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INQUIRY FOR THE CONVERSION OF THE $(\pi^+ - \pi^-)$ BOUND STATE INTO TWO π^0

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1. Introduction. Agenda of $\pi\pi$-interaction

At present, the stringent knowledge of $\pi\pi$-interaction is well understood to be of fundamental value in its own right as well as for the reliable treatment of the various phenomena, where pionic degrees of freedom prove to be substantial. Pion being the lightest and, properly speaking, simplest among strong-interacting particles, an inquiry into the pion-pion interaction spreads the way to visualization of the main features of hadron interactions in general, and in all their immense complexity [1–5]. At the same time, the pion-pion interactions are bound to be allowed for in describing the hot and dense hadronic systems abundant in pions which are known to be produced in colliding heavy ions [6] at high enough incident energies, the baryon number being rather negligible when compared with the number of genuine mesons. Even so, in treating the nuclear matter at large density and temperature, the phenomena non-linear in meson fields, that is the meson-meson interactions, are realized to play a crucial role, especially when the feasible phase transitions caused by the softening of the mesonic degrees of freedom are investigated [7]. Thus, to repose full confidence in the adequacy of our perception of such systems behaviour, the pion-pion interaction must be properly accounted for, in particular, when calculating the respective thermodynamic characteristics. Thereby, in all the cases, we must certainly conceive the pion-pion interaction to be provided by well specified trustworthy lagrangian, but not in the least simply just by the pion-pion scattering lengths.

Nowadays, in the lack of the pion interactions description strictly worked out from the first principles, we are in possession of the pion-pion interaction lagrangians [1–5] which are thought to be as good as effective, obtained in the framework of some plausible models, QCD-motivated at best. Consequently, there is to appeal to the experimental investigations from which the reliable information about the $\pi\pi$-interaction can be disentangled. Then, confronting the results of experimental data processing and theoretical calculations, we can test the validity of a certain $\pi\pi$-interaction description and subsequently improve the latter.

Up to now, the trustworthy cognizance concerning the $\pi\pi$-interaction has been acquired, strictly speaking, solely from the analysis of the data obtained in the $\pi N \to \pi\pi N$ reaction which was studied for the first time as far back as in 1965 [8] near the threshold ($\varepsilon_\pi \sim 200 - 300\ MeV$) and afterwards for manifold incident pion energies, up to

$\varepsilon_\pi \sim 1 - 2\ GeV$ as well (see, for instance, [9,10]). The results of profound processing these experimental data carried out in the series of investigations [9–11] make us visualize that the effective lagrangians asserted in [1–3] are thought to be expedient to describe the $\pi\pi$-interaction, at least at low and middle pion energies, $\varepsilon_\pi \sim m_\pi$. Unfortunately, the unavoidable involvement of strong pion-nucleon interactions in such a process puts a bound to the attainable reliability of the pure $\pi\pi$-interaction description because, on one hand, it is as good as impossible to get rid of the strong πN-interaction effect in the experimental measurements, and, on the other hand, one will scarcely maintain that a theoretical calculation can refine unambiguously the $\pi\pi$-interaction from the πN interactions in the treatment of the reaction $\pi N \to \pi\pi N$. Thus, the further development of the $\pi\pi$-interaction description by means of the far more complex effective lagrangians [4,5], or may be according to other approaches (see, for instance, [12]), calls for new experiments. For that matter, at first thought, the K_{e4}-decay, $K \to \nu e\pi\pi$, [13] might appear to be fruitful to learn directly the pure $\pi\pi$-interaction occurring in the final state, but one should realize that the semi-leptonic-decay vertex itself is not concisely known, the strong interactions being implicated therein as well, and needs to be approved in its own right [14]. Thus, as yet, the reaction $\pi N \to \pi\pi N$ was and remains, as a matter of fact, the unique source of the data to check our concept of the $\pi\pi$-interaction.

In the light of the aforesaid, the advent of the experiments dealing with the pure $\pi\pi$-interaction, without the imposition of other strong (or weak) interactions, proves to be extremely desirable.

2. Pionium treatment up to now

Long since, the inquiry into the properties of the $\pi^+\pi^-$ bound state, pionium, have been understood of being very instructive to study the pure $\pi\pi$-interaction, free of effect of any other strong or weak interactions [15]. The feasible measurement of the pionium lifetime having been first considered in the early investigations [15], the setting up of the corresponding experiments has been elaborated profoundly in Refs. [16,17], and the respective investigations are for now already under way [17], the results are liable to arrive in the nearest future.

Pionium typifies the bound hadron systems which owe their origin to electromagnetic interactions, but whose decay is, as a matter of fact, caused by strong interactions. All the time ago, as far back as in 1954, the handy semiquantitative approach to treat such

systems was set out [18], with the strong interaction corrections to the energy levels and wave functions of the π-atom, the $\pi^- P$ bound state, as well as the transition rate $\pi^- P \to \pi^0 N$ being expressed through the free pion-nucleon scattering lengths a_L^T and the π-atom wave function at the origin $\psi(0)$. Here, T, L indices denote various isotopic and angular states. Subsequently, following this method, the pionium lifetime (i.e. the $\pi^+\pi^- \to \pi^0\pi^0$ reaction rate) in the ground state was asserted in Refs. [15] to be the simple plain function

$$\tau^{-1} = \frac{16\pi}{9}\sqrt{\frac{2\Delta m}{m}}|a_0^0 - a_0^2|^2 \cdot |\psi(0)|^2 \tag{1}$$

of the s-wave $\pi\pi$-scattering lengths a_0^0, a_0^2, the pionium wave function at the origin $\psi(0)$, and the mass difference $\Delta m = m - m_0$, m being charged pion mass. Thus, if the original approach of Ref. [18] had been strictly valid in the pionium case, all we need to precisely calculate the pionium lifetime would have been the exact values of the quantities $|a_0^0 - a_0^2|$, $|\psi(0)|$, and Δm. It is to take cognizance of the fact that only the difference of the scattering lengths would have come into picture, regardless of the complete form of the genuine $\pi\pi$-interaction. This is due to the main original presumption of the approach of Ref. [18] that irrespective to the $\pi\pi$-interaction form the calculation of probability of the pionium decay into two π^0 is quite equivalent to the calculation of the annihilation probability of a free pair $\pi^+\pi^-$ with zero momenta into two π^0, $\pi^+\pi^- \to \pi^0\pi^0$, with the initial density of states being not the density of states of free particles, but the density of states of the particles in the bound state of pionium $|\psi(0)|^2$. Up to now, the authors of all the succeeding investigations [19–26] have been taking for granted that the pionium lifetime formula (1) as asserted according to [18] in Ref. [15] holds true strictly, and all the efforts were devoted to acquire somehow the precise values of the quantities a_L^T, $\psi(0)$, with the pure point-like Coulomb nonrelativistic $\psi(0)$ value and the free particle scattering lengths a_L^T values gained according to Refs. [1–4] being assumed as a starting point in all the calculations. Then, there was to calculate the corrections to that $\psi(0)$ value, especially due to strong interactions, and simultaneously the a_L^T-modifications on account of strong and electromagnetic interactions in the coupled $\pi^0\pi^0, \pi^+\pi^-$ channels.

In several investigations [19–24] various effective potentials were managed to describe this strong $\pi\pi$-interaction. The most profound calculations within such a potential approach were carried out in Ref. [21] and especially in [22], where the aforesaid corrections were thoroughly calculated in the framework of the model of the two-channels $\pi^0\pi^0, \pi^+\pi^-$

system, with the effective range approximation being used to account for the strong pion-pion interaction. Thereby, once an effective radius is chosen (equal in both channels), the strong potentials in the channels are determined merely just by the corresponding scattering lengths a_0^T. In such a calculation, the electromagnetic corrections are due the different masses of the pions in the different channels along with the Coulomb interaction imposition in the $\pi^+\pi^-$ channel. The coupled Schrödinger equations determining the pion wave functions in the coupled channels having been solved, the corrected, generalized scattering lengths, as well as the appropriately corrected $\psi(0)$ values are obtained, which must be substituted in the original formula (1) for τ to acquire its eventual corrected value. The scrutinized corrections to a_L^T values (and to $\psi(0)$ sa well) proved to amount no more than a few percents, being substantially less than the uncertainties in the a_L^T predictions following from Ref. [4], as the authors of [22] have inferred.

Unlike the effective potential approach of the Refs. [19–24], the investigation [25] utilized the Bethe-Solpeter equation to allow for the effect of strong interactions on the $\psi(0)$ value in the pionium lifetime (1) (via the pionium eigenstate energy shift ΔE), the corrections proving to be rather negligible.

The τ (1) value modification on account of pionium relativistic treatment, especially the allowance for the retardation effect in the $\pi^+\pi^-$ electromagnetic interaction, has been found $\sim 1\%$ in Ref. [26]. Thereby, the scattering lengths difference $a_0^0 - a_0^2$ was presumed to render the total strong interaction responsible of the $\pi^+\pi^- \to \pi^0\pi^0$ transition, likewise in all the aforecited investigations [19–24], in spite of treating the retardation effect in the $\pi^+\pi^-$ system which implies the $\pi^+\pi^-$ relative velocity to be comparable with light velocity c.

Profound as are all the afore discussed calculations of the quantities $a_0^T, \psi(0)$, we ought to realize that the expression (1) by itself, in so far as it originates from the very plausible, but semiquantitative approach [18], is, properly speaking, as good as semiquantitative in its turn. But this does not mean to say that any results obtained according to the method set out in Ref. [18] must be regarded as untenable and scarcely able to describe experimental data with high enough accuracy. There is to visualize that the validity and accuracy of this very approach are caused crucially by the form of the genuine strong interaction inducing the bound hadronic system decay in each certain treated case. The very germ of the idea set forth in Ref. [18] makes us comprehend that the approach of

[18] itself will hold true with high precision, if the hadron-hadron interaction is as good as point-like and constant, especially momentum-independent, which is thought to be well acceptable for the $P\pi^-$-interaction in the s-state in [18], but not in the least for the $\pi\pi$-interactions asserted and used in Refs. [1–4, 9–12]. Consequently, since the pionium properties are studied, we must refrain from pursuing the way paved in Ref. [18] and abandon, in turn, the handy expression (1) for the pionium lifetime.

3. Interactions inducing the pionium decay into two π^0

According to our lights, the general aim of the theoretical investigations of the pionium lifetime is to visualize whether a certain form of the $\pi\pi$-interaction is eligible to provide the experimental τ value. In the work presented, we set out the calculation of τ, with the $\pi\pi$-interaction being determined by the Weinberg lagrangian according to Refs. [1–3]. The probability of two-photon pionium annihilation, $\pi^+\pi^- \to 2\gamma$, being practically negligible when compared with the decay probability due to the strong interaction, will not be discussed henceforth.

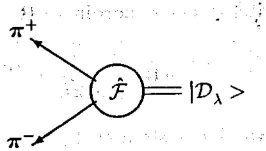
We treat pionium as the beforehand prepared $\pi^+\pi^-$ bound state which is stable when the strong interaction of pion fields is turned off. The coupling of this state, the pionium field, to the charged (complex) pion field is implemented via the virtual decay of the $\pi^+\pi^-$ bound state $|\mathcal{D}_\lambda>$, pionium or di-meson, into a free $\pi^+\pi^-$ pair :

$$\pi^+ + \pi^- \leftarrow |\mathcal{D}_\lambda> \tag{2}$$

In our nowaday consistently nonrelativistic approach, we presume that the formation of the initial $\pi^+\pi^-$ bound state $|\mathcal{D}_\lambda>$ is caused by pure-instantaneous potential interaction $U(\mathbf{y}_1, \mathbf{y}_2)$, where $\mathbf{y}_1, \mathbf{y}_2$ are the spatial coordinates of the $\pi^+(\mathbf{y}_1, t), \pi^-(\mathbf{y}_2, t)$ mesons composing the pionium, the time coordinates coinciding. Accordingly, the vertex operator

$$\hat{\mathcal{L}}_\mathcal{D} = -[\pi^+(\mathbf{y}_1, t)\pi^-(\mathbf{y}_2, t) + \pi^-(\mathbf{y}_1, t)\pi^+(\mathbf{y}_2, t)]\hat{\mathcal{F}}(\mathbf{y}_1, \mathbf{y}_2, t), \tag{3}$$

$$\hat{\mathcal{F}}(\mathbf{y}_1, \mathbf{y}_2, t) = \sum_\lambda [c_\lambda \mathcal{F}_\lambda(\mathbf{y}_1, \mathbf{y}_2, t) + c_\lambda^\dagger \mathcal{F}_\lambda^*(\mathbf{y}_1, \mathbf{y}_2, t)] \tag{4}$$



renders the virtual pionium state $|\mathcal{D}_\lambda>$ decay into a free $\pi^+\pi^-$ pair. Here, $\pi^\pm(\mathbf{y}, t)$ are the charged pion field operators, whereas $\hat{\mathcal{F}}(\mathbf{y}_1,\mathbf{y}_2,t)$ stands for the pionium field, the quantities c_λ, c^+_λ being the pionium production and distraction operators in the state λ. So far as the interaction $U(\mathbf{y}_1,\mathbf{y}_2)$ is instantaneous, the operators of all the fields in (3, 4) act at the same time point t. In our calculations, the common relations are adopted

$$\pi^+(x) = \frac{1}{\sqrt{2}}(\pi_1(x)+i\pi_2(x)),\ \ \pi^-(x) = -(\pi^+(x))^*,\ \ \pi^0(x) = \pi_3(x),$$

$$\pi^+(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\varepsilon_{\mathbf{p}}}}[a_{\mathbf{p}}e^{-it\varepsilon_{\mathbf{p}}+\mathbf{px}} + b^+_{\mathbf{p}}e^{it\varepsilon_{\mathbf{p}}-\mathbf{px}}], \tag{5}$$

with the operator $a_{\mathbf{p}}$ destructing π^+-meson and $b^+_{\mathbf{p}}$ producing π^--meson. The vertex functions $\mathcal{F}_\lambda(\mathbf{y}_1,\mathbf{y}_2,t)$ in (4) and the corresponding pionium eigenenergies E_λ in the states λ are well known (see, for instance, Refs. [27,28]) to be determined by the homogeneous Bethe-Solpeter equation

$$\mathcal{F}_\lambda(\mathbf{y}_1,\mathbf{y}_2,t) = U(\mathbf{y}_1,\mathbf{y}_2)\cdot\int dt'\int d\mathbf{y}'_1\int d\mathbf{y}'_2 D(y_1-y'_1)D(y_2-y'_2)\mathcal{F}_\lambda(\mathbf{y}'_1,\mathbf{y}'_2,t'), \tag{6}$$
$$y'_{10} = y'_{20} = t',$$

where

$$D(x) = \frac{1}{i(2\pi)^4}\int \frac{d^4k\cdot e^{ikx}}{k^2-m^2+i\delta} \tag{7}$$

is the usual pion propagator. In the presumed non-relativistic approach, the vertex function $\mathcal{F}_\lambda(\mathbf{y}_1,\mathbf{y}_2,t)$ proves to be reduced as follows (see, for instance, Refs. [27,28] and also [29])

$$\mathcal{F}_\lambda(\mathbf{y}_1,\mathbf{y}_2,t) = -i\mathcal{N}\cdot U(\mathbf{y}_1,\mathbf{y}_2)\cdot\Phi_\lambda(\mathbf{y}_1,\mathbf{y}_2,t), \tag{8}$$

where $\Phi_\lambda(\mathbf{y}_1,\mathbf{y}_2,t)$ is the non-relativistic $\pi^+\pi^-$ system wave function. The function $\mathcal{F}_\lambda$ being determined by the homogeneous equation (6), the normalization factor $\mathcal{N}$ emerges in (8) whose calculation we defer for a while (Sec. 4). The wave function $\Phi_\lambda(\mathbf{y}_1,\mathbf{y}_2,t)$ of such a nonrelativistic system is known (see, for instance, [30]) to be the product

$$\Phi_\lambda(\mathbf{y}_1,\mathbf{y}_2,t) = \psi_{nl}(\mathbf{z})\cdot\Psi_{\mathbf{P}}(\mathbf{R})\cdot e^{-itE_\lambda},\ \ E_\lambda = 2m+\frac{\mathbf{P}^2}{4m}+\varepsilon_{nl},\ \lambda=(nl,\mathbf{P}) \tag{9}$$

of the depending on the center of mass coordinate $\mathbf{R} = (\mathbf{y}_1+\mathbf{y}_2)/2$ wave function

$$\Psi_{\mathbf{P}}(\mathbf{R}) = \frac{1}{\sqrt{2E_\lambda}}e^{i\mathbf{RP}} \tag{10}$$

of the free motion of the two-pion system as a whole with the total momentum $\mathbf{P}$, and the intrinsic pionium wave function $\psi_{nl}(\mathbf{z})$ depending on the relative $\pi^+\pi^-$ coordinate

$\mathbf{z} = \mathbf{y}_1 - \mathbf{y}_2$. The functions ψ_{nl} simultaneously with the pionium energy levels ε_{nl} are determined by the Schrödinger equation [30]

$$-\frac{1}{m}\nabla^2\psi_{nl}(\mathbf{z}) + U(\mathbf{z})\psi_{nl}(\mathbf{z}) = \varepsilon_{nl}\psi_{nl}(\mathbf{z}) \tag{11}$$

with the relevant boundary conditions at $z=0, z\to\infty$. Here $m = 139.57MeV$ is the $\pi^\pm$-meson mass [31] . We utilize the units $c = h = 1$. For the pure-Coulomb point-like interaction

$$U(z) = -\frac{\alpha}{z} \tag{12}$$

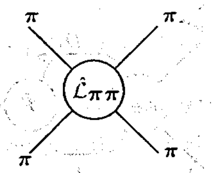
the ground state wave function $\psi_{10}\equiv\psi$, properly normalized, and energy $\varepsilon_{10}\equiv\varepsilon$ are known [30] to be

$$\psi(z) = \frac{1}{\sqrt{4\pi}}\cdot\sqrt{\frac{a^3}{2}}e^{-za/2},\ \ \varepsilon = -\frac{m\alpha^2}{4}, \tag{13}$$

where $a = m\alpha$ and $2/a$ is the "Bohr radius". Consequently, we denote $|\mathcal{D}_{10}>\equiv|\mathcal{D}>$. In what follows, we consider this pionium ground state decay. The $\pi\pi$-interaction of the type [1–3] including the dependence on the pion momenta being put to use in our further calculations, the finite pion size r_0 emerges to come into the picture, which we allow for in due course replacing (12) by the electrostatic potential between two homogeneously charged spheres, z being the distance between their centers, the explicit expression for which, a bit long, is set out in Ref. [32]. The magnitude of the quantity r_0 itself has been estimated in some theoretical and experimental investigations [33,34], whereby we have adopted $r_0 = 0.6fm$ as realistic. It might be well to note that the calculations with the generalized, but yet instantaneous potential accounting for the relativistic corrections up to $(1/c^2)$-order (the kind of the Breit potential [27,35]) would not provide the additional difficulties of principle.

In our present calculation, the $\pi\pi$-interaction inducing the $\pi^+\pi^-\to 2\pi^0$ transition is specified by the well known Weinberg lagrangian

$$\hat{\mathcal{L}}_{\pi\pi}(x) = -\frac{1}{(2f_\pi)^2}[\partial_\mu\pi(x)\partial^\mu\pi(x) - \beta m^2(\pi(x))^2]\pi^2(x) \tag{14}$$



elaborated and scrutinized in Refs. [1-3]. Here $f_\pi = 92.4MeV$ [31]. The dependence of the results of calculations on the parameters $\beta, \bar{m}$ in the term violating the chiral symmetry will be discussed in the last Section.

Let us recall that the validity of the lagrangian (14) has been inferred from processing the experimental data on the $N\pi \to N\pi\pi$ reaction, see Refs. [8–11], at least for not very high pion energies.

The difference of the masses of a charged pion, $m = 139.57MeV$, and a neutral one, $m_0 = 134.98MeV$, $\Delta m = m - m_0 = 4.59MeV$ being greater than the pionium binding energy ε, the initial $\pi^+\pi^-$ bound state $|\mathcal{D}>$ transition into the final two π^0 state turns out to be possible via the processes presented by (3, 14). All the effective interactions between the pion (charged and neutral) and the pionium fields are described by the total interaction lagrangian

$$\hat{\mathcal{L}}_{tot} = \hat{\mathcal{L}}_{\mathcal{D}} + \hat{\mathcal{L}}_{\pi\pi}, \tag{15}$$

which determines eventually the pionium lifetime τ.

4. Pionium decay amplitude

The matrix element

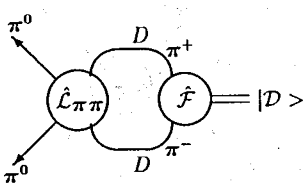
$$\mathcal{S}_{\pi^0\pi^0\mathcal{D}} = <\pi^0\pi^0|\hat{\mathcal{S}}|\mathcal{D}> \tag{16}$$

of the $\hat{\mathcal{S}}$-matrix dictated by the lagrangian (15) determines the initial pionium state $|\mathcal{D}>$ decay into two final π^0. To the first order in $\hat{\mathcal{L}}_{\pi\pi}$ (14), the $\mathcal{S}$-matrix element (16) takes the form (see, for instance, [27, 28])

$$\mathcal{S}^1_{\pi^0\pi^0\mathcal{D}} = -\int d\mathbf{R}\int d\mathbf{z}\int dt\int d^4x <\pi^0\pi^0|\hat{T}[\hat{\mathcal{L}}_{\mathcal{D}}(\mathbf{R},\mathbf{z},t)\cdot\hat{\mathcal{L}}_{\pi\pi}(x)]|\mathcal{D}>=$$

$$= \frac{i\mathcal{N}8}{(2f_\pi)^2\,2\sqrt{2E_\lambda\varepsilon_1\varepsilon_2}}\int d\mathbf{R}\int d\mathbf{z}\int dt\int d^4x U(\mathbf{z})\psi_\lambda(\mathbf{z})\{2\beta\bar{m}^2 - (\varepsilon_1\varepsilon_2 - \mathbf{p}_1\mathbf{p}_2) + \partial_{x\mu}\partial^\mu_x\}\times$$

$$\times D(\mathbf{R}+\mathbf{z}/2-\mathbf{x}, t-x_0)\cdot D(\mathbf{R}-\mathbf{z}/2-\mathbf{x}, t-x_0)\cdot e^{-itE_\lambda + i\mathbf{PR}}\cdot e^{ix_0(\varepsilon_1+\varepsilon_2) - i\mathbf{x}(\mathbf{p}_1+\mathbf{p}_2)}, \tag{17}$$



where $\hat{T}$ is the usual time-ordering operator and $\varepsilon_{1,2}, \mathbf{p}_{1,2}$ denote the energies and momenta of the final π^0. Certainly, when necessary, the high $\hat{\mathcal{L}}_{\pi\pi}$-order contributions in (16) could be allowed for in the usual way. These terms, if calculated, would render, in particular, the effect of strong $\pi\pi$-interaction on the pionium state. In the course of our today's calculations, we restrict ourselves by accounting for the first $\mathcal{L}_{\pi\pi}$-order. If anything, it may be well to recall that the analysis of the $N\pi \to N\pi\pi$ reaction was carried out in Refs. [8–11], as a matter of fact, in the same first order in $\mathcal{L}_{\pi\pi}$ approximation. For the ground state pionium decay at rest, the relations hold

$$\mathbf{P} = 0,\ E_{n\mathbf{P}} = E = 2m + \varepsilon,\ \varepsilon_1 = \varepsilon_2 = \varepsilon_0 = E/2,$$
$$\mathbf{p}_1 = -\mathbf{p}_2,\ |\mathbf{p}_1| = |\mathbf{p}_0| = p_0 = \sqrt{(E/2)^2 - m_0^2}\,, \tag{18}$$

and the Eq. (17) is reduced as follows

$$\mathcal{S}^1_{\pi^0\pi^0\mathcal{D}} = i(2\pi)^4\cdot\mathcal{T}_{\pi^0\pi^0\mathcal{D}}\cdot\delta(\mathbf{p}_1+\mathbf{p}_2)\delta(\varepsilon_2+\varepsilon_1-E)\,,$$

$$\mathcal{T}_{\pi^0\pi^0\mathcal{D}} = \frac{1}{(2\pi)^4}\frac{8\mathcal{N}}{(2f_\pi)^2E\sqrt{2E}}\int d\mathbf{z}\cdot U(\mathbf{z})\cdot\psi(\mathbf{z})\times \tag{19}$$

$$\times\int d^4q\frac{-2\beta\bar{m}^2 + m_0^2 - E^2/2 + q_0^2 - \mathbf{q}^2 - q_0E}{[q_0^2 - \mathbf{q}^2 - m^2 + i0]\cdot[(E-q_0)^2 - \mathbf{q}^2 - m^2 + i0]}\cdot e^{-i\mathbf{zq}}.$$

It is noteworthy that the quantities $\mathbf{q}^2, q_0^2$ emerge in the nominator in (19) due to the term

$$(\partial_\mu\pi\cdot\partial^\mu\pi)(\pi)^2$$

in the $\pi\pi$-interaction (14), this fact substantially affected the integrand behaviour in (19), especially at extremely large q values. Integrating over dq_0 and over the directions of the vectors $\mathbf{q}$ and $\mathbf{z}$ having been carried out, the Eq. (19) reduces to

$$\mathcal{T}_{\pi^0\pi^0\mathcal{D}} = \frac{-i8\mathcal{N}}{\pi(2f_\pi)^2E\sqrt{2E}}\int_0^\infty\frac{q\,dq}{\omega(q)}\cdot[1 - \frac{b}{q^2+c^2}]\cdot\int_0^\infty dz\cdot U(z)\cdot z\cdot\psi(z)\cdot sin(qz), \tag{20}$$

where the following notations are introduced:

$$\omega(q) = \sqrt{q^2+m^2},\ c^2 = m^2 - (E/2)^2,\ b = (-2\beta\bar{m}^2 + m_0^2 + m^2 - E^2)/2.$$

It is not difficult ot realize that the behaviour of the integrand in (20) at large momenta, $q \to \infty$, and subsequently the convergence of the integral in (20) itself are governed by the behaviour of the quantity $zU(z)\psi(z)sin(qz)$ when the z value tends to zero, $z \to 0$.

There is to calculate the contributions arising from two terms in brackets in the integrand (20): from "unit", 1, and from $b/(q^2+c^2)$. First, we take up integrating the term with "unit" and then set out the integral with the quantity $b/(q^2+c^2)$.

Not hard thing is to become convinced that the integral in (20) with "unit" in brackets would diverge logariphmically, if the pure point-like Coulomb values (12), (13) were adopted for the quantities $U(z), \psi(z)$ in (20). This divergency emerges because a pion size is neglected. To remove this puzzling, but spurious contradiction we allow for the finite pion size r_0, $r_0 a \ll r_0 m \ll 1$ in the course of calculating this integral, $U(z)$ being the electrostatic potential between two homogeneously charged spheres of the radius r_0 [32], as discussed already after Eq. (13). Then, on integrating over dq, the integral in (20) originating due to the "unit" in brackets transforms to (see Ref. [36])

$$-\int_0^\infty dz\cdot zU(z)\psi(z)\frac{d}{dz}K_0(mz) = -zU(z)\psi(z)K_0(mz)\Big|_0^\infty + \int_0^{2r_0} dzK_0(mz)\psi(z)\frac{d}{dz}[zU(z)] + \alpha\int_{2r_0}^\infty dzK_0(mz)\frac{d}{dz}\psi(z) + \int_0^{2r_0} dzK_0(mz)zU(z)\frac{d}{dz}\psi(z), \quad (21)$$

where $K_0(z)$ is the Mackdonald's cylindrical function (see [36]). The first term in the righthand side (21) vanishes due to $\psi(z)\sim e^{-az/2}\to 0$ when $z\to\infty$, and it disappears at $z=0$ owing to $zU(z)=0$ at $z=0$ because, in turn, the potential $U(z)$ has got at $z=0$ a finite value $U(0)$ for the charged particles of the finite size, in particular, for the afore adopted potential of the homogeneously charged spheres $U(0) = -6\alpha/(5r_0)$. Further, in our treatment, we are on the point to carry out all the calculations in the lowest α-order. All the expression (20) (as well as (21)) is proportional to $\alpha\sqrt{\alpha^3}$ due to the α-dependence of the functions U, ψ. Calculating the integrals in (20), (21), we retain only the terms which besides this α-dependence are inversely proportional to α, $\sim 1/\alpha$, and α-independent. Even so, we retain only the terms $\sim ln(r_0)$ in the asymptotic expansion in r_0, but drop out the terms $\sim r_0^n, n\geq 1$. Consequently, the second and third integrals in the righthand side in (21) are realized to be neglected. Indeed, at $z\geq 2r_0$ in the second integral, the function $\psi(z)$ behaves like (13), $\sim e^{-za'/2}$, the quantity a' being of the same order in α as $a, a'\approx a = m\alpha$ (see, for instance, Refs. [32,37]). Then, we have got $d\psi(z)/dz\sim \alpha m\psi(z)/2$, and, subsequently, this integral gets the additional factor of α and can be dropped out. Then, since the function $\psi(z)$ in the third integral in the righthand side of (21), i.e. for $z\leq 2r_0 \ll 2/a$, varies smoothly (see, for instance, Refs. [32,37]), $\psi(z)\sim\psi(0)(1+za'')$, where a'' is of the same order as a, the derivative $d\psi(z)/dz\approx\psi(0)\cdot a$, so the whole integral comes out to be $\sim \alpha r_0 a\psi(0)$ and can be omitted as well. Thus, eventually, there is to calculate the first integral in the righthand side of (21). Its upper limit turned out to be $2r_0$ because $\frac{d}{dz}[U(z)z]=0$ at $z\geq 2r_0$, $U(z)$ being the point-like Coulomb potential $-\alpha/z$ when $z\geq 2r_0$. For these z values, the relations $r_0 a\ll 1$, $r_0 m\ll 1$ being valid, the replacements hold true

$$\psi(z)=\psi(0), \quad K_0(mz) = -ln(mz/2) - C \quad (22)$$

with an accuracy up to order $\sim r_0 a$, $\sim r_0 m$. Here $C\approx 0.577$ is the Euler constant (see [36]). Then, with regard to the approximation (22), the first integral in the righthand side in (21) is calculated straightforward, and the whole expression (21) results in

$$\psi(0)[\alpha(ln(mr_0)+C)+\bar{U}], \quad \bar{U}=\int_0^{2r_0} dzU(z). \quad (23)$$

The quantity $\bar{U}$ is calculated according to Ref. [32] which gives $\bar{U}\approx -\alpha\cdot(3/2)$.

While treating the integral with the term $b/(c^2+q^2)$ within brackets in (20), the presence of an additional q^2 in the denominator provides this integral convergence even without allowance for the finite pion size r_0. This does mean to say the asymptotic expanding this integral in r_0 begins with the term $\sim r_0$ which is beyond our today's accuracy, as presumed above. Then, with the Eqs. (12, 13) being adopted, this integral in the lowest α-order transforms as follows:

$$\sqrt{\frac{a^3}{8\pi}}b\alpha\int_0^\infty\frac{dq\cdot q^2}{(q^2+(a/2)^2)(q^2+c^2)\omega(q)} = \sqrt{\frac{a^3}{8\pi}}\frac{b}{m}[\frac{\pi}{2}-\alpha]. \quad (24)$$

After all, with allowance for the results (23, 24), the transition amplitude (20) takes the form

$$T_{\pi^0\pi^0\mathcal{D}} = -\frac{i8\mathcal{N}}{\pi(2f_\pi)^2E\sqrt{2E}}\cdot\sqrt{\frac{a^3}{8\pi}}\cdot[\frac{b}{m^2}(\frac{\pi}{2}-\alpha)+\alpha(ln(mr_0)+C)+\bar{U}]. \quad (25)$$

The normalization factor $\mathcal{N}$ residing in the Eqs. (8, 17 - 20, 25) is to be determined by equating the energy E of the state $|\mathcal{D}>$ of pionium at rest and the expectation value in the $|\mathcal{D}>$-state of the operator of the $\hat{T}^{00}$-component of energy-momentum tensor of a charged (complex) pion field:

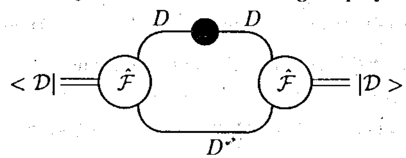
$$E = <\mathcal{D}|\hat{T}(\hat{T}^{00}\hat{S}_D)|\mathcal{D}>. \quad (26)$$

$$\hat{T}^{00}(\xi_0,\xi) = -[\frac{\partial\pi^-(\xi)}{\partial\xi_0}\cdot\frac{\partial\pi^+(\xi)}{\partial\xi_0}+\frac{\partial\pi^-(\xi)}{\partial\xi}\cdot\frac{\partial\pi^+(\xi)}{\partial\xi}+m^2\pi^-(\xi)\pi^+(\xi)]. \quad (27)$$

where the $\hat{S}_{\mathcal{D}}$-matrix is dictated by the lagrangian (3), so that

$$E = -\frac{1}{2} < \mathcal{D}|\,\hat{T}[\hat{T}^{00}(\xi_0,\xi) \int dy_1 dy_2 dt \hat{\mathcal{L}}_{\mathcal{D}}(y_1,y_2,t) \int dy_1' dy_2' dt' \hat{\mathcal{L}}_{\mathcal{D}}(y_1',y_2',t')]|\mathcal{D}>, \quad (28)$$

and, for clarity's sake, the expression is worth being displayed by the usual diagram



where the blob stands for the $\hat{T}^{00}$ operator. The values (12,13), asserted for the point-like pion, being adopted, the straightforward calculation of (28) results in

$$E = \frac{2\alpha^2 \mathcal{N}^2 a^3}{E\pi} \int_0^\infty \frac{dq \cdot q^2}{\omega(q)} \frac{E^2 + 4\omega^2(q)}{(q^2+(a/2)^2)^2 \cdot (4\omega^2(q) - E^2)}, \quad (29)$$

which apparently shows up no divergency when integrating over dq which is due to the integrand steep enough decrease at $q \to \infty$ on account, in turn, of the high power of q in the denominator of (29). When evaluating (29), we are to retain only the terms of the lowest α-order: the α-independent terms and terms $\sim \alpha$ (if they would have appeared), omitting the terms $\sim \alpha^n, n > 1$. Then, Eq. (29) reduces to

$$E = \frac{\mathcal{N}^2}{Em}, \quad \mathcal{N}^2 = 4m^3. \quad (30)$$

If anything, for verification's sake, the $\mathcal{N}$ value can be obtained by equating the expectation value of the particle number operator (related to zeroth component of charged pion field current)

$$\hat{N} = \sum_{\mathrm{p}} [a^+{}_{\mathrm{p}} a_{\mathrm{p}} + b_{\mathrm{p}}^+ b_{\mathrm{p}}]$$

(see Eqs. (5)) in the pionium state $|\mathcal{D}>$ and the number of pions $N = 2$, that is from the equation

$$< \mathcal{D}|\hat{T}(\hat{N}\hat{S}_{\mathcal{D}})|\mathcal{D}> = 2.$$

All the calculations having been carried out in due course, we arrive at the same $\mathcal{N}$ value (30). Let us take cognizance of the fact that the righthand side in Eqs. (27 - 29) proves to have got no terms $\sim \alpha$, its expansion in α starting with a term $\sim \alpha^2$. Evidently, it must be just so, because the quantity $E = 2m - m\alpha^2/4$ in the lefthand side does not include terms $\sim \alpha$.

5. The results of the pionium life-time calculation and concluding remarks

Thus, we have at our disposal the expression (25) for the transition amplitude with $\mathcal{N}$ defined by Eq. (30). Then, we acquire in the usual way (see, for instance, Ref. [27]) the total probability W of pionium conversion into two π^0, that is the inverse pionium lifetime τ

$$W = \frac{1}{\tau} = \frac{a^3 \cdot p_0 \cdot m^3 \cdot \tilde{b}^2}{(2f_\pi)^4 2\pi^2 E^2}[1 - \frac{4\alpha}{\pi}(1 - \frac{\bar{U}/\alpha + ln(mr_0) + C}{\tilde{b}})], \quad (31)$$

where $\tilde{b} = [-2\beta \cdot (\bar{m}/m)^2 + (m^0/m)^2 - 3]/2$ and all other quantities have been set forth above.

Let us now inquire into how the τ value (31) depends on the $\bar{m}, \beta$ values which reside in the chiral symmetry violating term in the lagrangian (14). Let firstly $\bar{m} = m_0$, then we gain for the β values $\beta = 1/2$, $\beta = 1/3$, $\beta = 1/4$ asserted in Refs. [1–3]:

$$\tau_{m_0,1/2} = 4.95 \cdot 10^{-15} sec, \quad \tau_{m_0,1/3} = 6.18 \cdot 10^{-15} sec, \quad \tau_{m_0,1/4} = 6.90 \cdot 10^{-15} sec.$$

Thus, the dependence of τ on β is thought to be sizeable, the deviations of these τ values from each other amounting to $\approx 15\%$. On the other hand, if we adopt $\bar{m} = m$ instead of $\bar{m} = m_0$, we shall have got

$$\tau_{m,1/2} = 4.71 \cdot 10^{-15} sec,$$

which deviates from $\tau_{m_0,1/2}$ by about 5%. Let us also note that the second term within the brackets in (31) amounts to $\approx 2\%$ to the whole W value.

Our result is thought to be not contradicting to the nowaday estimation $\tau = 2.9^{+\infty}_{-2.1} \cdot 10^{-15} sec$ set out in Ref. [17]. It might be instructive to recall that the results of τ calculation obtained in the previous investigations, surveyed in Section 2, appear to be somewhat smaller as compared to ours. For instance, the value $\tau = 2.72 \cdot 10^{-15} sec$ has been asserted in Ref. [21] and $\tau = 3.2 \cdot 10^{-15} sec$ in Ref. [23]. Especially, it is to stress that the eq. (1) with the a_L^T values from Refs. [1-3] corresponding to the very $\pi\pi$-interaction (14) gives $\tau(W) = 3.1 \cdot 10^{-15} sec$ instead of our values $\tau_{m,1/2} = 4.71 \cdot 10^{-15} sec$, or $\tau_{m_0,1/2} = 4.95 \cdot 10^{-15} sec$.

The investigation carried out makes us realize that the pionium lifetime (as well as its other properties) does depend crucially on the form of the genuine $\pi\pi$-interaction, but not much simply just on the free pions scattering lengths only. Thus, the pionium decay as being due to the most plausible concise Weinberg lagrangian (14) having been studied, the investigations pursuing other present-day trustworthy $\pi\pi$-interaction descriptions are very desirable and instructive. If the consistent τ calculation in the framework of a certain method of the $\pi\pi$-interaction description (see, for instance Refs. [4, 5, 12]) is carried out and, subsequently, its result is confronted to the experimental τ value, the validity of this method will come to light. In the course of our further pionium lifetime studying, we are on the point of inquiring into the various $\pi\pi$-interaction representations.

References

[1] S. Weinberg, Phys. Rev. Lett. 18 (1967) 188.
S. Weinberg, Physica A 96 (1979) 327.

[2] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.

[3] P. Chang and F. Gürsey, Phys. Rev. 164 (1967) 1752.
J. Schwinger, Phys. Lett. B 24 (1967) 473.
W. A. Bardin, L. Brown, B. W. Lee and H. T. Neih, Phys. Rev. Lett. 18 (1967) 1170.

[4] J. Gasser and H. Leutwyler, Ann. of Phys. 158 (1984) 142.
U.-G. Meissner, Rep. Prog. Phys. 56 (1993) 903.
G. J. Stephenson, K. Maltman and T. Goldman, Phys. Rev. D 43 (1991) 860.
J. Stern, H. Sazdjian and N. H. Fuchs, Phys. Rev. D 47 (1993) 3814.

[5] A. A. Andrianov, V. A. Andrianov, Yu. V. Novozhilov and V. Yu. Novozhilov, Phys. Lett. B 186 (1987) 401.

[6] A. Iyonov et al. ((EMU05) Collaboration), Nucl. Phys. A 544 (1991) 55c.
R. Rapp, J. Wambach, Phys. Rev. C 53 (1996) 3057.
G. G. Bunatian, J. Wambach, Phys. Lett. B 336 (1994) 290.
G. G. Bunatian, Sov. J. Nucl. Phys. 31 (1980) 631; 41 (1985) 560.

[7] G. G. Bunatian and I. N. Mishustin, Nucl. Phys. A 404 (1983) 525.

[8] Yu. A. Batusov, S. A. Buniatov, V. M. Sidorov and V. A. Yarba, Sov. J. Nucl. Phys. 1 (1965) 526.

[9] G. Ecker and J. Honerkampf, Nucl. Phys. B 62 (1973) 509; Nucl. Phys. B 52 (1973) 211.
H. Lehmann and H. Trute, Nucl. Phys. B 52 (1973) 280.
T. H. Truong, Phys. Lett. B 99 (1981) 154.
O. Jäckel, H.-W. Ortner and M. Dilling, Nucl. Phys. A 511 (1990) 733; Nucl. Phys. A 541 (1992) 675.

[10] B. R. Martin, D. Morgan and G.Shaw, Pion-Pion Interaction in Particle Physiks (Academic Press, New York, 1977).

[11] A. A. Bel'kov and S. A. Buniatov, Sov. J. Particles and Nuclei 13 (1982) 5.

[12] D. Lohse, J. W. Durso, K. Holinde and J. Speth, Nucl. Phys. A 516 (1990) 513.

[13] L. Rosselet, Phys. Rev. D 15 (1977) 574.

[14] E. D. Commins and P. H. Bucksbaum, Weak Interactions of Leptons and Quarks (Cam. Univ. Press, Cam., England, 1983).

[15] J. L. Uretsky and T. R. Palfrey, Phys. Rev. 121 (1961) 1798.
S. M. Bilenky, Hyuen Van Heu, L. L. Nemenov and F. G. Tkebuchava, Sov. J. Nucl. Phys. 10 (1969) 812.

[16] L. L. Nemenov, Sov. J. Nucl. Phys. 15 (1972) 1047; 41 (1985) 980.

[17] L. G. Afanasyev et al., Phys. Lett. B 308 (1993) 200; B 338 (1994) 478.

[18] S. Deser, M. L. Goldberger, K. Baumann and W. Thirring, Phys. Rev. 96 (1954) 774.
H. Bethe and F. Hoffman, Mesons and Fields, V. 2 (Row, Peterson and Co., Evanston, Illinois, 1955) p. 103.

[19] N. Byers, Phys. Rev. 107 (1957) 843.

[20] T. L. Trueman, Nucl. Phys. 26 (1961) 57.

[21] S. Wycech and A. M. Green, Nucl. Phys. A 562 (1993) 446.

[22] U. Moor, G. Rasche and W. S. Woolcock, Nucl. Phys. A 587 (1995) 747.

[23] G. V. Efimov, M. A. Ivanov and V. E. Lubovitsfij, Sov. J. Nucl. Phys. 44 (1986) 296.

[24] A. A. Bel'kov, V. N. Pervushin and F. G. Tkebuchava, Sov. J. Nucl. Phys. 44 (1986) 300.

[25] V. E. Lubovitskij and A. G. Rusetsky, Phys. Lett. B 389 (1996) 181.

[26] Z. Silagadze, JETPh Lett. 60 (1994) 689.

[27] E. M. Lifshiz, L. P. Pitaevsky, Relativistic Quantum Theory, Parts 1 and 2 (Nauka, Moscow, 1971).

[28] S. Shweber, An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Co., N. Y., 1961).

[29] A. B. Migdal, FInite Fermi System Theory and Atomic Nuclei Properties (Nauka, Moscow, 1965).

[30] L. D. Landau and E. M. Lifshiz, Quantum Mechanics (FM, Moscow, 1963).
L. I. Schiff, Quantum Mechanics (MeCraw-Hill Book Company, England, 1955).

[31] Rev. Part. Prop., Phys. Rev. D 50, Patr I (1994) 1177.

[32] G. C. Oades and G. Rasche, Nucl. Phys. B 20 (1970) 333.

[33] S. R. Amendolia, Phys. Lett. B 244 (1984) 469; Nucl. Phys. B 138 (1986) 168.

[34] V. Bernard, R. Brockman and W. Weise, Nucl. Phys. A 434 (1985) 685.

[35] A. I. Achiezer and V. B. Berestezky, Quantum Electrodynamic (FM, Moscow, 1959).

[36] I. S. Gradstein and I. M. Rizik, Tables of Integrals, Sums, Serieses and Products (FM, Moscow, 1963).

[37] I. P. Pustovalov, ZETP 36 (1959) 1806.

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