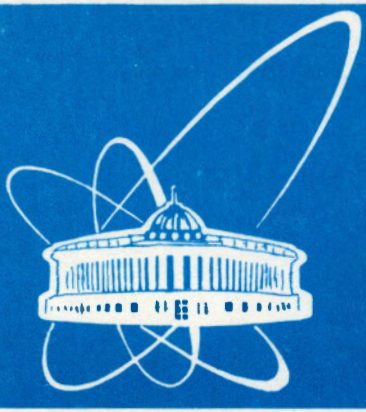

ОБЪЕДИНЕННЫЙ
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I.Amirkhanov, I.Puzynin, A.Tarasov, O.Voskresenskaya,
O.Zeinalova

A METHOD FOR RESEARCH ON BEHAVIOUR OF DIMESOATOMIC WAVE FUNCTIONS AT SMALL DISTANCES

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1. Introduction

The method for the measurement of the pionium ($\pi^+\pi^-$-atom) lifetime proposed in ref. [1], is essentially based on the assumption that the n-dependence of probability w_n on creation of $\pi^+\pi^-$-atom in nS-state is well known or, at least, may be calculated with a high degree accuracy.

The first consideration of this problem has been done in L. Nemenov's paper [2], where the following relation

$$w_n \sim n^{-3} \tag{1}$$

has been derived from a more general result of the author [2]

$$w_n \sim \left| \int M(\vec{r}) \psi_n(\vec{r}) d^3r \right|^2 , \tag{2}$$

$$M(\vec{r}) = \frac{1}{(2\pi)^3} \int M(\vec{p}) e^{-i\vec{p}\vec{r}} d^3p , \tag{3}$$

where $M(\vec{p})$ is the amplitude of production of free $\pi^+\pi^-$-pairs with relative momentum $\vec{p}$ in hadron – hadron or hadron – nucleus collisions and $\psi_n(r)$ is the wave function of nS-state of pionium.

In line with accounting the short range nature of amplitude $M(r)$ in his original derivation of (1) from (2) L. Nemenov also has made an assumption that the pure Coulomb wave functions describe quite well a distribution of pions in the pionium not only at large distances, but at small ones also. However, as it has been shown recently by E. Kuraev [3], this assumption is unjustified due to some noticeable influence of strong a $\pi^+\pi^-$-interaction on the behaviour of the pionium wave functions in the nearest of origin. Due to this reason a more careful analysis of this problem is needed.

Below we shall represent some preliminary results of the analysis based on the local potential model of the strong $\pi^+\pi^-$-interaction. In this model the "reduced" pionium wave functions

$$\Phi_n(r) = \sqrt{4\pi} r \psi_n(r) , \quad \int |\Phi_n(r)|^2 dr = 1 , \tag{4}$$

obey the Shrödinger equation

$$\Phi_n''(r) + m[U_c(r) + U_s(r)]\Phi_n(r) = m\varepsilon_n \Phi_n(r) , \tag{5}$$

where m is the pion mass, ε_n — binding energy, $U_c = \alpha/r$, U_s — Coulomb and strong potentials, respectively.

2. The scheme and results of perturbative analysis

First of all, let us apply the methods of the perturbation theory to the Schrödinger equation (5), treating the strong interaction potential U_s as perturbation, in order to obtain some qualitative estimations. Putting

$$\Phi_n(r) \approx \Phi_n^{(0)}(r) + \Phi_n^{(1)}(r) , \tag{6}$$

$$\varepsilon_n = \varepsilon_n^{(0)} + \varepsilon_n^{(1)} , \tag{7}$$

where

$$\varepsilon_n^{(0)} = \frac{m\alpha^2}{4n^2} , \quad \varepsilon_n^{(1)} = \int_0^\infty U_s(r) \Phi_n^{(0)2}(r) dr ; \tag{8}$$

$$\Phi_n^{(0)\prime\prime}(r) + m[U_c(r) - \varepsilon_n^{(0)}]\Phi_n^{(0)}(r) = 0 , \tag{9}$$

$$\Phi_n^{(1)\prime\prime}(r) + m[U_c(r)\varepsilon_n^{(0)}]\Phi_n^{(1)}(r) = m[\varepsilon_n^{(1)} - U_s(r)]\Phi_n^{(0)}(r) ; \tag{10}$$

and applying the general methods of solving the linear inhomogeneous equations [4, 5] one can obtain

$$\Phi_n^{(1)} = \Phi_n^{(0)} [c_n - \chi_n(r)] , \tag{11}$$

where

$$\chi_n(r) = \int_0^r \frac{dr_1}{|\Phi_n^{(0)}(r_1)|^2} \int_0^{r_1} |\Phi_n^{(0)}(r_1)|^2 [-\varepsilon_n^{(1)} + U_s(r_2)] dr_2 \tag{12}$$

and

$$c_n = \int_0^\infty dr |\Phi_n^{(0)}(r_1)|^2 \chi_n(r) . \tag{13}$$

If we define the ratio

$$R_n(r) = \frac{\Phi_n(r)}{\Phi_n^{(0)}(r)} \equiv \frac{\psi_n(r)}{\psi_n^{(c)}(r)} \tag{14}$$

as a measure of the influence of strong interactions on the values of the pionium wave functions, then in the first order of the perturbation theory

$$R_n(0) \approx 1 + c_n . \tag{15}$$

With explicit expressions for the pure Coulomb wave functions $\Phi_n^{(0)}$ we have proceeded in the calculation of c_n with $n = 1, 2, 3$.

The result looks like as follows:

$$c_n = \int\limits_0^\infty mU_s(r)rdr + \sum_{n=1} d_k^{(n)} \int\limits_0^\infty mU_s(r)r\Big(\frac{r}{r_B}\Big)^k \ln\Big(\frac{r_k^{(n)}}{r}\Big)dr\,, \tag{16}$$

where $d_k^{(n)} \sim 1$, $r_k^{(n)} \sim r_B \approx 400\, fm$.

Taking into account the relation

$$\int\limits_0^\infty mU_s(r)r^2dr \approx a \approx 0.15\, fm\,, \tag{17}$$

where a is the $\pi^+\pi^-$ scattering length, it is easy to see that the n-dependent contributions to c_n are numerically small (of order 10^{-3}) and may be neglected.

Putting $U_s = g/r\exp(-br)$ with the values of parameters [3] $g \approx 3$, $b = m_\rho \approx 3.8 fm^{-1}$, that corresponds to applying the ρ-exchange model for a describtion of the strong $\pi^+\pi^-$-interaction, one can obtain for n-independent part of c_n (see also [6])

$$\int\limits_0^\infty mU_s(r)rdr = \frac{gm}{m_\rho} \approx 0.55\,. \tag{18}$$

These estimations show that the strong $\pi^+\pi^-$-interaction can change noticeably the value of the pionium wave functions at origin and this effect cannot be ignored at evaluating the probabilities w_n (2). On the other hand, the large correction to the values of the wave functions, obtained in the first order of perturbation theory, means that the higher order corrections are not small and must be taken into account. Their calculation does not look a simple problem.

3. Numerical investigation

Due to this reason we have applied numerical methods for an accurate investigation of the pionium wave functions behaviour at small distances, using the following equation with boundary conditions and normalization:

$$\Phi_1 = \frac{d^2\chi}{d\rho^2} + U(\rho)\chi(\rho) - \varepsilon\chi(\rho) = 0\,, \tag{19}$$

$$\Phi_2 = \chi(0) = 0\,, \quad \Phi_3 = \chi(\infty) = 0\,, \tag{20}$$

$$\Phi_4 = \int\limits_0^\infty \chi^2(\rho)d\rho - 1 = 0\,, \tag{21}$$

where

$$\rho = \frac{r}{r_B} = \mu\alpha r\,, \quad \mu = \frac{1}{2}m_\pi\,, \quad \alpha = \frac{1}{137}\,. \tag{22}$$

So we have an operator equation

$$\Phi(z) = 0\,, \tag{23}$$

where

$$\Phi(z) = \begin{cases} \Phi_1(z) \\ \Phi_2(z) \\ \Phi_3(z) \\ \Phi_4(z) \end{cases}\,,$$

$$z = (\chi(\rho),\, \varepsilon)\,.$$

In order to calculate the normalized pionium wave functions, we have used an improved version of code [7], based on applying the continuous analogue of Newton's method, developed in ref. [8]. In this way we change the primary equation by the following differential equation with parameter t

$$\frac{d}{dt}\Phi\left(z(t)\right) = -\Phi\left(z(t)\right)\,, \tag{24}$$

$$z(0) = z_0\,, \quad 0 < t < \infty\,.$$

When $t \to \infty$, the solution of this equation tends to the solution of primary equation (23) $z(t) \to z$.

For numerical calculations the grid on t was used $\{t_i,\ i = 1\,, 2\,, \ldots\,;\ t_0\,;\ t_{i+1} - t_i = \tau\}$. For every t_i we obtained the solution $dz(t = t_i)/dt$ to equation (24)

$$\frac{dz(t_i)}{dt} = -\phi'\left(z(t_i)\right)^{-1}\Phi\left(z(t_i)\right)\,, \tag{25}$$

where ϕ' is Frechet derivative of function Φ, ϕ'^{-1} is an inverse operator.

For approximation of $dz(t_i)/dt$ we use a finite difference

$$\frac{dz(t_i)}{dt} \approx \Delta z_i = \tau_i^{-1}\left[z(t_{i+1}) - z(t_i)\right]\,. \tag{26}$$

Thus,

$$z(t_{i+1}) \cong \tau_i\Delta z_i + z(t_i)\,. \tag{27}$$

If $z(t_i)$ is known, then as a result of this iterative procedure with parameter τ_i we obtained a sequence of the approximate values of the solution to equation (24) $\{z_i\} \to z$. As a starting approximation z_0 we choose an analytical solution of the corresponding Coulomb problem. The iterative procedure was continued until residual $\parallel \Phi\left(z(t)\right) \parallel < 10^{-5}$.

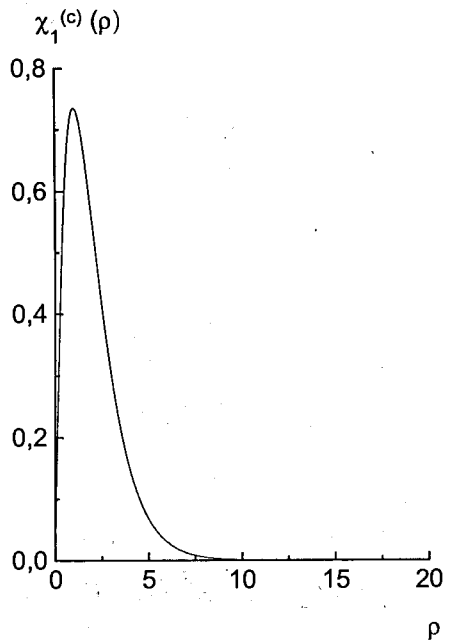

Fig. 1. A result of numerical calculation of Coulomb pionium eigenfunction $\chi_1^{(c)}(\rho)$.

The problem (25) was solved numerically in the interval $(a,\ b)$, where $a = 10^{-8}$, b is a sufficiently large number, depending on the principal quantum number n. The solution at the point b was adjusted logarithmically to the appropriate Coulomb eigenfunction

$$\frac{\chi'(b)}{\chi(b)} = \frac{\chi^{(c)\prime}(b)}{\chi^{(c)}(b)}. \tag{28}$$

To calculate, we used a nonuniform grid on ρ as follows: in the interval $(a,\ 0.02)$ — 301 nodes, in the interval $(0.02,\ 2)$ — 301 nodes and in the interval $(2,\ b)$ — 200 nodes.

The accuracy of the method was tested on Coulomb problem. Fig. 1 shows the numerically obtained solution of Coulomb problem for $n = 1$ for potential $U_c(\rho) = 2/\rho$. Fig. 2a gives the corresponding value of the wave function $\psi_1^{(c)}(\rho)$. The same functions for $n = 2,\ 3,\ 4$ are given in Figs. 2b-2d. The difference between the numerical and analytical values of function $\psi_n^{(c)}(\rho)$, $n = 1,\ 2,\ 3,\ 4$, is represented in Figs. 3a-3d.

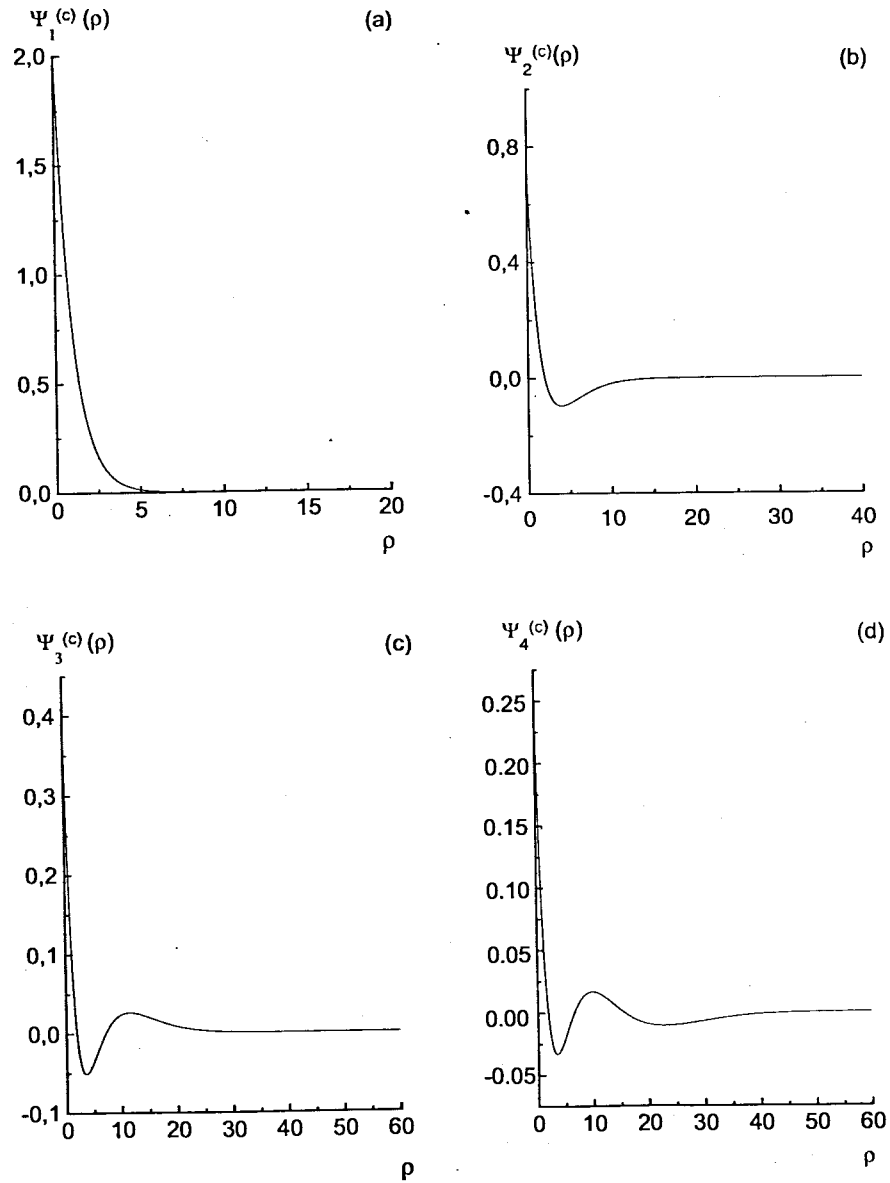

Fig. 2. Results of numerical calculations of Coulomb pionium nS-state wave functions $\psi_n^{(c)}(\rho)$ for $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).

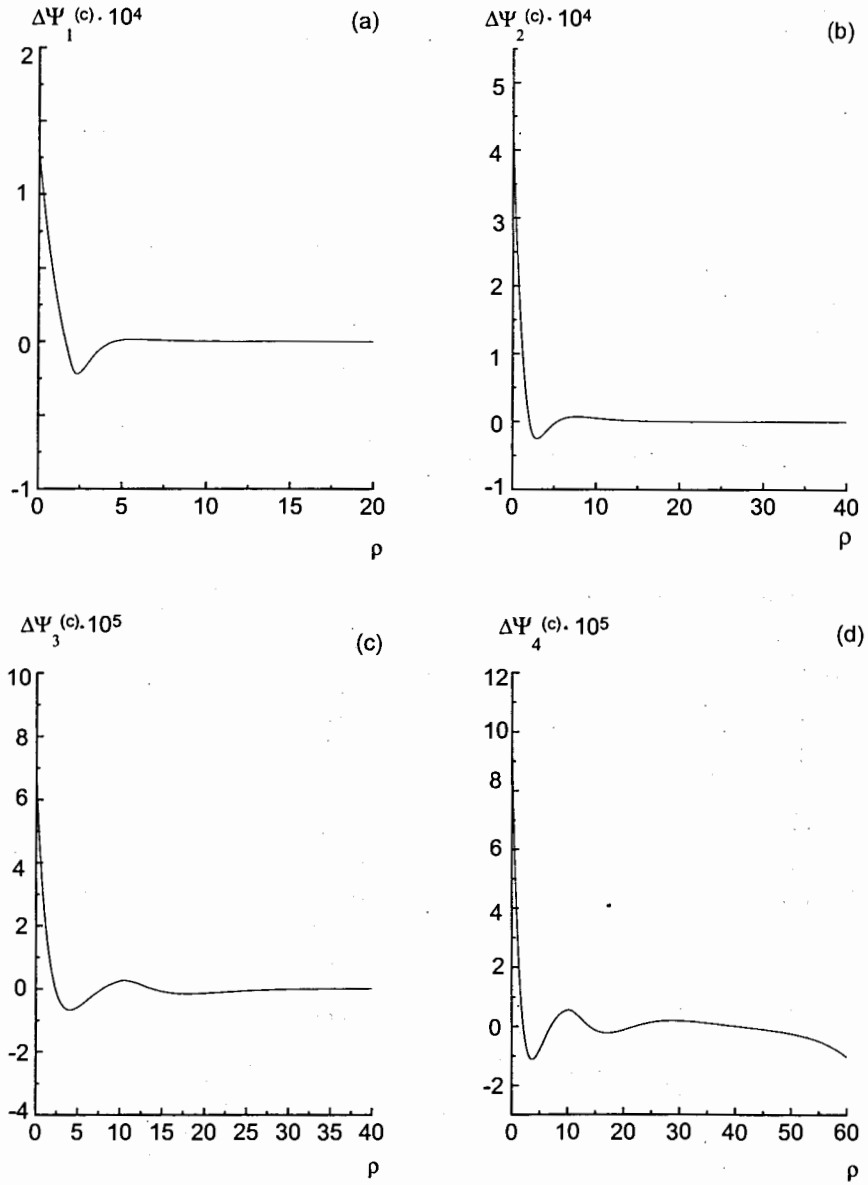

Fig. 3. Difference between numerical and analytical values of functions $\psi_n^{(c)}(\rho)$ for $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).

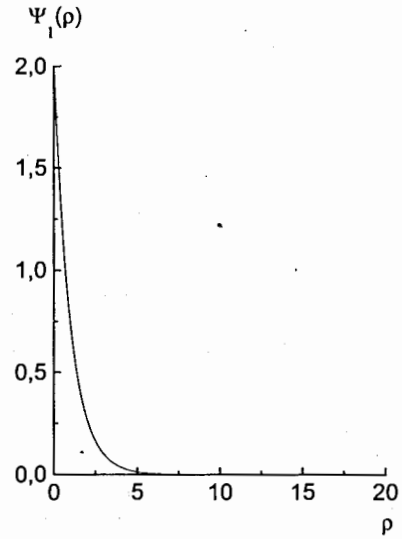

Fig. 4. Behaviour at small distances on the pionium wave function $\psi_1(\rho)$.

The input of parameters of the code has been chosen in this way to guarante an absolute accuracy of the calculations higher than 10^{-4}. To check up this accuracy, we have compared the numerical solution of Shrödinger equation with the pure Coulomb interaction with the analytical one.

At the next step we solved our problem for the sum of strong and Coulomb potentials as follows:

$$U(\rho) = \frac{2}{\rho}(1 - e^{-b\rho}) + \frac{a}{\rho}e^{-b\rho}, \tag{29}$$

where

$$b = \frac{m_\rho}{\alpha\mu} \approx 1.5 \cdot 10^3, \quad a = \frac{2\alpha_{\rho\pi\pi}}{\alpha} \approx 8 \cdot 10^2, \tag{30}$$

$$\alpha_{\rho\pi\pi} = \frac{g^2_{\rho\pi\pi}}{4\pi} \approx 3.$$

The results of the numerical calculations of the wave function $\psi(\rho)$ for $n = 1$ are represented in Fig. 4. At a long interval of ρ we cannot see a difference between it and the corresponding wave function of Coulomb problem, but when ρ is compatible with Fermi radius, then this difference is significant. This fact is shown for $n = 1$ in Fig. 5a. The difference also takes place for $n = 2$, 3, 4 (see Figs. 5b-5d).

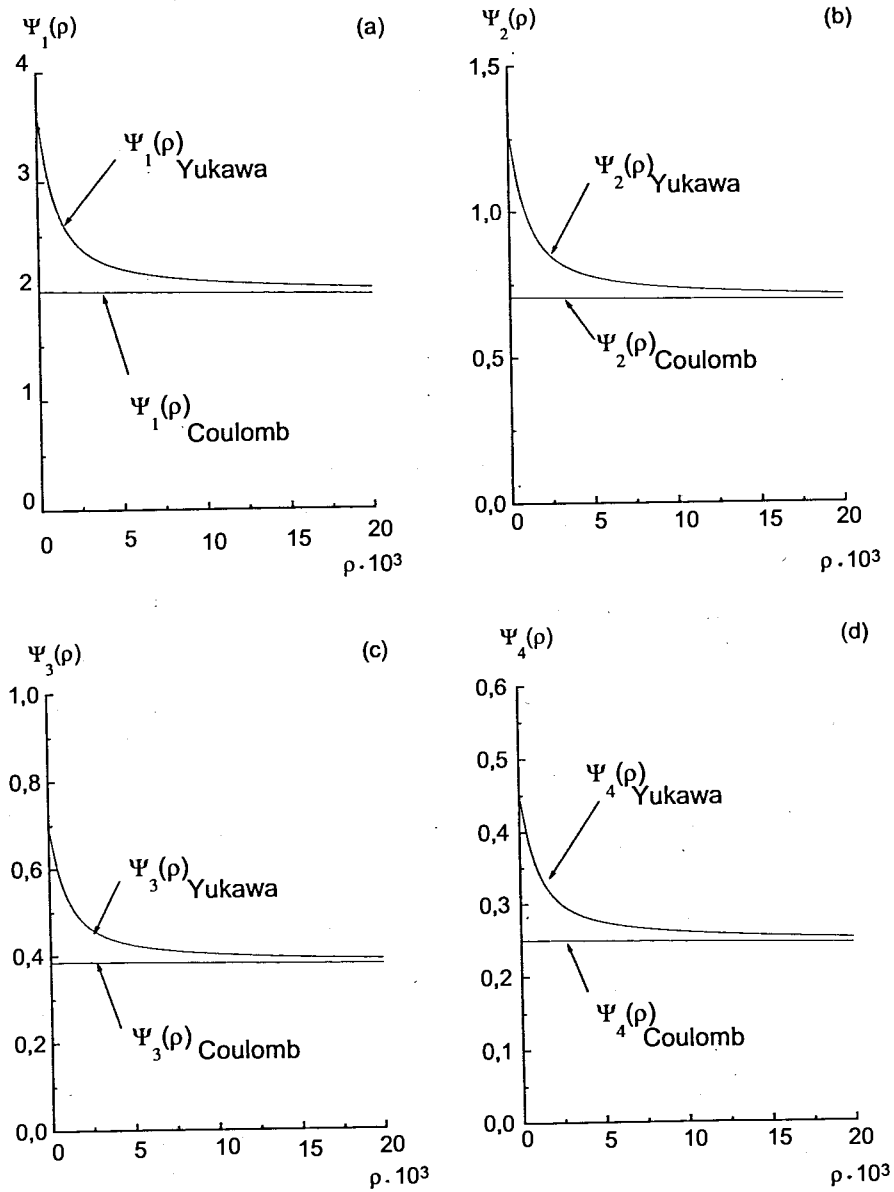

Fig. 5. Influence of strong pion-pion interaction on the pionium nS-state wave functions $\psi_n(\rho)$ at small distances : $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).

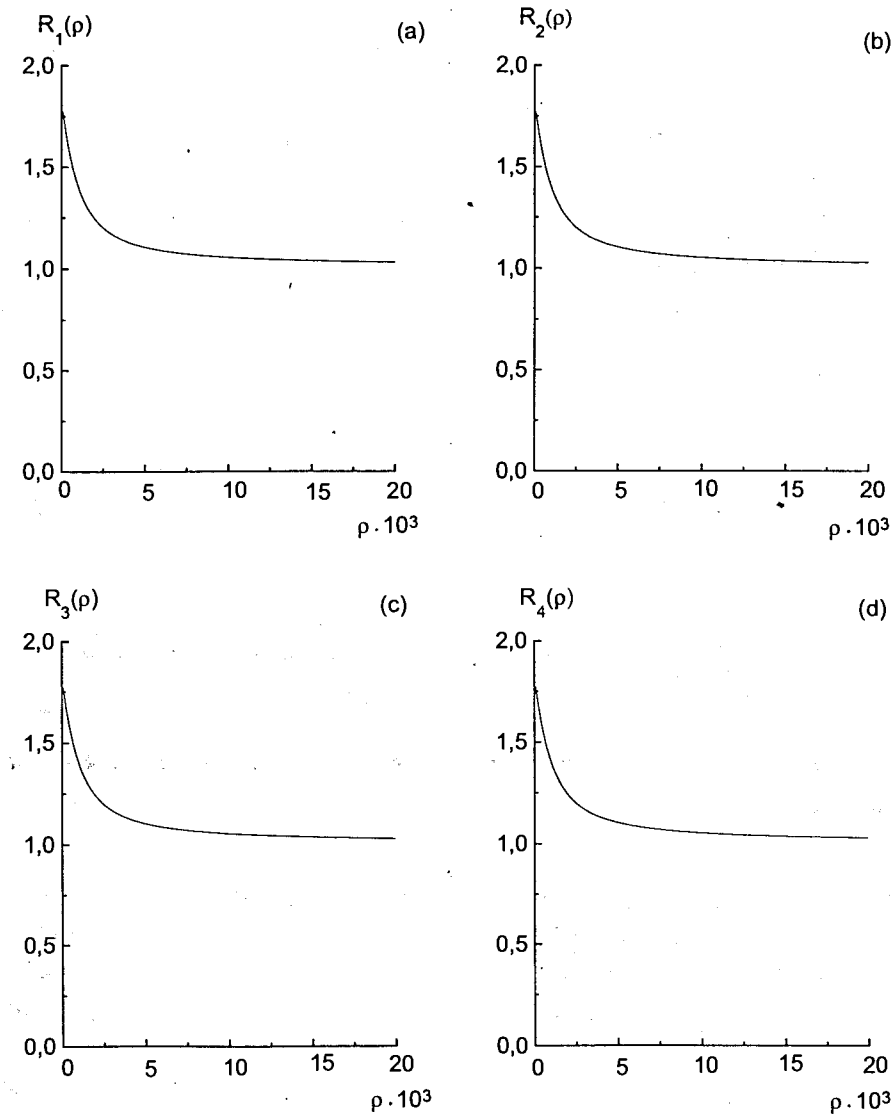

Fig. 6. The ratio $R_n(\rho)$ as a measure of the influence of strong interactions on the values of the pionium wave functions: $n = 1$ (a), $n = 2$ (b), $n = 3$ (c), $n = 4$ (d).

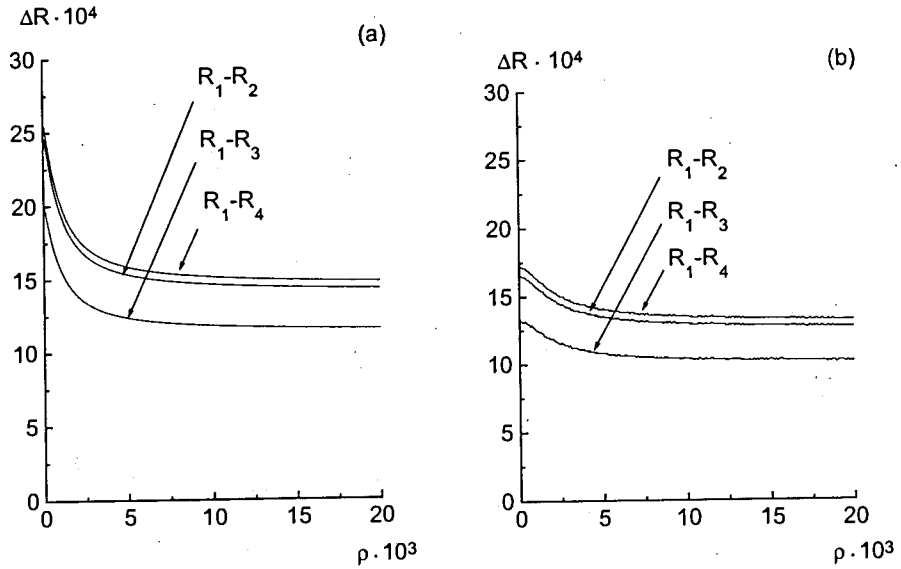

Fig. 7. The accuracy of n-independence of strong renormalization factors for potentials (a) $U(\rho)$ and (b) $\tilde{U}(\rho)$.

The most important conclusion from our numerical calculations is as follows: the functions

$$R_n(\rho) = \frac{\psi_n(\rho)}{\psi_n^{(c)}(\rho)} \,. \tag{31}$$

are practically independent of n (n = 1, 2, 3, 4). This is illustrated by Fig. 6.

Fig. 7a shows the differences between functions $R_n(\rho)$ for the considered n. One can see from the plots that the differences between functions R_n for $n = 1-4$ are of the order less or equal 10^{-3}. Thus, we can replace functions $R_n(\rho)$, $(n = 1-4)$ with accuracy 10^{-3} with $R_1(\rho)$

$$R_n(\rho) = R_1(\rho) + O(10^{-3}) \,. \tag{32}$$

The same results were obtained for potential

$$\tilde{U}(\rho) = \frac{2}{\rho}(1 - e^{-b\rho}) + \tilde{a}\, e^{-b\rho} \,, \tag{33}$$

where

$$b = \frac{m_\rho}{\alpha\mu} \approx 1.5 \cdot 10^3 \,, \quad \tilde{a} = \frac{1}{2}\, a \cdot b \approx 6 \cdot 10^5 \,. \tag{34}$$

As one can see from Fig. 7b, the estimation (32) is valid for potential (33).

So, we can make the following conclusions from our calculations: for the considered potentials (29) and (33) for the principal quantum numbers n = 1, 2, 3, 4 the following estimation is valid

$$\frac{\psi_n(\rho)}{\psi_n^{(c)}(\rho)} = \frac{\psi_1(\rho)}{\psi_1^{(c)}(\rho)} + O_n(10^{-3}) \,, \tag{35}$$

i.e. with accuracy 10^{-3} function $\psi_n(\rho)$ can be represented as

$$\psi_n(\rho) \approx \alpha(\rho)\psi_n^{(c)}(\rho) \,, \tag{36}$$

where $\alpha(\rho)$ is independent of n, ρ is compatible with Bohr radius.

4. Conclusion

Thus, the results of the numerical solution to equation (5) with Yukawa-type strong potential have confirmed the main conclusions of the perturbative consideration, namely: the ratios

$$R_n(r) = \frac{\psi_n(r)}{\psi_n^{(c)}(r)} \equiv R_n(\rho) \tag{37}$$

being numerically large (see Fig. 6) in the region $r \le r_s \sim 1\ fm$ and essential for the problem under consideration (n-dependence of values w_n), are practically n-independent (their n-independence is illustrated by Fig. 7).

This means that with a high degree accuracy one can substitute

$$\psi_n(r) = R(r)\psi_n^{(c)}(r) \tag{38}$$

in eq. (2) and, replacing $M(\vec{r}) \Rightarrow \widetilde{M}(\vec{r}) = M(\vec{r})R(r)$, obtain

$$w_n \sim \left| \int \widetilde{M}(\vec{r})\psi_n^{(c)} dr \right|^2 \sim \left|\psi_n^{(c)}(0)\right|^2 \left| \int \widetilde{M}(\vec{r}) dr \right|^2 \sim n^{-3} \,. \tag{39}$$

Therefore, we can conclude that strong interaction corrections to the pionium wave functions at small distances, being sufficiently large, do not change the n^{-3}-law (1) primarily derived in paper [2] assuming that the pionium wave functions are pure Coulomb.

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Амирханов И. и др.

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Метод исследования поведения димезоатомных волновых функций на малых расстояниях

Изучено уравнение Шредингера, описывающее сильное $\pi^+\pi^-$-взаимодействие в локальной потенциальной модели. Влияние сильного $\pi^+\pi^-$-взаимодействия на поведение волновых функций пиония на малых расстояниях для nS-состояний исследовано аналитически (пертурбативно) и численно. Показано, что эффект влияния сильного взаимодействия на «кулоновскую» волновую функцию пиония сводится к умножению ее на функцию, практически не зависящую от главного квантового числа n. В связи с чем n-зависимость вероятности рождения $\pi^+\pi^-$-атомов в nS-состоянии оказывается той же, что и в случае чисто кулоновского $\pi^+\pi^-$-взаимодействия.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Сообщение Объединенного института ядерных исследований. Дубна, 1998

Amirkhanov I. et al.

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A Method for Research on Behaviour of Dimesoatomic Wave Functions at Small Distances

The Schrödinger equation describing the local potential model of a strong $\pi^+\pi^-$-interaction was studied. The influence of the strong $\pi^+\pi^-$-interaction of the behaviour of pionium nS-state wave functions at small distances is studied both analytically (perturbatively) and numerically. It is shown that in the whole the accounting of strong interaction results in multiplying pure Coulomb pionium wave functions by some function which is practically independent of the value of the principal quantum number n. Due to this reason, the n-dependence of $\pi^+\pi^-$-atom production probability in nS-state remains the same as in the case of a pure Coulomb $\pi^+\pi^-$-interaction.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1998