ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E4-98-207

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GENERAL RELATIVITY AND GAUGE GRAVITY THEORIES OF HIGHER ORDER

Submitted to the Proceedings of the
«International Seminar on Mathematical Cosmology»,
March 30–April 4 1998, Potsdam, Germany

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1998

1 Construction concepts of gauge gravity theories of higher order

There are many different reasons for construction of gravity theories with higher derivatives. Partly it is gauge reasons but all gravity theories of higher order are non-Einstein theories.

At first the theory of such sort was proposed by Weyl in 1918 ([1]). In Weyl's theory the metric tensor $g_{\mu\nu}$ remained as a gravity field variable. One-parameter local gauge invariance was used by Weyl for electromagnetic field introduction. This idea was followed by modification of space-time V_4 connections and Lagrangian of theory. Weyl's Lagrangian was curvature tensor square. Hence Weyl's theory equations were of more higher order of $g_{\mu\nu}$ derivatives than Einstein's equations ([2]). But it was not gauge gravity theory.

Utiyama was first who tried to get the gravity theory from the local gauge invariance principle ([3]). He used Lorentz group as the gauge symmetry group. Modified connections were found to be Ricci connections, that is the gravity field variables was of higher order than Einstein's one. Therefore Utiyama introduced a'priori 16 new variables $h_{\mu\nu}$ (vierbeins) in addition to Ricci connections $\Delta_\mu(ik)$. He postulated that $g_{\mu\nu} = h^i_\mu h_{\nu i}$ and the gravity Lagrangian is Einstein's one (scalar curvature R) beyond the gauge scheme.

To avoid apriority in $h_{\mu\nu}$ introduction in 1961 Kibble proposed to use Poincaré group of space-time coordinate transformations as the local gauge symmetry group ([4]). Really he interpreted general covariant coordinate transformations $x^{\mu\prime} = f^\mu(x^\nu)$, where $f^\mu(x^\nu)$ is arbitrary function, as following local gauge Poincaré transformations:

$x^{\mu\prime} = x^\mu + \delta x^\mu$, where $\delta x^\mu = \epsilon^\mu_\nu x^\nu + \epsilon^\mu$,

$\epsilon^\mu_\nu, \epsilon^\mu$- 10 Poincaré group parameters which are functions of coordinates. After that Kibble proposed to consider $\xi^\mu = \epsilon^\mu_\nu x^\nu + \epsilon^\mu$ and $\epsilon^{\mu\nu}$ as independent functions. It is incorrect assumptions. But as a result he obtained 16 vierbeins h^i_μ and 24 connection coefficients $A_\mu(ik)$ as the gravity field variables.

Without any matter $A_\mu(ik)$ coincide with Ricci connections $\Delta_\mu(ik)$ but in matter presence $A_\mu(ik)$ can depend on the spin matter tensor $S^\mu_{\rho\sigma}$. It is need to note that Kibble introduced h^i_μ by other compensating procedure than Weyl-Yang-Mills-Utiyama one. He did not can obtain Einstein's Lagrangian R by any regular procedure, but did not use curvature tensor square as the Lagrangian following from the standard compensating procedure. Later the spin matter tensor $S^\mu_{\rho\sigma}$ was identified with torsion tensor of V_4, and so Kibble's theory turned out the gravity theory with torsion. It is non-Einstein theory.

After Utiyama's and Kibble's papers it became clear that GR is not a theory of Yang-Mills type. Even now there is not exist any gauge theory where GR can be obtained by the compensating procedure. But it is possible to obtain many different non-Einstein gravity theories by formalistic extension of connection coefficients and space-time structure. Some of them contain higher derivatives of $g_{\mu\nu}$. Good review of metric-affine gauge theories is ([5]).

There is an opinion that new geometrical objects (torsion, nonmetricity, etc.) are necessary for quantization of gravity. Hence the quantum gravity must be non-Einstein theory. Is that so?

2 GR as the gauge theory: Mathematical technique

Why GR must be a gauge theory? Modern theoretical physics goal is unification of all fundamental interactions: mechanics, electrodynamics, nuclear forces (weak and high) and gravity.

The ways of this problem decision being proposed now:

1. Single big symmetry group G_r generating all conservation laws for all interactions (for example, Grand Unification);
2. Single big wave function ψ which components correspond to each particle or field (for example, supersymmetry);
3. Single equation which components correspond to equations of each interaction (Kaluza-Klein theory and its extensions);
4. Single construction principle of each interaction theory under conservation individuality of each interaction.

In all cases the gauge invariance is used. Fourth way was proposed by me in 1967 and published in ([6],[7]). It does not use any compensating procedure. In this case each gauge field theory can be produced by choice:

- field variables;
- symmetry groups (space-time symmetry and internal symmetry);
- transformation properties of field variables under two types of symmetry groups;
- order of derivatives of the field variables in Lagrangian.

It is necessary and sufficiently for construction of variational and geometrical theory of any fundamental interaction.

Including GR!

Some of gauge gravity theories were obtained by this technique. For instance:

2.1 Einstein's GR as the gauge gravity theory

It is necessary to choose:

- field variables - $g_{\mu\nu}$ (metric tensor of V_4);
- symmetry group - $G_{\infty 4}$ ($x^{\mu'} = f^{\mu}(x^{\nu})$);
- transformation properties - $g_{\mu\nu}$ is symmetrical tensor of rank two, i.e. under $G_{\infty 4}$ its transformations are:

$$\delta_L g_{\mu\nu} = \xi^{\tau}(x)\partial_{\tau} g_{\mu\nu} + g_{\tau\nu}\partial_{\mu}\xi^{\tau}(x) + g_{\mu\tau}\partial_{\nu}\xi^{\tau}(x).$$

δ_L-Lie derivatives, $\xi^{\tau}(x)$-Killing's vector, $x^{\mu'} = x^{\mu} + \xi^{\mu}(x)t$, $\mu, \nu = 0, 1, 2, 3$;

- order of derivatives in Lagrangian -2.

Hence, Lagrangian has Einstein's form: $L = R$, and equations of theory are Einstein one:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

Here GR is constructed without any compensating idea, torsion, etc., but by the same regular procedure which is used for all other interactions.

2.2 $SO(3,1)$-gauge gravity

Corresponding choice must be done in the following way:

- field variables - $\Delta_{\mu}(ik)$ (Ricci connections);
- symmetry group - $G_{\infty 6}$ (local Lorentz group);
- transformation properties under $G_{\infty 6}$:

$$\delta\Delta_{\mu}(ik) = f^{ik}_{lm\ pq}\Delta_{\mu}(pq)\epsilon^{lm}(x) + \partial_{\mu}\epsilon^{ik}(x),$$

where $f^{ik}_{lm\ pq}$-the structure constants, $\epsilon^{ik}(x)$- parameters of $G_{\infty 6}$;

- order of derivatives in Lagrangian - 2.

Hence, Lagrangian of theory is $L = R_{\mu\nu}(ik)R^{\mu\nu}(ik)$, i.e. it has Weyl's form without Weyl's connections but with Riemannian curvature tensor.

Equations of the theory are similar to Maxwell and Yang-Mills one:

$$\hat{R}^{\mu\nu}(ik)_{;\nu} = 0.$$

2.3 GR + $SO(3,1)$-gravity

It is corresponding with the following choice:

- field variables - $g_{\mu\nu}$ and $\Delta_\mu(ik)$;
- symmetry groups - $G_{\infty 4}$ and $G_{\infty 6}$;
- transformation properties (the same that in pp.2.1-2.2):
- order of derivatives - 2 for each variable.

Hence, Lagrangian is:

$$L = R - \frac{\kappa}{4} R_{\mu\nu}(ik) R^{\mu\nu}(ik) \tag{1}$$

equation system is:

$$\hat{R}^{\mu\nu}(ik)_{;\nu} = 0 \tag{2}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa (R_{\mu\tau}(ik) R^{\tau}_{\nu}(ik) - \frac{1}{4} g_{\mu\nu} R_{\lambda\tau}(ik) R^{\lambda\tau}(ik)) \tag{3}$$

With respect to $g_{\mu\nu}$ in cases 2.2-2.3 we have the gauge gravity theory of higher order. But $g_{\mu\nu}$ and $\Delta_\mu(ik)$ are introduced as independent variables.

The main feature of this approach is absence of Poincaré local gauge group. Instead of it we have two groups $G_{\infty 4}$ and $G_{\infty 6}$ acting in fibre bundle space on V_4 (tangential fibre bundle). $G_{\infty 4}$ acts in the base of fibre bundle space (i.e. in Riemannian space-time V_4), and $G_{\infty 6}$ acts in the fibre bundle (tangential space).

The gravity field variable choice is a problem of paying attention to the physical sense of theory, devices properties and measurement methods being used.

If $\Delta_\mu(ik)$ is field variable, then $R^{\mu\nu}(ik)$ is analog of $F^a_{\mu\nu}$ of any gauge field. Lagrangian (1) is analog of any gauge field Lagrangian in Riemannian space-time:

$$L = R - \frac{\kappa}{4} F^a_{\mu\nu} F^{\mu\nu}_a,$$

$\Delta_\mu(ik)$ is analogous to vector-potential A^a_μ.

Stress-energy tensor of real gravity (tidal forces) is analogous to Maxwell one:

$$T^{(g)}_{\mu\nu} = \kappa (R_{\mu\tau}(ik) R^{\tau}_{\nu}(ik) - \frac{1}{4} g_{\mu\nu} R_{\lambda\tau}(ik) R^{\lambda\tau}(ik)) \tag{4}$$

$$T^{(em)}_{\mu\nu} = F_{\mu\tau} F^{\tau}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\tau} F^{\lambda\tau} \tag{5}$$

3 Hilbert equations and Wheeler-Misner geometrodynamics

In 1915 in his paper "Die Grundlagen der Physik" ([8]) Hilbert obtained a system of two variational equations for description of gravity and electrodynamics together. Each equation corresponded to its field variable. These variables were $g_{\mu\nu}$ (metric tensor of V_4) and q_s (vector- potential of electromagnetic field, i.e. A_μ in modern designation). Hilbert noted that in consequence of general relativity coordinate transformations 4 equations of the system are corollary of the rest. And so he decided to consider the electrodynamics equations as corollary of 10 gravity equations. From this statement it followed that electromagnetic phenomena are effects of gravity.

As invariant function under integration (world function H) Hilbert chose $H = K + L$, where $K = R$ (Riemannian scalar curvature), $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ (in modern designation).

Hilbert's system of equations can be written in the form:

$$(\sqrt{g} F^{\mu\nu})_{;\nu} = \hat{F}^{\mu\nu}_{;\nu} = 0 \tag{6}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(em)}_{\mu\nu} \tag{7}$$

(here $\kappa = 1, T^{(em)}_{\mu\nu}$-stress-energy tensor of electromagnetic field (equation (5))).

Such equation system coincides with the system of equations of Wheeler-Misner geometrodynamics ([9]), which was proposed in 1957.

In 1970 I showed that in Riemannian space-time V_4 any gauge fields are described by analogous equation system ([11]):

$$\hat{F}^{\mu\nu}_{a\ ;\nu} = 0 \tag{8}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(gf)}_{\mu\nu} \tag{9}$$

where $T^{(gf)}_{\mu\nu}$-stress-energy tensor of any gauge field which has Maxwell-type form:

$$T^{(gf)}_{\mu\nu} = F^a_{\mu\tau} F^{\tau}_{a\ \nu} - \frac{1}{4} g_{\mu\nu} F^a_{\lambda\tau} F^{\lambda\tau}_a \tag{10}$$

Wheeler and Misner found explicit relations between electric **E** and magnetic **H** strenghs, on the one hand, and gravity variables, on the other hand. It turned out that, indeed, the gravity variables can describe the electromagnetic effects. But dependence of **E** and **H** on $g_{\mu\nu}$ has very complex and nonlinear form, and corresponding formulae contain derivatives of Ricci curvature square.

These formulae were at first obtained by Rainich in 1924/25 ([10]) and, independently, by Wheeler and Misner in 1957([9]). They have the form:

$$\mathbf{E}_\mu = (\xi, 0, 0)\cos\alpha, \ \mathbf{H}_\mu = (\xi, 0, 0)\sin\alpha; \tag{11}$$

$$\alpha(x) = \int_0^x \alpha_\mu(x)dx^\mu + \alpha_0;$$

$$\alpha_\mu \equiv \frac{\sqrt{g}\epsilon_{\mu\alpha\beta\nu}R^{\alpha\rho;\beta}R^\nu_\rho}{R^\sigma_\tau R^\tau_\sigma}. \tag{12}$$

Here $\alpha(x)$- complexion of field; ξ-eigenvalue of Ricci tensor $R_{\sigma\tau}$ in the coordinate system, where $R_{\sigma\tau}$ has diagonal form ($\xi^4 = \frac{1}{4}R^\sigma_\tau R^\tau_\sigma$); $\epsilon_{\mu\alpha\beta\nu}$ - discriminant tensor.

The angle $\alpha(x)$ realizes a dual rotation of electric **E** and magnetic **H** fields. Moreover $\alpha(x)$ must be independent on integration way in V_4. This is additional constraint to metrics $g_{\mu\nu}$:

$$\alpha_{\mu,\nu} = \alpha_{\nu,\mu}. \tag{13}$$

This constraint contains second derivatives of curvature tensor and so derivatives of $g_{\mu\nu}$ up to fourth order.

Hence, in Wheeler's opinion unified classical theory of electrodynamics and gravity can be full described by equation (13) and Rainich conditions:

$$R^\alpha_\mu R^\nu_\alpha = \frac{1}{4}\delta^\nu_\mu R^\sigma_\tau R^\tau_\sigma, \tag{14}$$

under positive energy condition: $R^0_0 \leq 0$ and $R^\sigma_\tau R^\tau_\sigma \neq 0$.

In such form *geometrodynamics does not contain any coupling constant.*

It is necessary to note that perhaps it is better to replace the equation (13) by equation

$$R_{\mu\nu;\tau} = R_{\mu\tau;\nu}$$

of paper ([11]). The constant of integration α_0 must be chosen as $\alpha_0 = \pi/2$.

4 $SO(3,1)$-gauge gravity vacuum and GR. Hyperbolical instantons

Let us consider the gauge gravity theory with $g_{\mu\nu}$ and $\Delta_\mu(ik)$ as field variables.

Equation system (2)-(3) is analogous of the system (6)-(7) of geometrodynamics and (8)-(9) of any gauge field in Riemannian V_4. Such a theory takes into account the extend of real objects and describes the real gravity forces acting on them, i.e. tidal forces. In this theory vacuum can be defined by condition $T_{\mu\nu} = 0$ as it is in GR. Then gravity vacuum definiton is:

$$T^{(g)}_{\mu\nu} = \kappa(R_{\mu\tau}(ik)R^\tau_\nu(ik) - \frac{1}{4}g_{\mu\nu}R_{\lambda\tau}(ik)R^{\lambda\tau}(ik)) = 0. \tag{15}$$

Besides trivial solution $R_{\mu\nu}(ik) = 0$ equation (15) has nontrivial solutions. These solutions obey the duality equations, which in electrodynamics have the following form:

$$F_{\mu\nu} = \pm i^* F_{\mu\nu}$$

Nontrivial solutions of duality equations are named the instantons. They minimize the action integral $S = \int F_{\mu\nu}F^{\mu\nu}dV$ and transform it into the topological constant.

In the case of the gauge field the instantons are nontrivial solutions of the equations $T^{(gf)}_{\mu\nu} = 0$ and duality equations

$$F^a_{\mu\nu} = \pm i^* F^a_{\mu\nu}$$

They minimize the action integral $S = \int F^a_{\mu\nu}F^{\mu\nu}_a dV$ and transform it into the topological constant.

In the case of the gauge gravity the equation $T^{(g)}_{\mu\nu} = 0$ implies arising of the vacuum state of the real gravity and the transition to GR. All solutions of Einstein equations are the solutions of the gauge gravity equations. But instead of duality equations

$$R^{\tau\lambda}_{\mu\nu} = \pm i^* R^{\tau\lambda}_{\mu\nu} \tag{16}$$

we have twice dual equations

$$R^{\tau\lambda}_{\mu\nu} = \pm^* R^{*\tau\lambda}_{\mu\nu} \tag{17}$$

and therefore

$$R_{\tau\nu} = \pm^* R^*_{\tau\nu} \tag{18}$$

The duality equations (16) which are analog of electromagnetic conditions of duality have only trivial solutions in the case of gravity (Euclidean V_4).

Taking into account that equations (3) followed by $R = 0$ we can transform them to the form ([11])

$$R^\mu_\nu = -\kappa(R^{\mu\sigma\tau\lambda} -^* R^{*\mu\sigma\tau\lambda})(R_{\nu\sigma\tau\lambda} +^* R^*_{\nu\sigma\tau\lambda}) \tag{19}$$

Therefore $T^{(g)}_{\mu\nu} = 0$ if either

$$R^{\tau\lambda}_{\mu\nu} = +^* R^{*\tau\lambda}_{\mu\nu} \quad \text{and} \quad R_{\tau\nu} = +^* R^*_{\tau\nu} = R_{\tau\nu} - \frac{1}{2}g_{\tau\nu}R \longmapsto R = 0 \tag{20}$$

and we have not any new solution, or

$$R^{\tau\lambda}_{\mu\nu} = -^* R^{*\tau\lambda}_{\mu\nu} \quad \text{and} \quad R_{\tau\nu} = -^* R^*_{\tau\nu} \longmapsto R_{\tau\nu} = 0 \tag{21}$$

that is Einstein gravity.

Hence we have vacuum Einstein equations which solutions are gravity instantons by definition in the frame of (GR+$SO(3,1)$)-gauge gravity theory. Therefore all solutions of GR-equations describe the vacuum structure of the gauge gravity theory and Schwarzschild solution is one of them. The hyperbolical signature is not an obstacle to being instanton. All vacuum Einstein spaces are the hyperbolical instantons ([12]).

Thus it is shown that the gravity (including Einstein's GR) has to be consider the gauge field in the single scheme with other interactions and quantization procedure has to be analogous to that of any nonabelian gauge field. It is necessary to note that under condition $T_{\mu\nu} = 0$ we obtain always the Einstein gravity vacuum equation independently of the gauge field type. Thus all gauge field instantons can take part in creation of space-time vacuum structure.

*Acknowledgments.*It is a pleasure to thank Professors Novikov S.P., Kadyshevsky V.G., Shirkov D.V., Baldin A.M., Burov V.V., Filippov A.T. and Desiatnikov I.I. for their support of this investigation. This investigation was partically supported by RFFI.

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Received by Publishing Department
on July 10, 1998.

Коноплева Н.П. E4-98-207

Общая теория относительности
и калибровочные теории гравитации высшего порядка

Представлен краткий обзор современных калибровочных теорий гравитации и их взаимосвязей с общей теорией относительности Эйнштейна. Анализируются концепции, используемые при построении калибровочных теорий гравитации с высшими производными. ОТО рассматривается как калибровочная теория гравитации, соответствующая выбору $G_{\infty 4}$ в качестве локальной группы симметрии и симметричного тензора второго ранга $g_{\mu\nu}$ в качестве полевой переменной. Используя единый для всех фундаментальных взаимодействий математический аппарат (а именно, вариационный формализм для бесконечных групп Ли), можно получить теорию Эйнштейна как калибровочную теорию без каких-либо изменений. Все другие калибровочные подходы ведут к неэйнштейновым теориям гравитации. Но вышеупомянутая математическая техника позволяет строить и калибровочные теории гравитации высшего порядка (например, $SO(3,1)$-гравитацию) таким образом, что все вакуумные решения уравнений Эйнштейна являются решениями $SO(3,1)$-гравитационной теории. Структура уравнений $SO(3,1)$-гравитации становится аналогичной структуре уравнений геометродинамики Уилера–Мизнера.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1998

Konopleva N.P. E4-98-207

General Relativity and Gauge Gravity Theories of Higher Order

It is a short review of today's gauge gravity theories and their relations with Einstein General Relativity. The conceptions of construction of the gauge gravity theories with higher derivatives are analyzed. GR is regarded as the gauge gravity theory corresponding to the choice of $G_{\infty 4}$ as the local gauge symmetry group and the symmetrical tensor of rank two $g_{\mu\nu}$ as the field variable. Using the mathematical technique, single for all fundamental interactions (namely variational formalism for infinite Lie groups), we can obtain Einstein's theory as the gauge theory without any changes. All other gauge approaches lead to non-Einstein theories of gravity. But above-mentioned mathematical technique permits us to construct the gauge gravity theory of higher order (for instance $SO(3,1)$-gravity) so that all vacuum solutions of Einstein equations are the solutions of the $SO(3,1)$-gravity theory. The structure of equations of $SO(3,1)$-gravity becomes analogous to Weeler–Misner geometrodynamics one.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1998