^{\infty} e^{\phi(r)} dr = \frac{e^{\phi(r)}}{\phi'(r)}\Big|_0^{\infty} + \frac{\phi''}{[\phi']^3} e^{\phi(r)}\Big|_0^{\infty} + ..., \tag{2.19}$$

which can be obtained through integration by parts. Here we have the integrals

$$\mathcal{J}^{(\pm)} \simeq e^{-\frac{c}{a}} \int_0^{\infty} q(\pm r)\, e^{-\frac{r}{a}} dr \tag{2.20}$$

with $\phi(r) = \ln q(\pm r) - \frac{r}{a}$. It follows from (2.19) that

$$\mathcal{J}^{(\pm)} \simeq a \frac{q(0)}{1 - a\frac{q'(0)}{q(0)}} \left\{ 1 + a^2 \frac{\phi''(0)}{\left[1 - a\frac{q'(0)}{q(0)}\right]^2} + ... \right\}, \tag{2.21}$$

if the function $q(\pm r)\exp(-r/a)$ tends to zero as $r \to +\infty$. In particular, one can see that for a frequently oscillating $q(r)$ with $a|q'(0)/q(0)| \gg 1$ the additional small factor $\mp q(0)/q'(0)$ appears in the estimation (2.21). Moreover, for even functions $q(r)$, the "correction" term $\mathcal{J}$ becomes zero. In general, this is not the case for each $\mathcal{J}^{(\pm)}$ taken separately. In order to make the essential points more transparent let us consider as an example the form factors:

$$F_{SF(F)}(p) = \int_0^{\infty} f_{SF(F)}(r) \sin(pr)\, r dr = -\frac{d}{dp} I_{SF(F)}(p), \tag{2.22}$$

where

$$I_{SF(F)}(p) = \int_0^{\infty} f_{SF(F)}(r) \cos(pr)\, dr. \tag{2.23}$$

First, it is easily seen from (2.8) that $I_s = \sin pc/p$. Then, in calculating I_{as} by means of (2.12) we use $d^n \cos pr/dr^n = (-1)^{(n+1)/2} p^n \sin pr$ for $n = odd$ and the relation from [20] (p.66):

$$H(z) = \frac{1}{z}\left(\frac{z}{\sin z} - 1\right) = \frac{1}{\pi} \sum_{n=1,odd} D_n \left(\frac{z}{\pi}\right)^n, \qquad |z| < \pi. \tag{2.24}$$

Thus, we obtain:

$$I_{as} = ai \sin pc \sum_{n=1,odd} D_n \left(\frac{i\pi pa}{\pi}\right)^n = \pi a \frac{\sin pc}{\sinh \pi pa} - \frac{\sin pc}{p}, \qquad pa < 1. \tag{2.25}$$

Bearing in mind that for the even $\cos pr$-function $\mathcal{J} = 0$, one gets:

$$F_{SF}(p) = -\frac{d}{dp} \frac{\pi a \sin pc}{\sinh \pi pa}. \tag{2.26}$$

Then, applying eq.(2.21) to calculate the integral of interest with $q(r) = \exp(ipr)$ one can show that

$$F_F(p) = -\frac{d}{dp}\left[\pi a \frac{\sin pc}{\sinh \pi pa} + \frac{a}{1 + a^2p^2} e^{-(c/a)}\right]. \tag{2.27}$$

One should stress an important point, namely that the results (2.26) and (2.27) have been obtained for the SF- and F-integrals with the oscillating function $\cos pr$ under the condition $pa < 1$ which ensures the convergence of the series in (2.25). It means that the method used may be applied if the "wave length" p^{-1} is greater than the thickness a of a "surface layer" of the SF- and F-functions. Moreover, the quantity I_{as} is a small correction to the "sharp - edge" contribution I_s under the stronger condition $pa \ll 1$. In fact, we have

$$I_{as} \simeq -\frac{\pi^2}{6} p^2 a^2 \frac{\sin pc}{p} = -\frac{\pi^2}{6} p^2 a^2 I_s \tag{2.28}$$

retaining only the term with $n = 1$ in the series (2.25). In other words, as one should expect, the diffuseness effects which are accumulated in the terms with the derivatives of the decomposition (2.18) are not considerable if the "wave length" p^{-1} is much greater than a. On the other hand, if one evaluates the integral (2.9) by using the result from [21] (p. 505) we obtain

$$I_{as} = a \int_0^{\infty} \frac{\cos[p(c+az)] - \cos[p(c-az)]}{1 + e^z} dz = \pi a \frac{\sin pc}{\sinh \pi pa} - \frac{\sin pc}{p} \tag{2.29}$$

for any values of the effective parameter pa. The r.h.s. of (2.29) may be expanded in the series appearing in (2.25) only under the condition $pa < 1$. This analysis shows that the method based on the expansion (2.18) becomes impractical when we deal with frequently oscillating functions. Rather it is applicable for evaluations of the Fermi-type integrals with slowly varying functions (for instance, of the polynomial type).

Also, it is seen from (2.27) that the "correction" terms of the order $\exp(-c/a)$ may be comparable and in some cases larger than the oscillating contribution to the form factor. In these cases of rapidly varying functions $q(r)$ one needs to develop methods which calculate these contributions in a satisfactory way. In Sec.4 a method will be described in which the results are expressed through the hypergeometric functions and the corresponding series are, in fact, the decompositions in the small parameter $\exp(-c/a)$.

3 A general method for the calculation of the Fermi type integrals

3.1 Expansion of the "generalized" Fermi type integral using a Taylor series

Here we extend our consideration by introducing the integration limits $R_i < c$ and $R_f > c$, so that the "standard Fermi integral" is a special case of the integral we calculate (namely, for $R_i \to 0$ and $R_f \to \infty$). Such a generalization is not only of mathematical interest but it is also relevant (pertaining to the upper limit) to a problem of physical interest (see Sec.5). Henceforth in this Section we proceed in the same way as in certain treatments made for more specialized cases [23]. Namely, let us split the second integral in a form suitable for the the use of the well known formula for the geometrical progression. Respectively, one can write

$$I_F(R_i,R_f) = \int_{R_i}^{R_f} \frac{q(r)}{1+e^{(r-c)/a}}dr = \int_{R_i}^{c} \frac{q(r)}{1+e^{(r-c)/a}}dr + \int_{c}^{R_f} \frac{q(r)e^{-(r-c)/a}}{1+e^{-(r-c)/a}}dr =$$

$$\sum_{m=0}^{\infty}(-1)^m \left[\int_{R_i}^{c} q(r)e^{m(r-c)/a}dr + \int_{c}^{R_f} q(r)e^{-(m+1)(r-c)/a}dr\right]. \tag{3.1}$$

Further, separating out the first term of the first sum in eq.(3.1) and shifting the dummy index in the second sum (by setting $m+1=m' \to m$) we find

$$I_F(R_i,R_f) = \int_{R_i}^{c} q(r)dr + \sum_{m=1}^{\infty}(-1)^m \left[\int_{R_i}^{c} q(r)e^{m(r-c)/a}dr - \int_{c}^{R_f} q(r)e^{-m(r-c)/a}dr\right]. \tag{3.2}$$

We now assume that the function $q(r)$ can be expanded in a Taylor series around $r=c$

$$q(r) = \sum_{n=0}^{\infty} q^{(n)}(c)\frac{(r-c)^n}{n!}. \tag{3.3}$$

Substituting (3.3) into (3.2) and making the replacement $az = r-c$ we get:

$$I_F(R_i,R_f) = \int_{R_i}^{c} q(r)dr +$$

$$\sum_{n=0}^{\infty}\frac{q^{(n)}(c)}{n!}a^{n+1}\sum_{m=1}^{\infty}(-1)^m \left[(-1)^n \int_0^{(c-R_i)/a} z^n e^{-mz}dz - \int_0^{(R_f-c)/a} z^n e^{-mz}dz\right], \tag{3.4}$$

or

$$I_F(R_i,R_f) = \int_{R_i}^{c} q(r)dr + \sum_{n=0}^{\infty}\frac{q^{(n)}(c)}{n!}a^{n+1}\Big\{n!D_n +$$

$$\sum_{m=1}^{\infty}\frac{(-1)^m}{m^{n+1}}\left[(-1)^{n+1}\Gamma(n+1,m\frac{c-R_i}{a}) + \Gamma(n+1,m\frac{R_f-c}{a})\right]\Big\}, \tag{3.5}$$

where $\Gamma(\alpha,y)$ is the incomplete Γ-function defined by ([24], p.138):

$$\Gamma(\alpha,y) = \int_y^{\infty} e^{-t}t^{\alpha-1}dt. \tag{3.6}$$

When deriving eq.(3.5) we have used the relation:

$$[(-1)^n - 1]\sum_{m=1}^{\infty}(-1)^m \int_0^{\infty} e^{-mt}t^n dt = [1-(-1)^n]\int_0^{\infty}\frac{t^n}{1+e^t}dt = n!D_n, \tag{3.7}$$

where D_n is determined by (2.13). Then, using the decomposition

$$\Gamma(1+n,x) = n!e^{-x}\sum_{l=0}^{n}\frac{x^l}{l!}, \tag{3.8}$$

eq.(3.5) can be written as

$$I_F(R_i,R_f) = \int_{R_i}^{c} q(r)dr + \sum_{n=0}^{\infty} q^{(n)}(c)a^{n+1}\Big\{D_n +$$

$$\sum_{l=0}^{n}\frac{1}{l!}\left[(-1)^{n+1}\left(\frac{c-R_i}{a}\right)^l F(-e^{\frac{R_i-c}{a}},n+1-l) - \left(\frac{R_f-c}{a}\right)^l F(-e^{\frac{c-R_f}{a}},n+1-l)\right]\Big\}. \tag{3.9}$$

Here according to ([20], p.45) the function $F(z,s)$ is determined by

$$F(z,l) = \sum_{m=1}^{\infty}\frac{z^m}{m^l} = z\Phi(z,l,1), \tag{3.10}$$

where $\Phi(z,l,1)$ has the following integral representation:

$$\Phi(z,l,1) = \frac{1}{\Gamma(l)}\int_0^{\infty}\frac{t^{l-1}e^{-t}}{1-ze^{-t}}dt, \tag{3.11}$$

which is valid if either $|z| \le 1,\ z \ne 1$ and $Re\ l > 0$ or $z=1$ and $Re\ l > 1$ (see eq. (3) in [20], p.43). Here $\Gamma(l)$ is the ordinary Γ-function. Note a compact form:

$$I_F(R_i,R_f) = \int_{R_i}^{c} q(r)dr + \sum_{n=0}^{\infty} q^{(n)}(c)a^{n+1}\left\{D_n + (-1)^n D_n\left(\frac{c-R_i}{a}\right) + D_n\left(\frac{R_f-c}{a}\right)\right\}, \tag{3.12}$$

where

$$D_n(\beta) = \frac{2}{n!}\int_{\beta}^{\infty}\frac{t^n}{e^t+1}dt \qquad (\beta \ge 0). \tag{3.13}$$

Eq.(3.9) follows from (3.12) if one uses the geometric progression expansion in powers e^{-t} for the denominator $[e^t+1] = e^{-t}[1+e^{-t}]^{-1}$ in the integrand of (3.13).

3.2 Integrals with the SF-function

It is convenient to use form (1.3) of the SF function. Thus we have only to calculate the integral which corresponds to the second term in (1.3). In this case no separation of the interval of integration is needed and we obtain after some algebra

$$\mathcal{J}^{(+)}(R_i, R_f) \equiv \int_{R_i}^{R_f} \frac{q(r)}{1+e^{(r+c)/a}} dr = \sum_{n=0}^{\infty} q^{(n)}(c) \frac{a^{n+1}}{n!} \cdot$$

$$\left\{ \sum_{s=0}^{n} \frac{1}{s!} \left[\left(\frac{R_f+c}{a}\right)^s F\left(-e^{-(R_f+c)/a}, n+1-s\right) - \left(\frac{R_i+c}{a}\right)^s F\left(-e^{-(R_i+c)/a}, n+1-s\right) \right] \right\}, \tag{3.14}$$

The above result can be combined with the corresponding one of the previous section and therefore we obtain immediately the expansion of the integral with the SF distribution. However, it is more expedient to write the results obtained in a unified way, that is to write in a simple formula the expansion of both the F and SF-function, by introducing ϵ, which is equal to 1 in the case of the SF function and 0 in the case of the usual F one. Thus, we write:

$$I(R_i, R_f, \epsilon) = \int_{R_i}^{R_f} q(r) f(r) dr = I_F(R_i, R_f) - \epsilon \mathcal{J}^{(+)}(R_i, R_f), \tag{3.15}$$

where

$$f(r) = \frac{1}{1+e^{(r-c)/a}} - \epsilon \frac{1}{1+e^{(r+c)/a}}, \tag{3.16}$$

and the final expression for the integral is written in both cases:

$$I(R_i, R_f, \epsilon) = \int_{R_i}^{c} q(r) dr + \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{1}{n! l!} a^{n+1} q^{(n)}(c) \Bigg\{ n! D_n \delta_{l,0} +$$

$$\left(\frac{R_f-c}{a}\right)^l F\left(-e^{-\frac{R_f-c}{a}}, n+1-l\right) - (-1)^n \left(\frac{c-R_i}{a}\right)^l F\left(-e^{-\frac{c-R_i}{a}}, n+1-l\right) +$$

$$\epsilon \left[\left(\frac{R_f+c}{a}\right)^l F\left(-e^{-\frac{R_f+c}{a}}, n+1-l\right) - \left(\frac{R_i+c}{a}\right)^l F\left(-e^{-\frac{R_i+c}{a}}, n+1-l\right) \right] \Bigg\}. \tag{3.17}$$

In the special case in which $R_i \to 0$ and $R_f \to \infty$ the above formula is simplified, as follows:

$$I(0, \infty, \epsilon) = \int_0^{\infty} q(r) f(r) dr = \int_0^{c} q(r) dr + \sum_{n=0}^{\infty} q^{(n)}(c) a^{n+1} \left[D_n + B_n(c/a) \right], \tag{3.18}$$

where D_n are given by (2.13) and $B_n(c/a, \epsilon)$ is defined as follows:

$$B_n(c/a, \epsilon) = \sum_{l=0}^{n} \left[\epsilon(-1)^l - (-1)^n \right] \frac{1}{l!} \left(\frac{c}{a}\right)^l F(-e^{-c/a}, n+1-l). \tag{3.19}$$

The preceding results have been obtained by expanding $q(r)$ around the point $r = c$.

In certain cases, alternative expansions may be more appropriate. For example, if we expand $q(r)$ around the point $r = 0$

$$q(r) = \sum_{n=0}^{\infty} q^{(n)}(0) \frac{r^n}{n!}. \tag{3.20}$$

we obtain the following final result:

$$I(R_i, R_f, \epsilon) = \int_{R_i}^{R_f} \frac{q(r) dr}{1+e^{(r-c)/a}} - \epsilon \int_{R_i}^{R_f} \frac{q(r) dr}{1+e^{(r+c)/a}} = \int_{R_i}^{c} q(r) dr +$$

$$\sum_{n=0}^{\infty} q^{(n)}(0) a^{n+1} \Bigg\{ \sum_{l=0}^{n} \frac{1}{(n-l)!} \Bigg[D_l \left(\frac{c}{a}\right)^{n-l} + \cdot$$

$$\left(\frac{R_i}{c}\right)^{n-s} \left[\epsilon F\left(-e^{-\frac{c+R_i}{a}}, l+1\right) - (-1)^l F\left(-e^{-\frac{c-R_i}{a}}, l+1\right) \right] -$$

$$\left(\frac{R_f}{c}\right)^{n-l} \left[\epsilon F\left(-e^{-\frac{c+R_f}{a}}, l+1\right) - F\left(-e^{-\frac{R_f-c}{a}}, l+1\right) \right] \Bigg] \Bigg\}. \tag{3.21}$$

This is again simplified in the case, when $R_i \to 0$ and $R_f \to \infty$. We obtain, by changing the free index from n to m:

$$I(0, \infty, \epsilon) = \int_0^{\infty} q(r) f(r) dr = \int_0^{c} q(r) dr + \sum_{m=0}^{\infty} q^{(m)}(0) a^{m+1} \cdot$$

$$\left\{ \sum_{l=0}^{m} \frac{1}{(m-l)!} D_l \left(\frac{c}{a}\right)^{m-l} + \left[\epsilon - (-1)^m\right] F\left(-e^{-c/a}, m+1\right) \right\}. \tag{3.22}$$

In the special case in which $q(r) = r^n$, all the terms in the sum over m are zero, because of the derivatives of $q(r)$, except the one with $m = n$, since in this case $q^{(n)}(0) = n!$. Therefore, we find:

$$I_n(0, \infty) = \int_0^{\infty} r^n f(r) dr = \frac{c^{n+1}}{n+1} \Bigg\{ 1 + (n+1)! \left(\frac{a}{c}\right)^{n+1} \cdot$$

$$\left[\sum_{l=0}^{n} \frac{1}{(n-l)!} D_l \left(\frac{c}{a}\right)^{n-l} + \left[\epsilon - (-1)^n\right] F\left(-e^{-c/a}, m+1\right) \right] \Bigg\}. \tag{3.23}$$

The following remarks can be made regarding this expression:

Firstly, in the case of the Fermi distribution ($\epsilon = 0$) it reduces to the result which follows from the general expression of the "Fermi integral" $F_n(k)$, $k = (c/a)$ quoted by Elton (see Appendix of ref.[2]) since $\int_0^{\infty} r^n f_F(r) dr = a^{n+1} F_n(k)$. We note that generalizations to non-integral values of n etc in moment calculations have been discussed in literature ([25], [26], [27]). Secondly, in the case of the Symmetrized Fermi distribution ($\epsilon = 1$), there are no exponential terms when n is even. Thus, the use of the symmetrized Fermi distribution has the advantage that all its even moments are free of exponential terms, which simplifies their treatment. It is seen that this result is in agreement with that found in Sec. 2, since when n is even $q(r) = r^n$ is symmetric while when n is odd $q(r)$ is antisymmetric.

4 Treatment on the basis of Fourier transforms and the properties of the hypergeometric functions

4.1 The hypergeometric series for the typical Fermi integrals

The previous results have been based on the assumption that the function $q(r)$ may be expanded in a power series at a vicinity of the radius $r = c$. In this section we shall relax this assumption and consider the exponential Fourier transform [3]:

$$q(r) = \mathcal{F}\{\bar{q}(p); r\} = (1/2\pi)\int_{-\infty}^{\infty} \bar{q}(p)e^{irp}dp. \qquad (4.1)$$

In calculating the Fermi type integrals with such functions $q(r)$, for which the fourier transform exists one can use the following representation for the Gauss hypergeometric function $F(a,b;c;z)$ ([21], p.319):

$$\int_0^{\infty}(1-e^{-x})^{\nu-1}(1-\beta e^{-x})^{-\rho}e^{-\mu x}dx = B(\mu,\nu)F(\rho,\mu;\nu+\mu;\beta), \qquad (4.2)$$

where

$$Re\mu > 0, \quad Re\nu > 0, \quad |arg(1-\beta)| < \pi,$$

and $B(x,y)$ is the beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Let us set $q(r) = e^{ipr}$ and calculate the integral $\int_0^{\infty} q(r)f_F(r)dr$. Obviously, this is the case when in the more general expression (4.2) one sets $\mu = 1 - ipa, \nu = 1, \rho = 1$ and $\beta = -e^{c/a}$. Therefore, one can obtain [28], [29]:

$$A_F(p) = \int_0^{\infty}\frac{e^{ipr}}{1+e^{(r-c)/a}}dr = aB(1-ipa,1)e^{c/a}F(1,1-ipa;2-ipa;-e^{c/a}). \qquad (4.3)$$

Furthermore, because for the applications in question $\exp(c/a) > 1$ (or even $e^{c/a} \gg 1$) it is pertinent to transform (4.3) into

$$A_F(p) = \frac{\pi a}{i\sinh\pi pa}e^{ipc} + ip^{-1}.F(1,ipa;1+ipa;-e^{-(c/a)}) \qquad (4.4)$$

When deriving eq.(4.4) we have used one of the Kummer relations ([20], p.116, eq.(2)) for the hypergeometric series

$$F(1,b;b+1;-z) = B_1 z^{-1}F(1,1-b;2-b;-z^{-1}) + B_2 z^{-b}, \quad (|\arg z| < \pi), \qquad (4.5)$$

where

$$B_1 = \frac{\Gamma(b+1)\Gamma(b-1)}{\Gamma^2(b)}, \qquad B_2 = \Gamma(b+1)\Gamma(1-b),$$

and the formula

$$\Gamma(b)\Gamma(1-b) = \frac{\pi}{\sin\pi b}. \qquad (4.6)$$

[3] What follows is easily extended to the sine- and cosine Fourier transforms and the Laplace one.

Thus, the Fourier transform of the Fermi distribution has been expressed in terms of functions of well-known properties. One should emphasize that the exact result (4.4) reflects explicitly the interplay between the physical parameters involved, viz., the radius c, the diffuseness parameter a and the "incident frequency" p. In many applications the latter plays the role of momentum transfer.

Formula (4.4) enables one to separate all at once the oscillating part of the form factor $A_F(p)$ (the first term in the r.h.s. of (4.4)) from a comparatively smooth p-dependence which is determined by its second term. Note that the separation has been achieved without those constraints inherent to the previous approaches (see Sect.2 and 3). We see that the corresponding oscillations at $pc > 1$ (the "edge" effect) have an exponential falloff generated by the factor $[\sinh\pi pa]^{-1} \sim \exp(-\pi pa)$ at $pa \geq 1$ (the "surface diffuseness " effect).

Further, by using the definition

$$F(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ... \qquad (4.7)$$

of the Gauss series, the smooth contribution to $A_F(p)$ can be splitted into the pole term p^{-1} and an expansion in descending powers of an "effective" parameter $\exp(-\frac{c}{a}) < 1$. The former is cancelled at $p = 0$ with the same term which stems from $-i\pi a[\sinh\pi pa]^{-1}\exp(ipc)$, while the latter may not be disregarded even for the values of $c/a \gg 1$. In fact, at high frequences with $\pi ap \sim \frac{c}{a}$ all these exponentially small contributions get comparable to one another and the formula gives a systematic way to calculate each of them.

Now, we apply this result to evaluate the integral considered in Sec.3:

$$I_F(R_i,R_f) = \int_{R_i}^{R_f}\frac{q(r)}{1+e^{(r-c)/a}}dr = I_F(R_i,\infty) - I_F(R_f,\infty) \qquad (4.8)$$

with finite lower R_i and upper R_f limits which satisfy the condition $R_i < c < R_f$. Here

$$I_F(R,\infty) = \int_R^{\infty}\frac{q(r)}{1+e^{(r-c)/a}}dr = \frac{1}{2\pi}\int_{-\infty}^{\infty}dp\bar{q}(p)A_F(p,R) \qquad (4.9)$$

with the function $q(r)$ being replaced by its exponential Fourier transform. Again the problem reduces to the following:

$$A_F(p,R) = \int_R^{\infty}\frac{e^{ipr}}{1+e^{(r-c)/a}}dr = e^{ipR}\int_0^{\infty}\frac{e^{ipr}}{1+e^{(r-c+R)/a}}dr. \qquad (4.10)$$

By using (4.3) one gets

$$A_F(p,R) = aB(1-ipa,1)e^{ipR}e^{(c-R)/a}F(1,1-ipa;2-ipa;-e^{(c-R)/a}). \qquad (4.11)$$

Two cases should be considered, namely: $R < c$ and $R > c$.

Case i) $R < c$:

In this case it is convenient to convert the hypergeometric function of (4.11) into the corresponding hypergeometric series (cf., the transition from (4.3) to (4.4)). Thus we have

$$A_F(p,R) = e^{ipR}\left\{\frac{\pi a}{\sin(\pi i p a)}e^{ip(c-R)} + \frac{i}{p}F(1, ipa; 1+ipa; -e^{-(c-R)/a})\right\} =$$

$$e^{ipR}\left\{\frac{\pi a}{\sin(\pi i p a)}e^{ip(c-R)} - \frac{1}{ip} + \frac{a}{1+ipa}e^{-(c-R)/a} + O\left(e^{-2(c-R)/a}\right)\right\}, \tag{4.12}$$

or omitting the terms of higher order in $e^{-(c-R)/a}$ we obtain

$$A_F(p,R) = \pi a H(\pi i p a)e^{ipc} + \int_R^c e^{ipr}dr + e^{ipR}\frac{a}{1+ipa}e^{-(c-R)/a}, \tag{4.13}$$

where the function $H(z) = \sin^{-1} z - z^{-1}$ is the function considered in Sec.2. Substituting (4.13) into (4.8) and preserving the exponential Fourier transform in r-space we arrive at the expression

$$I_F(R,\infty) = \pi a\mathcal{F}\{\bar{q}(p)H(\pi i p a); c\} + \int_R^c q(r)dr + a\mathcal{F}\{\frac{\bar{q}(p)}{1+ipa}; R\}\ e^{-(c-R)/a}. \tag{4.14}$$

Case ii) $R > c$:

In this case eq.(4.11) includes the hypergeometric series directly from the beginning and therefore

$$A_F(p,R) = \frac{a}{1-ipa}e^{ipR}e^{-(R-c)/a}F(1, 1-ipa; 2-ipa; -e^{-(R-c)/a}). \tag{4.15}$$

If the parameters involved meet the inequality $e^{-(R-c)/a} \ll 1$ we find

$$A_F(p,R) = \frac{a}{1-ipa}e^{ipa}e^{-(R-c)/a} \tag{4.16}$$

and finally making the same substitutions as in the case i) we get for $I_F(R,\infty)$:

$$I_F(R,\infty) = a\mathcal{F}\{\frac{\bar{q}(p)}{1-ipa}; R\}\ e^{-(R-c)/a}. \tag{4.17}$$

Combining eq.(4.14) and eq.(4.17) we get

$$I_F(R_i,R_f) = \pi a\mathcal{F}\{\bar{q}(p); c\} + \int_{R_i}^c q(r)dr +$$

$$a\mathcal{F}\{\frac{\bar{q}(p)}{1+ipa}; R_i\}\ e^{-(c-R_i)/a} - a\mathcal{F}\{\frac{\bar{q}(p)}{1-ipa}; R_f\}\ e^{-(R_f-c)/a}. \tag{4.18}$$

As an illustration of this method we evaluate the generalized n-th moment for the F-distribution:

$$< r^n > \Big|_{R_i}^{R_f} = \int_{R_i}^{R_f}\frac{r^n}{1+e^{(r-c)/a}}dr. \tag{4.19}$$

To this point note that

$$< r^n > \Big|_{R_i}^{R_f} = (-i)^n\left[A_F^{(n)}(0, R_i) - A_F^{(n)}(0, R_f)\right]. \tag{4.20}$$

where $A_F^{(n)}(0,R)$ denotes the n-order derivative of the integral (4.10) at the point $p=0$. Finally the following result is obtained:

$$< r^n > \Big|_{R_i}^{R_f} = \frac{c^{n+1} - R_i^{n+1}}{n+1} + a\sum_{l=0}^{n}\frac{n!}{l!}a^{n-l}\left\{c^l D_{n-l} + (-1)^l R_i^l e^{-(c-R_i)/a} - R_f^l e^{-(R_f-c)/a}\right\}. \tag{4.21}$$

with $(n-l)$ odd.

We point out that the formulae hold if one neglects the exponentially small contributions to the series (4.12) and (4.15). If only one of the limits $R_{i,f}$ is close to c then one needs to employ the general expressions for these series.

In the case when $R_i \to 0$ and $R_f \to \infty$ eq.(4.21) yields the ordinary n-th moment:

$$< r^n >_F = an!\sum_{l=0}^{n}\frac{c^l}{l!}a^{n-l}D_{n-l} + \frac{c^{n+1}}{n+1} + n!a^{n+1}e^{-c/a}. \tag{4.22}$$

It follows from (4.22) that:

$$< r^2 >_F = c^3/3 + 2a^2cD_1 + 2a^3e^{-c/a}. \tag{4.23}$$

$$< r^4 >_F = c^5/5 + 4a^2c^3\left[D_1 + 2\left(\frac{a}{c}\right)^2 D_3\right] + 24a^5e^{-c/a}. \tag{4.24}$$

4.2 A "closed form" expression for the generalized symmetrized Fermi integral

According to the decomposition (1.3) the integral of interest

$$I_{SF}(R_i,R_f) = \int_{R_i}^{R_f} q(r)f_{SF}(r)dr \tag{4.25}$$

can be written as

$$I_{SF}(R_i,R_f) = I_F(R_i,R_f) - \mathcal{J}^{(+)}(R_i,R_f), \tag{4.26}$$

where

$$\mathcal{J}^{(+)}(R_i,R_f) = \int_{R_i}^{R_f}\frac{q(r)}{1+e^{(r+c)/a}}dr. \tag{4.27}$$

Similarly as in subsection 4.1 the evaluation of the generalized form factors (4.25) can be reduced to the integrals $A_F(p,R)$ and

$$\mathcal{A}^{(+)}(p,R) = \int_R^{\infty}\frac{e^{ipr}}{1+e^{(r+c)/a}}dr. \tag{4.28}$$

Using eq.(4.2) we find:

$$\mathcal{A}^{(+)}(p,R) = -\frac{ae^{ipR}}{1-ipa}e^{-(c+R)/a}F(1, 1-ipa; 2-ipa; -e^{-(c+R)/a}) \tag{4.29}$$

or if the higher order terms (i.e., $O(\epsilon^{-2(c+R)/a})$) are omitted then we have

$$\mathcal{A}^{(+)}(p,R) = -\frac{a}{1-ipa}e^{-(c+R)/a}. \tag{4.30}$$

Combining eq.(4.12) with eq.(4.29) one gets:

$$A_{SF}(p;R) = A_F(p,R) - \mathcal{A}^{(+)}(p,R) =$$

$$e^{ipR}\left\{\frac{\pi a}{i\sinh \pi pa}e^{ipc} + \frac{i}{p} + \frac{a}{1+ipa}e^{-(c-R)/a}F(1,1+ipa;2+ipa;-e^{-(c-R)/a})\right.$$

$$\left. - \frac{a}{1-ipa}e^{-(c+R)/a}F(1,1-ipa;2-ipa;-e^{-(c+R)/a})\right\}. \tag{4.31}$$

In deriving this formula we have used the relation:

$$F(1,b;b+1;z) = 1 + \frac{b}{b+1}zF(1,b+1;b+2;z). \tag{4.32}$$

Putting in (4.31) $R=0$ we find for the "standard" form factor the following expression:

$$I_{SF}(p) = ReA_{SF}(p;0) = \pi a\frac{\sin pc}{\sinh \pi pa}. \tag{4.33}$$

The approach described in this section is an alternative way to evaluate the integrals in question.The following comments can be made: (a) It is relied on the well known results of the theory of special functions and can be presented in a mathematically compact form. (b) We have managed to bypass the too strong assumption (2.12). (c) The corrections of any order in $\exp(-c/a)$ may be evaluated in a systematic manner.

5 Applications and discussion

In this section we consider certain specific cases and we also give the results of numerical calculations related to nuclear physics problems.

First, let us estimate the effect of the exponentially small contributions to the relation between the parameters c and a, which follows from the normalization of the nucleon density $\rho(r) = \rho_0 f(r)$, where $f(r)$ is given by (3.16) for nucleus of A nucleons [2], [31]:

$$4\pi\rho_0\int_0^\infty f(r)r^2dr = A. \tag{5.1}$$

Using formula (3.23) we obtain

$$c^3 + (\pi a)^2 c + 6(\epsilon-1)a^3F\left(-e^{-c/a},3\right) = r_0^3A \tag{5.2}$$

or neglecting the corrections of the $\exp(-2c/a)$-order and higher

$$c^3 + (\pi a)^2 + 6(1-\epsilon)a^3e^{-c/a} = r_0^3A, \tag{5.3}$$

where $r_0^3 = 3/(4\pi\rho_0)$. Eq.(5.3) with $\epsilon=0$ follows also from (4.23). It is clear again that for the SF function there are no exponential terms. In such a case, or if they are negligible

in the case of F function the above third order equation, which is of the same form as in the case of the trapezoidal distribution [32] can be solved for c, which is then expressed in terms of a and r_0 (that is ρ_0):

$$c = \left(\frac{1}{2}\right)^{1/3} r_0A^{1/3}\left[(1+b)^{1/3} + (1-b)^{1/3}\right], \tag{5.4}$$

where

$$b = \left[1 + \frac{4}{27}\left(\frac{\pi a}{r_0A^{1/3}}\right)^6\right]^{1/2}.$$

In the case of the Fermi distribution ($\epsilon=0$) an improved expression for c may be obtained, if the exponential terms (which are assumed to be small) are not completely neglected but are estimated using an approximate expession for c: $c = c_{ap}$, such as $c = r_0A^{1/3}$ or expression (5.4). Then the improved expression for c is given again by (5.4), but instead of r_0 the quantity

$$r_0' = r_0\left[1 - (6/A)(\epsilon-1)a^3F\left(-e^{-c_{ap}/a},3\right)\right]^{1/3} \tag{5.5}$$

appears. It is easily shown that the normalized Fermi distribution corrected by the small terms of the $\exp(-c/a)$-order looks like

$$\rho_F(r) = \frac{\rho_0^c}{1+\exp\dfrac{r-c}{a}}, \qquad \rho_0^c = \rho_0^F\left[1+\delta\right], \tag{5.6}$$

where

$$\rho_0^F = \frac{3A}{4\pi c^3}\left[1+\frac{\pi^2a^2}{c^2}\right]^{-1}, \qquad \delta = -6\frac{a^3}{c^3}\,e^{-c/a}. \tag{5.7}$$

We also note that the central density $\rho(0)$ of the nucleus may be expressed in terms of the half-density radius c and of the diffuseness parameter a, as follows, by using (3.16):

$$\rho(0) = \frac{3}{4\pi r_0^3}\left[\frac{1}{1+e^{-c/a}} - \frac{\epsilon}{1+e^{c/a}}\right], \tag{5.8}$$

where

$$r_0^3 = \frac{c^3}{A}\left[1 + (\pi a/c)^2 + 6(\epsilon-1)(a/c)^3F\left(-e^{-c/a},3\right)\right]. \tag{5.9}$$

Finally, the m.s. radius of the nuclear density

$$< r^2 >_F = \frac{\int r^2\rho(r)d\vec{r}}{\int \rho(r)d\vec{r}} = \frac{\int_0^\infty r^4f(r)dr}{\int_0^\infty r^2f(r)dr} \tag{5.10}$$

is expressed in terms of c and a as follows:

$$< r^2 >_F = \frac{c^2}{5}\frac{3 + 10(\pi a/c)^2 + 7(\pi a/c)^4 + 3\cdot 5!(\epsilon-1)(a/c)^5F\left(-e^{-c/a},5\right)}{1 + (\pi a/c)^2 + 3!(\epsilon-1)(a/c)^3F\left(-e^{-c/a},3\right)}. \tag{5.11}$$

It is observed that in the limit $a \to 0$ the above expression reduces to the well-known expression of the m.s. radius of the uniform distribution $< r^2 >_u = (3/5)c^2$. Furthermore, in case of the symmetrized Fermi distribution ($\epsilon = 1$) we obtain the following exact expression (see, e.g., [33]):

$$< r^2 >_{SF} = \frac{3}{5} c^2 \left[1 + \frac{7}{3} \left(\frac{\pi a}{c} \right)^2 \right]. \qquad (5.12)$$

For the Fermi distribution such an expression holds approximately, as long as the exponential terms are small, which is the case even for light nuclei. A carefull calculation for ^{40}Ca, ^{12}C (with the parameters c and a from [2]) and for ^{6}Li, ^{4}He (with parameters c and a from [34]) has shown that the corresponding corrections to the r.m.s. radii of the nuclear densities are: $^{40}Ca: 3.3\,10^{-5}$; $^{12}C: 8.1\,10^{-5}$; $^{4}Li: 6.9\,10^{-3}$ and $^{4}He: 1.2\,10^{-3}$, i.e., do not exceed 0.05 %.

Finally we consider, for the above nuclei the influence of the exponential terms in the values of r_0^3. These values are given in Table 1.

Table 1

Nucleus	r_0^3 With exp. terms	r_0^3 Without exp. terms
^{40}Ca	1.496	1.496
^{12}C	1.379	1.379
^{6}Li	1.312	1.285
^{4}He	0.919	0.915

It is seen that the effect of the exponential terms in the value of r_0 depends on the nucleus, but is still very small, although somewhat larger in comparison with that in the r.m.s. radii.

We consider now the generalized Fermi-type integral $\int_0^{R_f} f(r) r^2 dr$. The physical interest in integrals of this or other similar forms, such as $\int_0^{R_f} f(r) r^4 dr$, originates from the equation which determines the value R_0: R_M [35] of an harmonic oscillator (HO) potential

$$V_{HO}(r) = -D + D\frac{r^2}{{R_M}^2}, \qquad (5.13)$$

which approximates a given Woods-Saxon (or symmetrized Wood-Saxon) nucleon-nucleus or Λ-nucleus potential; $V_{WS}(r) = -Df(r)$ (that is with $f(r)$ given by (3.16) in a sort of "best approximation in the mean" (in the nuclear interior and to some extent in the region of nuclear surface)):

$$\left[\frac{2}{15} + \frac{1}{4} f^2(R_M) \right] R_M^3 = \int_0^{R_M} f(r) r^2 dr. \qquad (5.14)$$

More precisely, the value $R_0 = R_M$, determined by the above equation minimizes the integral [35]:

$$J_0(R_0) = \int_0^{R_0} |V_{WS}(r) - V_{HO}(r)|^2 dr, \qquad (5.15)$$

provided that

$$f(R_M) < \left[\frac{2}{5} + \frac{3}{2} f^2(R_M) \right] + \frac{R_M}{4} \frac{df^2(R_0)}{dR_0} \Big|_{R_0 = R_M}. \qquad (5.16)$$

The above procedure may be used in determining the variation with the mass number of the core nucleus $A_c = A$ of the harmonic oscillator energy level spacing for a nucleon: $h\omega_N$ or for a Λ-particle: $h\omega_\Lambda$, since the spring constant is given by $k = \mu\omega^2 = (2D/R_0^2)$ and therefore

$$h\omega = \left[\frac{h^2}{\mu} 2D \right]^{1/2} \frac{1}{R_M}. \qquad (5.17)$$

where μ is the reduced mass of the nucleon (or Λ-particle)-core system. Such a treatment has also been considered recently for atomic clusters [36], [37].

In order to find the value of R_M which is needed, one has to solve eq.(5.14) and therefore to calculate the integral $\int_0^{R_f} f(r) r^2 dr$ for various values of R_f and choose that one for which eq. (5.14) is satisfied. This can be done either by means of a subroutine for the computation of integrals or by means of the relevant formula of section 3. The latter procedure is in a way preferable since it can lead to an approximate analytic solution of eq.(5.14) and therefore to a formula for the variation of $h\omega$ with the mass number, in terms of the particle mass and the parameters of the Woods-Saxon (or symmetrized Woods-Saxon) potential. In such a procedure it is of interest to know the magnitute of the exponential terms, in order to be sure that their omission or approximate evaluation is justified. This is expected to be the case from the results of ref [35]. We further elaborate on this point here. According to eq.(4.21) the integral in question is equal to:

$$I_2(0, R_f) = \int_0^{R_f} r^2 f(r) dr = (c^3/3) \left[1 + (\pi a/c)^2 + C \right]. \qquad (5.18)$$

$$C = 6\, \frac{a^3}{c^3} \left\{ e^{-c/a} - e^{-(R_f - c)/a} \left[1 + \frac{R_f}{a} + \frac{R_f^2}{2a^2} \right] \right\}. \qquad (5.19)$$

Substitution of (5.18) with $R_f = R_M$ into (5.14) leads to the following equation for the determination of R_M

$$\left[(2/5) + (3/4) f^2(R_M) \right] (R_M/c)^3 = 1 + (\pi a/c)^2 + C. \qquad (5.20)$$

We consider as an example the hypernucleus $^{13}_{\Lambda}C$ and we use a Woods-Saxon Λ-nucleus potential with parameters [35] $D = 28.3\ MeV$, $r_0 = 1.205\ fm$ and $a = 0.35\ fm$ which have been determined by fitting to ground-state energies of the Λ-particle in hypernuclei using as half-depth radius c the expression (5.4) (see ref.[35] for more details) We note that for $^{13}_{\Lambda}C$ the value of c is 2.613 fm.

In Table 2 the values of the integral $I_2(0, R_f)$ are given for various values of $R_f > c$, along with the contribution of the non exponential and exponential terms, as well as the percentage contribution of the latter. It is seen that as R_f decreases the exponential terms become more important. Fortunately for $R_f = R_M$ their contribution is small. The magnitude of these terms depends also on the hypernucleus considered, being larger for the lighter nuclei, and also on the potential parameters.

Table 2

Values of $R_f > c$	Values of $I_2(0, R_f)$	Values of non-exponential terms	Contribution of exponential terms	Percentage contribution
5.0	6.988	6.999	-0.011	0.16
4.0	6.873	6.999	-0.126	1.83
R_M=3.693	6.741	6.999	-0.258	3.80
3.0	5.846	6.999	-1.153	19.71
2.7	5.027	6.999	-1.972	39.22
2.613	4.740	6.999	-2.248	47.44

If the potential parameters of ref. [38] are used, that is $D = 28.0\ MeV$, $r_0 = 1.128\ fm$, $c = A^{1/3}1.128(1 + \frac{0.439}{1.128}A^{-2/3})\ fm$, $a = 0.6\ fm$, we obtain somewhat larger exponential terms (see table 3). In this case, for ${}^{13}_{\Lambda}C$, $c = 2.774\ fm$.

Table 3

Values of $R_f > c$	Values of $I_2(0, R_f)$	Values of non-exponential terms	Contribution of exponential terms	Percentage Contribution
5.0	9.946	10.403	-0.457	4.59
R_M=4.273	9.258	10.403	-1.145	12.72
4.0	8.820	10.403	-1.583	17.94
3.0	6.0254	10.403	-4.378	72.66
2.9	5.653	10.403	-4.749	84.01
2.775	5.177	10.403	-5.206	100.55

The fact that the exponential terms and also $(1/4)f^2(R_f)$ are usually small for $R_f = R_M$ makes it possible to obtain to a good approximation an analytic solution [35] of the equation (5.14), by omitting these terms:

$$R_0 = R_M^{(0)} = \left(\frac{5}{2}\right)^{1/3} c\left[1 + \left(\frac{\pi a}{c}\right)^2\right]^{1/3}. \qquad (5.21)$$

Furthermore, improved analytic expressions can be derived, if instead of omitting these terms we estimate them by using an approximate expression for R_M ($R_M \simeq R_M(0)$)., e.g.

$$R_M^{(1)} = c\left[\frac{1 + (\pi a/c)^2 + C}{(2/5) + (3/4)f^2(R_M^{(0)})}\right]^{1/3}. \qquad (5.22)$$

This procedure may be iterated until self-cosistency is achieved to a desirable accuracy. It should be noted that exponential terms exist in this case even for a symmetrized Woods-Saxon potential ($c = 1$). In Table 4 the various values of $R_M^{(n)}$ which are obtained by means of the above mentioned iteration procedure with the corresponding values of $\hbar\omega$ are shown in the case of ${}^{13}_{\Lambda}C$ using the potential parameters of $c = 2.7\ fm$, $r_0 = 1.423\ fm$ of ref.[35] for the Woods-Saxon potential.

Table 4

n	R_M^n fm	$\hbar\omega$, MeV
0	4.2730	
1	4.0954	11.316
2	4.0447	11.458
3	4.0277	11.506
4	4.0216	11.524
5	4.0194	11.529
6	4.0186	11.532
7	4.0183	11.533
8	4.0182	11.533
9	4.0182	11.533

From the analysis of this section and from the remarks made in the previous ones, it is clear that the exponential terms are not neglibigle in certain cases.

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