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E.B.Balbutsev, P.Schuck\*

MULTIPHONON STATES IN A SOLVABLE MODEL

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\*CNRS-IN2P3 Universite Joseph Fourier, Institut des Sciences Nucleaires,  
53 avenue des Martyrs, F-38026 Grenoble Cedex, France

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# 1 Introduction

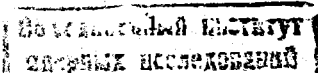
The description of nuclear collective motion in terms of the Wigner Function Moments (WFM) was first suggested in 1981 [1]. It was applied successfully to study isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with different realistic forces [2]. Their energies, excitation probabilities and widths were described. However, all calculations were made in the small amplitude approximation.

This work is the first attempt to apply the WFM method to the description of large amplitude motion. The interest to investigate collective motion going beyond the usual RPA (small amplitude approach) has been spurred after the experimental discovery of high-energy structures in heavy-ion grazing collisions and their interpretation in terms of multiphonon excitations of giant quadrupole resonances [3]. Also the recent possibilities to study double Giant Dipole Resonances with good precision [4]-[9] gives a new impetus to the theoretical study of multiphonon states. In the past the problem of large amplitude collective vibrations has been treated along various lines. The best known are the boson expansion method, an extended review of which can be found in [10], the generator coordinate method [11] and the time-dependent Hartree-Fock (TDHF) method [11] together with its adiabatic version ATDHF [11, 12, 13].

The basic problem in the theory of collective dynamics is to identify the collective variables and to derive the dynamical equations governing these variables. The practical value of any method essentially depends on the possibility of selecting a small number of collective degrees of freedom coupled weakly with all remaining degrees of freedom. The selection of proper collective variables is certainly a great problem and requires some physical intuition. Very useful general theorems have been provided in the papers [14, 15, 16]. There it was shown that any time-dependent description of a quantum system, derivable from the variational principle

$$\delta \int_{t_0}^{t_1} \langle \Psi | (i\hbar\partial_t - H) | \Psi \rangle dt = 0 \quad (1)$$

for  $\Psi$  in an arbitrary manifold in the Hilbert space of normalized wave functions, is equivalent to a classical Hamiltonian system. The variation in the full Hilbert space leads to the time-dependent Schrödinger equation. Any restriction of variations results in a non-



linearity of the equations. This property is a feature of the classical time evolution and one of the reasons requiring requantization. Let us consider the subspace  $\Psi(\rho, \pi)$  of the Hilbert space which is specified by two sets of parameters  $\rho \equiv [\rho_1, \dots, \rho_k]$ ,  $\pi \equiv [\pi_1, \dots, \pi_k]$ . Consequently, each element of this subspace can be written as  $\Psi(\mathbf{r}, t) \equiv \Psi(\mathbf{r}; \rho(t), \pi(t))$ . The parameters  $\rho, \pi$  can always be constructed as canonically conjugated variables. Assuming norm-conserving variations of  $\Psi(\mathbf{r}; \rho, \pi)$  in (1), one derives the Hamilton equations of motion for  $\rho(t), \pi(t)$ :

$$\partial_t \rho = \partial_\pi \mathcal{H}, \quad \partial_t \pi = -\partial_\rho \mathcal{H},$$

where  $\mathcal{H} = \langle \Psi | H | \Psi \rangle$  plays the role of a classical Hamiltonian. Increasing the number of parameters  $\rho, \pi$  is equivalent to a gradual change from a classical to a quantum theory.

Our approach is based on the TDHF equation for the Wigner function (Fourier transform of the one body density matrix). The collective variables are introduced as the various phase space moments of the Wigner function. The corresponding dynamical equations for them are obtained by averaging the TDHF equation with respective weights. Naturally, they are classical. It is possible (at least in the case of the two models considered below) to solve the inverse problem and to explicitly find the collective Hamiltonian  $\mathcal{H}$  which generates these equations. It appears to be equal to the average value of the microscopic Hamiltonian  $H$ . Hence, our approach satisfies the above mentioned theorems.

It was shown earlier [2] that several lowest rank moments are sufficient to describe the crudest characteristics of nuclei: center of mass, shape, giant resonances. To describe more subtle characteristics one needs to study higher moments. Naturally the general rule reads: the more the required information is detailed, the higher rank moments must be involved in the consideration. The knowledge of all possible (infinite number) moments is equivalent to the knowledge of the entire Wigner function. In other words, the higher the moments we take into account, the higher is the order of the quantum corrections which is considered.

In this paper two simple models of a harmonic oscillator single particle potential with separable residual interactions are investigated to demonstrate the possibilities of the WFM method. The systems of nonlinear dynamical equations for the monopole and quadrupole moments of the nucleus are derived and analyzed.

The first model, describing nonlinear monopole vibrations, is rather schematic and

simple. However, it turns out to be very useful because, being analytically solvable, it allows one to clarify some problems of the quantization of classical motion. TDHF is an initial value problem and as such it cannot be applied to bound states in the same way as the Schrödinger theory. The solutions to this problem rely on various quantization procedures [16, 17, 18]. However, some additional arguments (similar to the stationarity conditions of Kan [15]) are required in our approach to fix the initial conditions.

The second model, describing coupled nonlinear monopole and quadrupole vibrations, is more complicated. It is attractive because the equations of motion are derived rigorously and can be solved (numerically) without any approximations. Furthermore, it can be generalized such as to make it quite realistic.

The paper is organized as follows. In section 2 the general outline of the WFM formalism is described. This formalism is applied to the models of a harmonic oscillator with monopole-monopole and quadrupole-quadrupole residual interactions in sections 3 and 4 respectively: the equations of motion for collective variables are derived and solved, the energies and excitation probabilities of multiphonon excitations are calculated. The anharmonicity of the quadrupole multiphonon spectrum is studied in section 5 by means of quantization of classical equations of motion. Concluding remarks and the summary of the main results are contained in section 6.

## 2 Formulation of the method

The basis of our method for the description of collective nuclear dynamics is the equation of motion for the one-body density matrix  $\rho(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1 | \hat{\rho}(t) | \mathbf{r}_2 \rangle$ :

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}], \quad (2)$$

where  $\hat{H}$  is the self consistent one-body Hamiltonian depending implicitly on the density matrix. Equation (2) with an explicit form of the one-body Hamiltonian appears in the Hartree-Fock theory; it is also currently used within the so-called energy-density-functional approach leaving more possibilities for the choice of the one-body Hamiltonian and in addition giving some grounds to believe that equation (2) is rather general [19].

It is convenient to modify equation (2) by applying the Wigner transform of the density

matrix [20]

$$f(\mathbf{r}, \mathbf{p}, t) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \quad (3)$$

and of the Hamiltonian

$$H^W(\mathbf{r}, \mathbf{p}) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \left( \mathbf{r} + \frac{\mathbf{s}}{2} \right) \hat{H} \left( \mathbf{r} - \frac{\mathbf{s}}{2} \right). \quad (4)$$

Using (3,4) one arrives at [11, 21]

$$\frac{\partial f}{\partial t} = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} (\nabla_{\mathbf{r}}^H \cdot \nabla_{\mathbf{p}}^I - \nabla_{\mathbf{p}}^H \cdot \nabla_{\mathbf{r}}^I) \right) H^W f, \quad (5)$$

where the upper index on the nabla operator stands for the function on which this operator acts. If the Hamiltonian is a sum of a kinetic term and a local potential  $V(\mathbf{r})$ , its Wigner transform is just the classical version of the same Hamiltonian:

$$H^W = p^2/2m + V(\mathbf{r}). \quad (6)$$

Then equation (5) becomes:

$$\frac{\partial f}{\partial t} + \frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} f = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \nabla_{\mathbf{r}}^V \cdot \nabla_{\mathbf{p}}^I \right) V f. \quad (7)$$

The generalization for non-local potentials is straightforward and can be found in [22].

## 2.1 Wigner function moments

Now we apply the WFM method to derive a closed system of dynamical equations for cartesian tensors of second rank. This method was suggested in [1, 23] and is described in detail in Ref. [2]. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [24]. It is shown in [2], that by integrating equation (7) over the phase space  $\{\mathbf{p}, \mathbf{r}\}$  with the weights  $x_i x_j \dots x_k p_{k+1} \dots p_{i_{n-1}} p_{i_n}$ , where  $k$  runs from 0 to  $n$ , one can obtain a closed finite subsystem of dynamical equations for multipole moments and other integral characteristics of a nucleus. By choosing a row of integers for  $n$  (0, 1, 2, 3, and so on) one obtains subsystems for different multipoles.

Here we consider the case  $n = 2$ . Integrating equation (7) over the phase space  $\{\mathbf{p}, \mathbf{r}\}$  with the weights  $x_i x_j, p_i x_j, p_i p_j$  yields:

$$\int d^3r x_i x_j \frac{\partial n(\mathbf{r}, t)}{\partial t} + \int d^3r x_i x_j \frac{\partial (n(\mathbf{r}, t) u_s(\mathbf{r}, t))}{\partial x_s} = 0, \quad (8)$$

$$m \int d^3r x_j \frac{\partial}{\partial t} (n(\mathbf{r}, t) u_i(\mathbf{r}, t)) + \int d^3r n(\mathbf{r}, t) x_j \frac{\partial V}{\partial x_i} + \int d^3r x_j \frac{\partial}{\partial x_s} \mathcal{A}_{si}(\mathbf{r}, t) = 0, \quad (9)$$

$$\frac{\partial}{\partial t} \int d^3r \mathcal{A}_{ij}(\mathbf{r}, t) + \int d^3r n(\mathbf{r}, t) \left[ u_i(\mathbf{r}, t) \frac{\partial V}{\partial x_j} \right]_{ij} + \int d^3r \frac{\partial}{\partial x_s} \mathcal{A}_{sij}(\mathbf{r}, t) = 0, \quad (10)$$

where summation over repeated indices is assumed and  $[\dots]_{ij}$  means symmetrization with respect to the indices  $i$  and  $j$  ( $[a_i b_j]_{ij} = a_i b_j + a_j b_i$ ). We have introduced the notations:

$$n(\mathbf{r}, t) = 4 \int \frac{d^3p}{(2\pi\hbar)^3} f(\mathbf{r}, \mathbf{p}, t),$$

$$m n(\mathbf{r}, t) u_i(\mathbf{r}, t) = \int \frac{d^3p}{(2\pi\hbar)^3} p_i f(\mathbf{r}, \mathbf{p}, t),$$

$$\mathcal{A}_{i_1 i_2 \dots i_k}(\mathbf{r}, t) = m^{1-k} \int \frac{d^3p}{(2\pi\hbar)^3} p_{i_1} p_{i_2} \dots p_{i_k} f(\mathbf{r}, \mathbf{p}, t).$$

By definition  $n(\mathbf{r}, t)$  is the nucleon density,  $u(\mathbf{r}, t)$  is the mean velocity of nucleons,  $\mathcal{A}_{ij}(\mathbf{r}, t)$  is the kinetic energy tensor (or pressure tensor). Integrating by parts the last terms in (8)-(10) and introducing the notation

$$J_{ij}(t) = \int d^3r x_i x_j n(\mathbf{r}, t)$$

for the inertia tensor,

$$L_{i,j}(t) = m \int d^3r x_i u_j(\mathbf{r}, t) n(\mathbf{r}, t)$$

for the mixed momentum-position tensor and

$$\Pi_{ij}(t) = \frac{1}{2} \int d^3r \mathcal{A}_{ij}(\mathbf{r}, t)$$

for the integral kinetic energy tensor we have:

$$\frac{d}{dt} J_{ij}(t) - \frac{1}{m} (L_{i,j} + L_{j,i}) = 0, \quad (11)$$

$$\frac{1}{2} \frac{d}{dt} L_{i,j}(t) + \frac{1}{2} \int d^3r x_i n(\mathbf{r}, t) \frac{\partial V}{\partial x_j} - \Pi_{ij}(t) = 0, \quad (12)$$

$$\frac{d}{dt} \Pi_{ij}(t) + \frac{1}{2} \int d^3r n(\mathbf{r}, t) \left[ u_i(\mathbf{r}, t) \frac{\partial V}{\partial x_j} \right]_{ij} = 0. \quad (13)$$

The last integral of equation (10) with the third rank tensor  $\mathcal{A}_{sij}$  has disappeared due to the obvious boundary condition  $\mathcal{A}_{sij}(\mathbf{r}, t) \rightarrow 0$  at  $\mathbf{r} \rightarrow \infty$ , which follows from the boundary condition for the Wigner function  $f(\mathbf{r}, \mathbf{p}, t) \rightarrow 0$  at  $\mathbf{r} \rightarrow \infty$ . We thus have derived a system of three dynamical equations for the three collective variables  $J_{ij}(t), L_{i,j}(t)$  and  $\Pi_{ij}(t)$ . To close the system of equations one needs to represent the integrals involving

derivatives of the potential  $V(\mathbf{r})$  in terms of these three variables. This problem can be solved rigorously only in the case of a potential  $V_H$  with quadratic coordinate dependence (what is the subject of this paper). For more realistic potentials some approximations are needed.

We suggest the following procedure. Considering the potential  $V_H$  as the zero order approximation to the realistic potential  $V_R$ , we expand the difference  $V_R - V_H$  in a Taylor series and truncate it on the  $r^n$  term. The integration of the potential will generate the tensors of different ranks from 1 up to  $n$ . Hence, to have a closed system of equations we are forced to write the subsystems of dynamical equations for tensors of all ranks from 1 to  $n$ , the subsystems being coupled. The more terms of the Taylor series are taken into account, the higher rank tensors must be included in the calculations. So, the required minimal rank of tensors is determined by measuring the deviation of the realistic potential from the harmonic oscillator one. The desired maximum rank is determined by the physics of the phenomenon to be studied: in general the more detailed information is required, the higher rank tensors must be considered.

Both equations (11) and (13) evidently are symmetric with respect to the indices  $i, j$ , whereas eq. (12) has no specific symmetry. We can provide easily the symmetric and antisymmetric equations by obvious combinations of (12) with different indices:

$$\frac{1}{4} \frac{d}{dt} (L_{i,j} + L_{j,i}) + \frac{1}{4} \int d^3r n(\mathbf{r}, t) \left[ x_j \frac{\partial V}{\partial x_i} \right]_{ij} - \Pi_{ij}(t) = 0, \quad (14)$$

$$\frac{d}{dt} (L_{i,j} - L_{j,i}) = \int d^3r n(\mathbf{r}, t) \left\{ x_j \frac{\partial V}{\partial x_i} - x_i \frac{\partial V}{\partial x_j} \right\}. \quad (15)$$

By definition, the left-hand side of the equation (15) is the angular momentum component:

$$M_{i,j} = L_{i,j} - L_{j,i} = m \int d^3r n(\mathbf{r}, t) \{ x_i u_j(\mathbf{r}, t) - x_j u_i(\mathbf{r}, t) \}.$$

When  $V(\mathbf{r}, t)$  is a self-consistent potential, the right-hand side of (15) is equal to zero, what demonstrates angular momentum conservation.

Taking into account equation (11) it is possible to rewrite equation (14) in a more convenient form :

$$\frac{m}{4} \dot{J}_{ij}(t) + \frac{1}{4} \int d^3r n(\mathbf{r}, t) \left[ x_j \frac{\partial V}{\partial x_i} \right]_{ij} - \Pi_{ij}(t) = 0. \quad (16)$$

Concluding this section, we will give the recipe of deriving the integrals of motion from the equations of the system (16, 13). Let us suppose that these equations can be

written as Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} - \frac{\partial \mathcal{L}}{\partial q_\nu} = 0. \quad (17)$$

Then there must exist at least one integral of motion:

$$\sum_\nu \dot{q}_\nu \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} - \frac{\partial \mathcal{L}}{\partial q_\nu} \right) = \frac{d}{dt} \left( \sum_\nu \dot{q}_\nu \frac{\partial \mathcal{L}}{\partial \dot{q}_\nu} - \mathcal{L} \right). \quad (18)$$

Following this way one finds from (16):

$$\frac{m}{4} \dot{J}_{ij} \dot{J}_{ij} + \frac{1}{4} \dot{J}_{ij} \int d^3r n(\mathbf{r}, t) \left[ x_j \frac{\partial V}{\partial x_i} \right]_{ij} - \dot{J}_{ij} \Pi_{ij}(t) = 0. \quad (19)$$

Subtracting from this expression equation (13), multiplied by  $J_{ij}$ , one obtains:

$$\frac{d}{dt} \left( \frac{m}{8} \dot{J}_{ij}^2 - J_{ij} \Pi_{ij} \right) + \frac{1}{4} \dot{J}_{ij} \int d^3r n(\mathbf{r}, t) \left[ x_j \frac{\partial V}{\partial x_i} \right]_{ij} - \frac{1}{2} J_{ij} \int d^3r n(\mathbf{r}, t) \left[ u_j \frac{\partial V}{\partial x_i} \right]_{ij} = 0. \quad (20)$$

To make the next step one needs the detailed expression for the potential  $V(\mathbf{r})$ . For example, in the case of the harmonic oscillator  $\frac{\partial V}{\partial x_i} \sim x_i$ : Substituting this relation into (20) and using the definitions of  $J_{ij}$  and  $L_{i,j}$ , one finds (with the help of equation (11)) that the integrals in (20) cancel. Hence, in the case of harmonic potentials the system (16),(13) has six integrals of motion:

$$J_{ij}(t) \Pi_{ij}(t) - \frac{m}{8} \dot{J}_{ij}(t)^2 = c_{ij}, \quad i, j = 1, 2, 3. \quad (21)$$

### 3 Harmonic oscillator with residual monopole-monopole interaction

#### 3.1 Equations of motion

Let us first consider the classical and quantum-mechanical aspects of the exactly solvable one-dimensional model whose microscopic Hamiltonian is:

$$H = \sum_{i=1}^A \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 x_i^2 \right) + \frac{1}{2} \kappa \sum_{i \neq j}^A (x_i^2 - x_0^2 / \Lambda) (x_j^2 - x_0^2 / \Lambda), \quad (22)$$

where  $x_0^2$  is the value of the tensor  $J_{11}$  for an oscillator ground state. This model was studied for various values of the constant  $x_0$ . It is known as the Suzuki model [25] when

$x_0^2$  is equal to the equilibrium value of the tensor  $J_{11}$ . Its solution in a time dependent Hartree approximation was found by Blaizot, Schulz and Reinhardt [26, 27]. The case of  $x_0 = 0$  was studied in [28].

The mean field of the model is

$$V(x, t) = \frac{1}{2}m\omega^2 x^2 + \kappa(J - x_0^2)(x^2 - x_0^2/A), \quad (23)$$

where  $J = J_{11}(t)$ . For this potential eq. (15) is fulfilled identically and equations (16), (13) become:

$$\begin{aligned} \frac{m}{4}\dot{j} + \frac{1}{2}J[m\omega^2 + 2\kappa(J - J_0)] - \Pi &= 0, \\ \dot{\Pi} + j\left[\frac{m}{2}\omega^2 + \kappa(J - J_0)\right] &= 0 \end{aligned} \quad (24)$$

with  $J_0 \equiv x_0^2$ ,  $\Pi = \Pi_{11}(t)$ . The time dependence of tensors is omitted for the sake of simplicity. The second equation of this system can be reduced to the integral of motion

$$\Pi + \frac{m}{2}\omega^2 J + \frac{\kappa}{2}(J - J_0)^2 = c_0, \quad (25)$$

whose physical meaning is that the total energy of the system is a conserved quantity. It is indeed easy to see that it is equal to the Hartree-Fock average of the microscopic Hamiltonian (22), i.e.  $c_0 = \langle \Psi | H | \Psi \rangle$ . Another integral of motion can be found using equation (20). For the potential (23) the integrals in this equation cancel and we get:

$$J(t)\Pi(t) - \frac{m}{8}\dot{J}(t)^2 = c, \quad (26)$$

where the constant  $c$  is determined from the Initial Conditions (IC). With the help of equation (26) one is able to reduce the system (24) to the single equation:

$$\frac{m}{4}\dot{j} + \frac{m}{2}\omega^2 J + \kappa(J - J_0)J - \frac{c}{J} - \frac{m}{8J}\dot{J}^2 = 0. \quad (27)$$

Performing the change of variable  $J = J_0 r^2$  this equation can be written in the form:

$$\dot{r} + \omega^2 \left[ r + 2\bar{\kappa}(r^3 - r) - \frac{c'}{r^3} \right] = 0, \quad (28)$$

where  $\bar{\kappa} = \kappa \frac{J_0}{m\omega^2}$  and  $c' = 2c/(m\omega^2 J_0^2)$ . If to suppose here  $c' = 1$  this equation becomes identical to equation (5.8) of ref. [26] (their variable  $\eta^2 = r^2/A$ ) and to equation (3.17) of ref. [27] (their variable  $r^2 = r_{0ur}^2 \tilde{\omega}/\omega_0$ , their  $\omega_0$  is identical to our  $\omega$ , the parameter

$\tilde{\omega}$  is fixed by their self-consistency requirement that leads to the relation  $\tilde{\omega} = \omega_0$ ). The choice of  $c'$  is not accidental - this value is linked to the method of solution, especially due to the boundary condition in [26] and the self-consistency condition in [27], where it is supposed that the ground state has to be that of the harmonic oscillator. On the contrary, our method allows to find a more general solution. Solving equations (25) and (26) with respect to  $\Pi$ , one can rewrite the energy in a more traditional form, as the sum of kinetic and potential energies:

$$c_0 = \frac{m}{8J}\dot{J}^2 + \frac{c}{J} + \frac{m}{2}\omega^2 J + \frac{\kappa}{2}(J - J_0)^2 \quad (29)$$

or in terms of  $r$ :

$$c_0 = \frac{m}{2}J_0\{r^2 + \omega^2[r^2 + c'/r^2 + \bar{\kappa}(r^2 - 1)^2]\}. \quad (30)$$

Again, if we here suppose  $c' = 1$  this expression becomes identical to expression (5.9) of [26] and to expression (3.30) of [27]. The  $r$ -dependence of the potential

$$V(r) = \frac{m}{2}\omega^2 J_0[r^2 + c'/r^2 + \bar{\kappa}(r^2 - 1)^2]$$

for various values of  $\bar{\kappa}$  is illustrated schematically in fig. 1.

### 3.2 Equilibrium state and small amplitude approximation

By definition, at equilibrium the kinetic energy is equal to zero and the potential energy is at its minimum. The equation determining the extremums of  $V(r)$  is:

$$g(r) - c' = 0, \quad (31)$$

where  $g(r) = r^4[1 + 2\bar{\kappa}(r^2 - 1)]$ . The function  $g(r)$  for various values of  $\bar{\kappa}$  is sketched in fig. 2.

It is seen that in the case of  $\bar{\kappa} > 0$  the polynom (31) has only one positive root for  $c' > 0$  that corresponds to the minimum of the potential (see fig.1). It describes the stable equilibrium state that is more compressed ( $J_{eq} < J_0$ ), than that of the harmonic oscillator, when  $c' < 1$  and less compressed ( $J_{eq} > J_0$ ) when  $c' > 1$ . Using an analogy with an equilibrium deformation, one can say that the system has a positive static compression for  $c' < 1$  and a negative static compression (dilatation) for  $c' > 1$  if to assume that the oscillator ground state has a zero static compression.

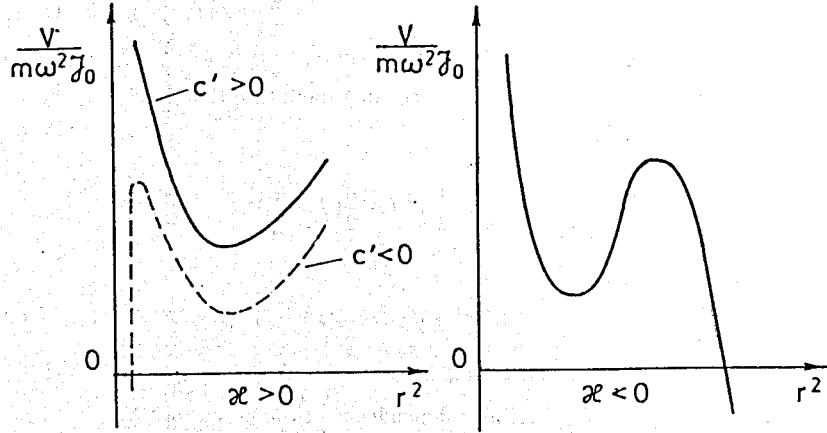


Fig.1. The  $r$ -dependence of the potential  $V(r) = \frac{m}{2} J_0 \omega^2 [r^2 + c'/r^2 + \bar{\kappa}(r^2 - 1)^2]$  for various values of  $\bar{\kappa}$  and  $c'$ .

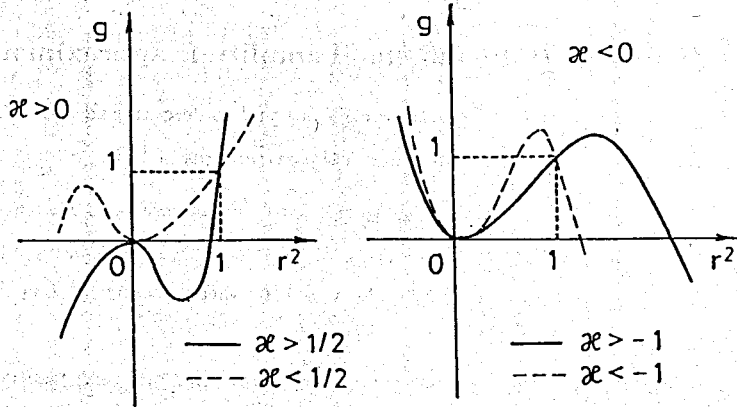


Fig.2. The function  $g(r) = r^4[1 + 2\bar{\kappa}(r^2 - 1)]$  for various values of  $\bar{\kappa}$ .

There is no necessity to analyse the situation with  $c' < 0$  (see however next section) because this integral of motion cannot be negative in the state of equilibrium. Really substituting  $\dot{J} = 0$  into (26) we find:

$$c_{eq} = J_{eq} \Pi_{eq}. \quad (32)$$

$J$  and  $\Pi$  being positive by definition, hence  $c_{eq}$  and  $c'_{eq}$  are positive definite. By the way, using in (32) the relation  $\Pi_{eq} = \frac{m}{2} J_0 \omega^2 r_{eq}^2 [1 + 2\bar{\kappa}(r_{eq}^2 - 1)]$  following from (24), one can reproduce equation (31).

In the case of  $\bar{\kappa} < 0$  the polynomial (31) has two positive roots if  $0 \leq c' < (1 - 2\bar{\kappa})^3 / (27\bar{\kappa}^2)$ . The smaller root corresponds to the minimum of the potential well and the bigger one corresponds to the maximum of the barrier. The latter equilibrium state is metastable due to the finite value of the barrier height. For  $\bar{\kappa} \leq -1$  the equilibrium state has a positive static compression independent of the value of  $c'$ . For  $\bar{\kappa} > -1$  the equilibrium state has a positive static compression when  $c' < 1$  and a negative one when  $c' > 1$ . The potential has no extrema, when  $c' \geq (1 - 2\bar{\kappa})^3 / (27\bar{\kappa}^2)$ , possessing only an inflection point at  $r^2 = (-c'/\bar{\kappa})^{1/3}$ .

To find the energy of small vibrations around the equilibrium state we apply the linearization procedure. Writing equation (28) in terms of the new variable  $y = r - r_{eq}$  and neglecting nonlinear terms in  $y$  we find:

$$\ddot{y} + y\omega^2 [1 + 3c'/r_{eq}^4 + 2\bar{\kappa}(3r_{eq}^2 - 1)] + \omega^2 [r_{eq} - c'/r_{eq}^3 + 2\bar{\kappa}(r_{eq}^3 - r_{eq})] = 0. \quad (33)$$

This equation is transformed into

$$\ddot{y} + 4\omega^2 [1 + \bar{\kappa}(3r_{eq}^2 - 2)]y = 0 \quad (34)$$

after taking into account eq. (31) satisfied by  $r_{eq}$ . The corresponding eigenfrequency is:

$$\tilde{\Omega} = 2\omega \sqrt{1 + \bar{\kappa}(3r_{eq}^2 - 2)}. \quad (35)$$

Assuming here  $x_0 = 0$ , we reproduce expression (9.45) in [28].

Equation (31) can be solved analytically when  $c' = 1$ . One positive extremum lies at  $r^2 = 1$ . It corresponds to the maximum of the barrier for  $\bar{\kappa} < -1$  and to the minimum of the potential for  $\bar{\kappa} > -1$ . Only this minimum was analysed in [26] and [27]. From formula (35) one gets the corresponding expression for the RPA frequency:

$$\tilde{\Omega} = 2\omega \sqrt{1 + \bar{\kappa}}. \quad (36)$$

Another positive extremum lies at  $r^2 = -(1 + \sqrt{1 - 8\bar{\kappa}})/(4\bar{\kappa})$ . It corresponds to the maximum of the barrier for  $0 > \bar{\kappa} > -1$  and to the minimum of the potential for  $\bar{\kappa} < -1$ . Formula (35) gives the corresponding expression of the RPA frequency in this potential well:

$$\tilde{\Omega}^2 = \omega^2(1 - 8\bar{\kappa} - 3\sqrt{1 - 8\bar{\kappa}}). \quad (37)$$

The strength constant  $\bar{\kappa} = -1$  is the critical one. With this  $\bar{\kappa}$  the potential has neither a minimum nor a maximum and the point  $r^2 = 1$  turns out to be its inflection point.

### 3.3 Analysis of the exact solution

To find an exact expression for the function  $J(t)$  it is convenient to use equation (29). Its solution can be expressed in terms of the Jacobian elliptic function [29]:

$$J(t) = \eta_1 + (\eta_2 - \eta_1)\text{sn}^2(\bar{\omega}t + \chi). \quad (38)$$

Here  $\bar{\omega} = \omega\sqrt{\bar{\kappa}(\eta_1 - \eta_3)/J_0}$  and  $\eta_i$  are the real roots of the polynomial

$$P(J) = -J^3 + a_2J^2 + a_1J + a_0 \quad (39)$$

with  $a_2 = 2J_0 - m\omega^2/\kappa$ ,  $a_1 = 2c_0/\kappa - J_0^2$ ,  $a_0 = -2c/\kappa$ . The roots satisfy the condition  $\eta_1 > \eta_2 > \eta_3$  for  $\kappa > 0$  and  $\eta_1 < \eta_2 < \eta_3$  for  $\kappa < 0$ . The phase  $\chi$  is determined by IC. The function  $\text{sn}(\phi)$  is periodic with the period  $\Delta\phi = 4\mathbf{K}$ , where

$$\mathbf{K} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad (40)$$

is a complete elliptic integral of the first kind with  $k^2 = \frac{\eta_2 - \eta_1}{\eta_3 - \eta_1}$ . There exists an analytical expression for the Fourier expansion of this function [30]:

$$\text{sn}\bar{\omega}t = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1 - q^{2n-1}} \sin(2n-1) \frac{\bar{\omega}\pi}{2\mathbf{K}} t.$$

Here  $q = \exp(-\pi\mathbf{K}'/\mathbf{K})$ ,  $\mathbf{K}'(k) = \mathbf{K}(k')$ ,  $k' = \sqrt{1 - k^2}$ . This formula involves only the frequencies proportional to odd numbers of the basic frequency  $\Omega = \frac{\bar{\omega}\pi}{2\mathbf{K}}$ . It is obvious that  $\text{sn}^2$  includes the frequencies  $n\Omega$  with  $n$  even only. So, the Fourier expansion of the function  $J(t)$  will involve only one basic frequency  $2\Omega$  and its satellites  $4\Omega$ ,  $6\Omega$ , etc. Numerically the frequency  $2\Omega = \frac{\bar{\omega}\pi}{\mathbf{K}}$  can be rather different from the result of the harmonic problem

(35). So, the effect of including the anharmonic term  $\sim J^2(t)$  into the system (24) is the variation of the basic frequency  $\tilde{\Omega} \rightarrow 2\Omega$  and the appearance of satellites  $n2\Omega$ , which are interpreted (using the quantum mechanical terms) as the levels of multiphonon states. The equidistance of such spectrum is evident, characteristic for bounded classical motion.

It is necessary to note the dependence of  $\Omega$  on IC (also characteristic for classical motion). The roots of the polynomial (39) depend on  $c$ ,  $c_0$ . These constants together with the phase  $\chi$  are determined by  $J(0)$ ,  $\dot{J}(0)$  and  $\Pi(0)$ . Examples of such a dependence are demonstrated in table 1.

A very interesting situation arises at a sufficiently large value of  $\dot{J}(0)$ , when the constant  $c'$  becomes negative. If  $\bar{\kappa} > 1/2$  and the  $0 > c' > (1 - 2\bar{\kappa})^3/(27\bar{\kappa}^2)$ , the polynomial (31) has two positive roots (fig.2) with  $r^2 < 1$ : the bigger one corresponds to the minimum of the potential well, and the smaller one corresponds to the top of the barrier. In this case the time dependent single particle potential (23) is always repulsive. Nevertheless, the system can possess a collective dynamical potential whose bottom is lower than that of the equilibrium state ( $E_{eq}$ ). The top of the barrier will be higher than  $E_{eq}$  if the condition

$$\bar{\kappa} > 2/(4 - 3r_{eq}^2) \quad (41)$$

is fulfilled. Such a potential is shown in figure 1 by the dashed curve. The eigenfrequencies calculated for this potential well are shown in table 1.

Table 1

Dependence of eigenfrequencies on initial conditions.

(I)  $\bar{\kappa} = 2$ ,  $E_{HF} = 24$  Mev,  $r_{eq}^2 = 1$ . (II)  $\bar{\kappa} = 2$ ,  $E_{HF} = 24$  Mev,  $r_{eq}^2 = 0.752$ .

(III)  $\bar{\kappa} = -0.5$ ,  $E_{HF} = 9.5$  Mev,  $r_{eq}^2 = 1$ . (IV)  $\bar{\kappa} = -2$ ,  $E_{HF} = 15$  Mev,  $r_{eq}^2 = 0.5$ .

I	$c'$	0	0.01	0.1	0.5	1
	$\hbar\Omega$	9.25	14.13	17.91	21.84	23.98
II	$c'$	-0.0001	-0.04	-0.08	-0.1	-0.11
	$\hbar\Omega$	16.89	15.96	14.55	13.31	12.18
III	$c'$	0.94	0.945	0.95	1	1.005
	$\hbar\Omega$	5.53	7.04	7.58	9.50	9.60
IV	$c'$	.47	.5	.7	.75	.755
	$\hbar\Omega$	10.66	14.68	18.90	19.42	19.46

The limits of variation of  $c'$  are determined by the input excitation energy  $E$ : at some value of  $c'$  the energy  $E_{eq} + E$  turns lower than the bottom of the potential well or higher than the top of the potential barrier.



### 3.4 Quantization

Solving nonlinear equations of motion one expects to find anharmonicity effects. We have observed already the main effect of anharmonicity - the satellites of the basic frequency that form the equidistant spectrum of multiphonon states. However, such a result is contradictory to the practice of quantum mechanical calculations, where one usually has some deviation from precise equidistance. Hence, to obtain the anharmonicity of the spectrum it is necessary to quantize this model.

Its quantization is elementary because we have already the expression for the energy of vibrations (29). Choosing  $q = J$  and  $p = \frac{mJ}{4J}$  as the canonically conjugate variables, one can represent the Hamilton function in the form

$$H = \frac{p^2}{2m^*} + V(q) \quad (42)$$

with

$$V(q) = \frac{m}{2}\omega^2 q + \frac{c}{q} + \frac{\kappa}{2}(q - J_0)^2, \quad m^* = \frac{m}{4q}. \quad (43)$$

It is easy to see that equation (27) coincides with the Hamilton equations  $\dot{q} = \frac{\partial H}{\partial p}$ ,  $\dot{p} = -\frac{\partial H}{\partial q}$  what justifies our choice of canonical variables.

The quantum Hamiltonian can be produced following the Pauli [31], [32] prescription. This operation, however, does not complete the construction of the quantum Hamiltonian because it is necessary to solve the initial condition problem. Our quantum Hamiltonian will contain the constant  $c$  which is determined by the IC. Thus, the variety of initial conditions of the classical problem will generate a variety of quantum Hamiltonians. However, the Hamiltonian which ideally describes the dynamics of the nucleus should be unique.

We suppose that the solution of this problem can be found by taking into account the principal difference between the classical and quantum descriptions of excitations. Being an integral of motion (energy), the classical Hamiltonian is changed each time the initial conditions are changed. Hence, strictly speaking, all excited states and the equilibrium (ground) state are described by different Hamiltonians. Absolutely another situation prevails in a quantum case. Here all states (ground and excited) are obtained as the eigenstates of only one Hamiltonian. The ground state is the only state that is described by the same Hamiltonian in both cases. So, it is natural to use for quantization the classical Hamiltonian derived for IC which correspond to the ground state. That

means, that for our model we have to take the equilibrium value of the constant  $c$ . This statement agrees with the conclusion of A. Klein [28] that "the value of  $c$  is related to the equilibrium value of  $q$ ". Furthermore, it bears a strong resemblance to the stationarity conditions of Kan [15]. So, the picture of excitations in the classical case is much richer than in the quantum one.

Two methods will be used to analyse the spectrum. The first one is the Bohr-Sommerfeld quantization rule:

$$\int_{q_1}^{q_2} P(q) dq = \pi \hbar \left(n + \frac{1}{2}\right), \quad (44)$$

where  $q_1$  and  $q_2$  are the classical turning points,  $P(q) = \sqrt{2m^*(E - V)}$ .

Another method was suggested by Cambiaggio [18]. Its idea is in the self-consistency prescription. One calculates the Fourier spectrum of the action (Lagrangian) for different input energies  $E$ . The selection of the energies goes as follows. When one finds the value of  $E$  that satisfies the relation  $E_p = E - E_{eq}$ , where  $E_p$  is the energy of the first Fourier peak, one selects this energy and calls it  $E_1$ . Then one continues to calculate the Fourier spectrum of the action with the input energies  $E > E_1$ . Again, when one finds the value of  $E$  that satisfies the new relation  $E_p = E - E_1$ , one selects this energy and calls it  $E_2$ . The procedure is repeated until the limit value of  $E$  is achieved.

The results depend strongly on the values of  $c_{eq}$  and  $\bar{\kappa}$ . In an accordance with the results of the previous analysis three domains of  $\bar{\kappa}$  values must be considered separately:  $\bar{\kappa} > 0$ ,  $-1 < \bar{\kappa} < 0$  and  $\bar{\kappa} < -1$ .

Let us consider first the case:  $\bar{\kappa} > 0$ . The potential well here has infinite walls and a minimum at the point  $J = J_0$  (for  $c'_{eq} = 1$ ) that corresponds to an equilibrium state of the harmonic oscillator, i.e. the inclusion of the residual interaction does not change the equilibrium state of the system that is characterized by the inertia tensor  $J_{eq} = J_0$  and by the energy  $E_{eq} = m\omega^2 J_0$ . The spectrum, being infinite, has very small anharmonicity. The calculations with  $\bar{\kappa} = 2$  show that the levels  $E_n$  are positioned equidistantly with a good accuracy up to a rather large  $n$ . For example, the difference  $E_1 - E_0 = 23.984$  practically coincides with  $E_{RPA} = 23.971$  Mev. A small anharmonicity can be noticed at  $n \approx 100$ . So the difference  $E_{101} - E_{100} = 26.017$  Mev demonstrates the anharmonicity  $Anh = (E_{101} - E_{100} - E_{RPA})/E_{RPA} \approx 8\%$ .

The second case ( $-1 < \bar{\kappa} < 0$ ) is more interesting. Here the potential (43) has a mini-

imum at the same point  $J = J_0$  (for  $c'_{eq} = 1$ ) that also corresponds to an equilibrium state of the harmonic oscillator. However this state is metastable because now the potential has the finite height barrier whose top lies at the point  $J = -J_0(1 + \sqrt{1 - 8\bar{\kappa}})/(4\bar{\kappa}) > J_0$ . So the inclusion of the residual interaction with  $-1 < \bar{\kappa} < 0$  changes the equilibrium state of the system qualitatively without changing its quantitative characteristics  $J_{eq}$  and  $E_{eq}$ . The barrier height decreases from  $\infty$  to 0 when  $\bar{\kappa}$  is changed from 0 to  $-1$ . Hence, the anharmonicity can be rather large when  $\bar{\kappa}$  is close to  $-1$ . For example, at  $\bar{\kappa} = -0.5$ ,  $c'_{eq} = 1$  the barrier height is  $\approx 50$  Mev. The potential well has four bound states, and the deviation of the spectrum from the equidistant one is appreciable right from the beginning (table 2). Taking  $c'_{eq} = 1.05$ , one obtains the barrier with the height  $\approx 22$  Mev. The potential has only two bound states and the anharmonicity is slightly increased (table 2).

The third case ( $\bar{\kappa} < -1$ ) is of special interest because here the potential has a maximum at the point  $J = J_0$  (for  $c'_{eq} = 1$ ). Its minimum lies at  $J = J_{eq} = -J_0(1 + \sqrt{1 - 8\bar{\kappa}})/(4\bar{\kappa}) < J_0$ . The well depth (or barrier height) increases from 0 to  $\infty$  when  $\bar{\kappa}$  is changed from  $-1$  to  $-\infty$ . Hence, a remarkable anharmonicity can be observed in the vicinity of  $\bar{\kappa} = -1$ . For example, at  $\bar{\kappa} = -2$  the well depth is  $\approx 67$  Mev. Here there are four bound states and the deviation of the spectrum from the equidistant one is strong, exactly as in the previous case, already for the low lying states (table 2).

**Table 2**

The differences between multiphonon levels calculated by Bohr-Sommerfeld (B) and Cambiaggio (C) methods.

(IIa)  $c' = 1$ ,  $\bar{\kappa} = -0.5$ ,  $E_{RPA} = 9.786$  Mev,  $r_{eq}^2 = 1$ . (IIb)  $c' = 1.05$ ,  $\bar{\kappa} = -0.5$ ,  $E_{RPA} = 8.975$  Mev,  $r_{eq}^2 = 1.053$ . (III)  $c' = 1$ ,  $\bar{\kappa} = -2$ ,  $E_{RPA} = 14.891$  Mev,  $r_{eq}^2 = 0.64$ .

		$E_1 - E_0$	$E_2 - E_1$	$E_3 - E_2$	$E_4 - E_3$
IIa	B	9.489	9.151	8.749	8.243
	C	9.496	9.166	8.777	8.299
IIb	B	8.557	8.032	-	-
	C	8.572	8.073	-	-
III	B	14.339	13.686	12.858	11.657
	C	14.355	13.723	12.939	11.870

It is seen from the table that the results found by the Bohr-Sommerfeld and Cambiaggio methods are quite close, the difference between them increases together with the

anharmonicity. Such a behaviour is naturally explained by the fact, that both methods are approximate ones.

## 4 Harmonic oscillator with residual quadrupole-quadrupole interaction

### 4.1 Equations of motion

The model we will consider here is a harmonic oscillator with a quadrupole-quadrupole residual interaction. Its microscopic Hamiltonian is

$$H = \sum_{i=1}^A \left( \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{r}_i^2 \right) + \frac{1}{2} \kappa \sum_{i \neq j} \sum_{\mu=-2}^2 q_{2\mu}(\mathbf{r}_i) q_{2\mu}^\dagger(\mathbf{r}_j), \quad (45)$$

with the quadrupole operator  $q_{2\mu} = 4\sqrt{\pi/5} r^2 Y_{2\mu}$ . The corresponding mean field potential can be written as

$$V(\mathbf{r}, t) = \frac{1}{2} m \omega^2 r^2 + \kappa \sum_{\mu=-2}^2 Q_{2\mu}(t) q_{2\mu}^\dagger(\mathbf{r}), \quad (46)$$

where the quadrupole moment  $Q_{2\mu}(t)$  is defined as

$$Q_{2\mu}(t) = \int d^3r q_{2\mu}(\mathbf{r}) n(\mathbf{r}, t) = 4 \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} q_{2\mu}(\mathbf{r}) f(\mathbf{r}, \mathbf{p}, t). \quad (47)$$

Using the relations

$$\begin{aligned} Q_{20} q_{20}^\dagger &= (J_{11} + J_{22} - 2J_{33})(x_1^2 + x_2^2 - 2x_3^2), \\ Q_{21} q_{21}^\dagger + Q_{2-1} q_{2-1}^\dagger &= 12(J_{13} x_1 x_3 + J_{23} x_2 x_3), \\ Q_{22} q_{22}^\dagger + Q_{2-2} q_{2-2}^\dagger &= 3(J_{11} - J_{22})(x_1^2 - x_2^2) + 12J_{12} x_1 x_2, \end{aligned}$$

one can rewrite this potential in terms of inertia tensors:

$$V(\mathbf{r}, t) = \frac{1}{2} m \omega^2 r^2 + 2\kappa \sum_{r=1}^3 \sum_{s=1}^3 (3J_{rs} x_s - J_{ss} x_r) x_r, \quad (48)$$

It is easy to see that for this potential the integral

$$\int d^3r n(\mathbf{r}, t) x_i \frac{\partial V}{\partial x_j} = m \omega^2 J_{ij} + 4\kappa \sum_{s=1}^3 (3J_{is} J_{js} - J_{ij} J_{ss}) \quad (49)$$

is symmetric with respect  $i, j$ . As a result, the right hand side of equation (15) is equal to zero. Therefore, this model conserves the angular momentum.

Let us further note an interesting observation. With the model potential (46) our system of the dynamical equations (11), (13), (16) for the variables

$$\int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} x_i x_j f(\mathbf{r}, \mathbf{p}, t), \quad \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} p_i p_j f(\mathbf{r}, \mathbf{p}, t),$$

$$\int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} [x_i p_j]_{ij} f(\mathbf{r}, \mathbf{p}, t)$$

coincides with the system (65) derived in the paper of Schuck [33] for the variances-covariances

$$D(x_i x_j) = \langle (\hat{x}_i - x_i^c)(\hat{x}_j - x_j^c) \rangle, \quad D(p_i p_j) = \langle (\hat{p}_i - p_i^c)(\hat{p}_j - p_j^c) \rangle,$$

$$2D(x_i p_j) = 2D(p_j x_i) = \langle [(\hat{x}_i - x_i^c)(\hat{p}_j - p_j^c)]_{ij} \rangle,$$

where  $\hat{x}_i, \hat{p}_i$  are quantum operators and  $x_i^c, p_i^c$  are their classical counterparts. The equivalence of both systems is not very surprising because, being written for essentially the same variables, they do not involve any approximations in the case of the potential (46).

Now substituting expression (48) into equations (16), (13), we finally get the following system of equations for the collective variables  $J_{ij}$  and  $\Pi_{ij}$ :

$$\frac{m}{4} \ddot{J}_{ij} + \frac{m}{2} \omega^2 J_{ij} + 2\kappa \sum_{s=1}^3 (3J_{is}J_{js} - J_{ij}J_{ss}) - \Pi_{ij} = 0,$$

$$\dot{\Pi}_{ij} + \frac{m}{2} \omega^2 J_{ij} + \kappa \sum_{s=1}^3 [3J_{is}(J_{js} + \frac{1}{m}M_{s,j}) - J_{ij}J_{ss}]_{ij} = 0. \quad (50)$$

For the sake of simplicity the time dependence of tensors is again omitted. As angular momentum is conserved, the variables  $M_{i,j}$  do not depend on time and are given by the initial conditions. In this work we study the case without rotation and take  $M_{i,j} = 0$ .

This system has several integrals of motion. The first one is obtained by summing the second equation of (50) over  $i$ , setting  $j = i$ :

$$\sum_{i=1}^3 \left[ \Pi_{ii} + \frac{m}{2} \omega^2 J_{ii} + \kappa \sum_{s=1}^3 (3J_{is}^2 - J_{ss}J_{ii}) \right] = c_0. \quad (51)$$

Its physical meaning is the conservation of the total energy of the system. It is easy to see that it is equal to the Hartree-Fock average of the microscopic Hamiltonian (45), i.e.

$c_0 = \langle \Psi | H | \Psi \rangle$ . Others integrals can be found by using equation (20) which takes now the form:

$$\frac{d}{dt} \left( \frac{m}{8} J_{ij}^2 - J_{ij} \Pi_{ij} \right) = 3\kappa \sum_{s=1}^3 \left[ J_{ij} (J_{is} J_{js} + J_{is} J_{js}) - 2J_{ij} J_{is} J_{js} \right]. \quad (52)$$

Summing it over  $i, j$  one has:

$$\sum_{i,j=1}^3 \left( J_{ij} \Pi_{ij} - \frac{m}{8} J_{ij}^2 \right) = const. \quad (53)$$

If the nucleus has a triplanar symmetry, its inertia tensor is diagonal ( $J_{ij} = \delta_{ij} J_{ii}$ ), and equation (52) gives three integrals of motion:

$$J_{ii}(t) \Pi_{ii}(t) - \frac{m}{8} J_{ii}(t)^2 = c_i, \quad i = 1, 2, 3. \quad (54)$$

The integrals (54) for variances-covariances were found in another way by Schuck [33].

It is known [34] that a cartesian tensor of a second rank may be represented by a sum of three irreducible tensors: 1) a zero rank tensor (monopole moment), 2) a first rank pseudotensor (angular momentum) and 3) a second rank tensor (quadrupole moment):

$$T_{ij} \Rightarrow T_{00} + T_{1\mu} + T_{2\mu}.$$

Taking the linear combinations of eqs. (50), we are able to represent them through the irreducible tensors

$$Q_{00} = J_{11} + J_{22} + J_{33}, \quad \mathcal{K}_{00} = \Pi_{11} + \Pi_{22} + \Pi_{33},$$

$$Q_{20} = J_{11} + J_{22} - 2J_{33}, \quad \mathcal{K}_{20} = \Pi_{11} + \Pi_{22} - 2\Pi_{33},$$

$$Q_{2\pm 1} = \mp \sqrt{6}(J_{13} \pm iJ_{23}), \quad \mathcal{K}_{2\pm 1} = \mp \sqrt{6}(\Pi_{13} \pm i\Pi_{23}),$$

$$Q_{2\pm 2} = \sqrt{3/2}(J_{11} - J_{22} \pm 2iJ_{12}), \quad \mathcal{K}_{2\pm 2} = \sqrt{3/2}(\Pi_{11} - \Pi_{22} \pm 2i\Pi_{12}).$$

Generally speaking, all equations (50) are coupled. However, by a proper choice of the initial conditions the system (50) can be reduced to three cases which correspond to the components  $\mu = 0, 2, 1$  of the quadrupole moment and are known as  $\beta$ -mode,  $\gamma$ -mode and the transverse-shear mode (we will call it  $\alpha$ -mode).

### 4.1.1 $\beta$ -mode

It is easy to see that one of the possible solutions of the system (50) is:

$$J_{11}(t) - J_{22}(t) = 0, \quad J_{12}(t) = J_{13}(t) = J_{23}(t) = 0,$$

$$\Pi_{11}(t) - \Pi_{22}(t) = 0, \quad \Pi_{12}(t) = \Pi_{13}(t) = \Pi_{23}(t) = 0$$

with the rest of variables different from zero. This solution keeps the axial and triplanar symmetry of a nucleus ( $\beta$ -mode). In this case the system (50) is converted into

$$\begin{aligned} \frac{m}{4}\ddot{Q}_{00} + \frac{m}{2}\omega^2 Q_{00} + \kappa Q_{20}^2 - \mathcal{K}_{00} &= 0, \\ \frac{m}{4}\ddot{Q}_{20} + \frac{m}{2}\omega^2 Q_{20} + \kappa Q_{20}(2Q_{00} - Q_{20}) - \mathcal{K}_{20} &= 0, \\ \dot{\mathcal{K}}_{00} + \frac{m}{2}\omega^2 \dot{Q}_{00} + \kappa Q_{20} \dot{Q}_{20} &= 0, \\ \dot{\mathcal{K}}_{20} + \frac{m}{2}\omega^2 \dot{Q}_{20} + \kappa Q_{20}(2\dot{Q}_{00} - \dot{Q}_{20}) &= 0. \end{aligned} \quad (55)$$

The third equation of this system is reduced to the integral of motion

$$\mathcal{K}_{00} + \frac{m}{2}\omega^2 Q_{00} + \frac{\kappa}{2} Q_{20}^2 = \text{const} \quad (56)$$

that is just a particular case of (51). It is also easy to see that (55) has the particular solution  $Q_{20} = \mathcal{K}_{20} = 0$  corresponding to the simple harmonic oscillator and to pure monopole vibrations with the frequency  $\Omega = 2\omega$ .

### 4.1.2 $\gamma$ -mode

Taking

$$J_{12}(t) = J_{13}(t) = J_{23}(t) = 0, \quad \Pi_{12}(t) = \Pi_{13}(t) = \Pi_{23}(t) = 0,$$

we find a second solution of system (50) which keeps the triplanar symmetry of a nucleus but spoils its axial symmetry ( $\gamma$ -mode). The corresponding set of equations is

$$\begin{aligned} \frac{m}{4}\ddot{Q}_{00} + \frac{m}{2}\omega^2 Q_{00} + \kappa(Q_{20}^2 + 3J_-^2) - \mathcal{K}_{00} &= 0, \\ \frac{m}{4}\ddot{Q}_{20} + \frac{m}{2}\omega^2 Q_{20} + \kappa\{Q_{20}(2Q_{00} - Q_{20}) + 3J_-^2\} - \mathcal{K}_{20} &= 0, \\ \frac{m}{4}\ddot{J}_- + \frac{m}{2}\omega^2 J_- + 2\kappa J_-(Q_{00} + Q_{20}) - \Pi_- &= 0, \\ \dot{\mathcal{K}}_{00} + \frac{m}{2}\omega^2 \dot{Q}_{00} + \kappa(Q_{20} \dot{Q}_{20} + 3J_- \dot{J}_-) &= 0, \\ \dot{\mathcal{K}}_{20} + \frac{m}{2}\omega^2 \dot{Q}_{20} + \kappa\{Q_{20}(2\dot{Q}_{00} - \dot{Q}_{20}) + 3J_- \dot{J}_-\} &= 0, \\ \dot{\Pi}_- + \frac{m}{2}\omega^2 \dot{J}_- + \kappa\{J_-(2\dot{Q}_{00} + \dot{Q}_{20}) + Q_{20} \dot{J}_-\} &= 0, \end{aligned} \quad (57)$$

where  $J_- = J_{11} - J_{22} = (Q_{22} + Q_{2-2})/\sqrt{6}$  and  $\Pi_- = \Pi_{11} - \Pi_{22} = (\mathcal{K}_{22} + \mathcal{K}_{2-2})/\sqrt{6}$ .

The fourth equation can be reduced to the integral of motion

$$\mathcal{K}_{00} + \frac{m}{2}\omega^2 Q_{00} + \frac{\kappa}{2}(Q_{20}^2 + 3J_-^2) = \text{const}, \quad (58)$$

that is a particular case of (51).

Analyzing the above set of equations, one can find three particular solutions. For the first one we have  $J_-(t) = \Pi_-(t) = 0$ , and the system (57) is reduced to (55). The other two solutions correspond to  $J_-(t) = \pm Q_{20}(t)$ ,  $\Pi_-(t) = \pm \mathcal{K}_{20}(t)$ . The equality  $J_- = Q_{20}$  is equivalent to  $J_{22} = J_{33}$ , and the equality  $J_- = -Q_{20}$  leads to  $J_{11} = J_{33}$ . Hence, these particular solutions represent nuclear vibrations which keep the axial symmetry along the first and the second coordinate axes, respectively. The same kind of motion (with the third axis as the symmetry axis) is presented by the system (55). From the physical point of view, all three axes are equivalent to one another, so that the corresponding systems of equations are equivalent. Indeed, assuming  $J_- = \pm Q_{20}$ ,  $\Pi_- = \pm \mathcal{K}_{20}$  in eq. (57) and substituting the variables  $2Q_{20}$ ,  $2\mathcal{K}_{20}$  by  $-Q_{20}$ ,  $-\mathcal{K}_{20}$ , this system can be reduced to (55).

Formally there exists one more solution for each of the systems (55) and (57) if we take  $Q_{20}(t) = Q_{00}(t)$  as well as  $\mathcal{K}_{20}(t) = \mathcal{K}_{00}(t)$ . However, it has not much physical meaning because the equality  $Q_{20} = Q_{00}$  is equivalent to  $J_{33} = 0$ , i.e. we are dealing with a two-dimensional object.

### 4.1.3 $\alpha$ -mode

The most intricate solution is found when

$$J_{12}(t) = J_{23}(t) = 0, \quad \Pi_{12}(t) = \Pi_{23}(t) = 0$$

with the rest of variables being nonzero. Both triplanar and axial symmetries are abandoned in this case keeping the symmetry with respect to the reflection in the plane  $x_2 = 0$  alone. The corresponding system of equations is

$$\begin{aligned} \frac{m}{4}\ddot{Q}_{00} + \frac{m}{2}\omega^2 Q_{00} + \kappa(Q_{20}^2 + 12J_{13}^2 + 3J_-^2) - \mathcal{K}_{00} &= 0, \\ \frac{m}{4}\ddot{Q}_{20} + \frac{m}{2}\omega^2 Q_{20} + \kappa\{Q_{20}(2Q_{00} - Q_{20}) - 6J_{13}^2 + 3J_-^2\} - \mathcal{K}_{20} &= 0, \\ \frac{m}{4}\ddot{J}_- + \frac{m}{2}\omega^2 J_- + 2\kappa\{J_-(Q_{00} + Q_{20}) + 3J_{13}^2\} - \Pi_- &= 0, \\ \frac{m}{4}\ddot{J}_{13} + \frac{m}{2}\omega^2 J_{13} + \kappa J_{13}(2Q_{00} - Q_{20} + 3J_-) - \Pi_{13} &= 0, \end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{K}}_{00} + \frac{m}{2} \omega^2 \dot{Q}_{00} + \kappa (Q_{20} \dot{Q}_{20} + 3J_- \dot{J}_- + 12J_{13} \dot{J}_{13}) &= 0, \\
\dot{\mathcal{K}}_{20} + \frac{m}{2} \omega^2 \dot{Q}_{20} + \kappa \{ Q_{20}(2\dot{Q}_{00} - \dot{Q}_{20}) + 3J_- \dot{J}_- - 6J_{13} \dot{J}_{13} \} &= 0, \\
\dot{\Pi}_- + \frac{m}{2} \omega^2 \dot{J}_- + \kappa \{ J_-(2\dot{Q}_{00} + \dot{Q}_{20}) + Q_{20} \dot{J}_- + 6J_{13} \dot{J}_{13} \} &= 0, \\
\dot{\Pi}_{13} + \frac{m}{2} \omega^2 \dot{J}_{13} + \frac{\kappa}{2} \{ J_{13}(3\dot{J}_- - \dot{Q}_{20} + 4\dot{Q}_{00}) + J_{13}(3J_- - Q_{20}) \} &= 0, \quad (59)
\end{aligned}$$

where  $J_{13} = (Q_{2-1} - Q_{21})/\sqrt{24}$  and  $\Pi_{13} = (\mathcal{K}_{2-1} - \mathcal{K}_{21})/\sqrt{24}$ .

The fifth equation is reduced to the integral of motion

$$\mathcal{K}_{00} + \frac{m}{2} \omega^2 Q_{00} + \frac{\kappa}{2} (Q_{20}^2 + 3J_-^2 + 12J_{13}^2) = \text{const} \quad (60)$$

which is a particular case of (51).

As to mathematics, one nontrivial particular solution is admitted here:  $J_-(t) = -Q_{20}(t)$ ,  $\Pi_-(t) = -\mathcal{K}_{20}(t)$ . However, as we know, the equality  $J_- = -Q_{20}$  leads to  $J_{11} = J_{33}$ . From the physical point of view the inevitable consequence of the last relation is the equality  $J_{13} = 0$ , i.e. we return to the  $\beta$ -mode.

## 4.2 Analysis of the equations of motion

### 4.2.1 Stationary solution

Investigating the stationary solutions of the systems (55, 57, 59), we can draw some conclusions about the shape of nuclei. Let us study the most intricate case which is the system (59). Assuming the time derivatives in (59) to be zero one obtains four relations:

$$m \omega^2 Q_{00} + 2\kappa (Q_{20}^2 + 12J_{13}^2 + 3J_-^2) = 2\mathcal{K}_{00}, \quad (61)$$

$$m \omega^2 Q_{20} + 2\kappa \{ Q_{20}(2Q_{00} - Q_{20}) - 6J_{13}^2 + 3J_-^2 \} = 2\mathcal{K}_{20}, \quad (62)$$

$$m \omega^2 J_- + 4\kappa \{ J_-(Q_{00} + Q_{20}) + 3J_{13}^2 \} = 2\Pi_-, \quad (63)$$

$$m \omega^2 J_{13} + 2\kappa J_{13}(2Q_{00} - Q_{20} + 3J_-) = 2\Pi_{13}. \quad (64)$$

The first equation is a generalization of the well known virial theorem. This theorem tells that the average kinetic ( $\bar{T}$ ) and average potential ( $\bar{V}$ ) energies of the system are connected by the relation

$$2\bar{T} = k\bar{V} \quad (65)$$

if the potential  $V$  is proportional to  $r^k$  [35]. When the potential is the sum of several terms  $V_n$  proportional to different degrees of coordinates ( $V_n \sim r^n$ ) this relation is transformed

into

$$2\bar{T} = \sum_n n \bar{V}_n. \quad (66)$$

In our case the potential consists of two terms with  $n = 2$  and  $n = 4$ .

The rest of equations couples parameters of nuclear deformation in coordinate space to the ones of the Fermi Surface (FS) deformation (momentum space). A rather nontrivial conclusion can be derived from these relations. They tell that it will not be possible to produce any quadrupole deformation ( $Q_{20} \neq 0$ ,  $J_{13} \neq 0$ ,  $J_- \neq 0$ ) without simultaneous FS deformation ( $\mathcal{K}_{20} \neq 0$ ,  $\Pi_{13} \neq 0$ ,  $\Pi_- \neq 0$ ) and vice versa. To demonstrate this we transform equations (62-64) using the self-consistent value of the strength constant [33, 36]

$$\kappa = \kappa_{\text{Bohr}} = \frac{-m \omega^2}{4A \langle r^2 \rangle}. \quad (67)$$

Taking into account the relation  $Q_{00} = A \langle r^2 \rangle$ , we can present eq. (67) as  $m\omega^2 + 4\kappa Q_{00} = 0$  which allows one to simplify equations (62-64) :

$$\kappa (3J_-^2 - Q_{20}^2 - 6J_{13}^2) = \mathcal{K}_{20}, \quad (68)$$

$$2\kappa (J_- Q_{20} + 3J_{13}^2) = \Pi_-, \quad (69)$$

$$\kappa J_{13}(3J_- - Q_{20}) = \Pi_{13}. \quad (70)$$

If we assume  $\Pi_{13} = \Pi_- = \mathcal{K}_{20} = 0$  then eq. (70) is reduced to:  $3J_- = Q_{20}$ . Using this result in (69), we arrive at the relation  $Q_{20}^2 + 9J_{13}^2 = 0$  which can be satisfied only under the condition  $J_{13} = Q_{20} = J_- = 0$ . So, the Bohr self-consistency condition - the shape of the potential well follows the shape of the density - can be reformulated as: any variation of the density shape leads inevitably to the proper deformation of FS shape. We do not say "one shape follows another" because they can be deformed "in phase" or "out of phase". One can show it by analyzing eqs. (68-70). Let us consider an axially symmetric nucleus where  $J_{13} = J_- = 0$ . In this case eq. (70) is identical to  $\Pi_{13} = 0$ , eq. (69) turns into  $\Pi_- = 0$  (i.e.  $\Pi_{11} = \Pi_{22}$ ) and eq. (68) reads:

$$\mathcal{K}_{20} = -\kappa Q_{20}^2. \quad (71)$$

Keeping in mind that  $\kappa < 0$ , we find from (71) that  $\mathcal{K}_{20} > 0$  (i.e.  $\Pi_{11} > \Pi_{33}$ ) independently of the sign of  $Q_{20}$ . So, the nucleus' FS will always be oblate, in spite of the fact that the shape of the nucleus can either be prolate or oblate.

The fact that there is always a FS deformation in conjunction with a non-zero real space deformation in our model is, of course, a consequence that we implicitly never redistribute the particles after level crossings (adiabatic motion). Did we put (by hand or by an additional pairing interaction) the particles always in the lowest available single particle states (adiabatic motion), the FS would acquire only minor deviations from sphericity, however strong the real space deformation becomes [11, 37]. This conclusion concerns the equilibrium state and does not contradict the well-known statement that FS deformation is small for adiabatic processes [11] (see the next section).

To have a feeling on the order of magnitude of the FS deformation we express the ratio  $\mathcal{K}_{20}/\mathcal{K}_{00}$  in terms of the deformation parameter  $\beta$ . The desired relation is derived with the help of the formulae (71, 61, 67) and (80):

$$\frac{\mathcal{K}_{20}}{\mathcal{K}_{00}} = \frac{Q_{20}^2}{2Q_{00}^2 - Q_{20}^2} \approx \frac{Q_{20}^2}{2Q_{00}^2} = \frac{5}{2\pi}\beta^2. \quad (72)$$

This formula demonstrates quite well that the effect of the FS deformation is negligible for the ordinary nuclear deformations ( $\beta \leq 0.3$ ) in an accordance with the conclusion of [11]. However this effect becomes remarkable for  $\beta \geq 0.4$ .

#### 4.2.2 Small amplitude approximation

Let us consider the system (59) in the small amplitude approximation. Applying the infinitesimal variations of variables  $Q_{\lambda 0} = Q_{\lambda 0}^{eq} + \delta Q_{\lambda 0}$ ,  $J_- = J_-^{eq} + \delta J_-$ ,  $J_{13} = J_{13}^{eq} + \delta J_{13}$ ,  $\mathcal{K}_{\lambda 0} = \mathcal{K}_{\lambda 0}^{eq} + \delta \mathcal{K}_{\lambda 0}$ ,  $\Pi_- = \Pi_-^{eq} + \delta \Pi_-$ ,  $\Pi_{13} = \Pi_{13}^{eq} + \delta \Pi_{13}$  and neglecting the terms quadratic in  $\delta$ , one obtains four independent subsystems. One of them is the system for the monopole tensors

$$\begin{aligned} \frac{m}{4}\delta\ddot{Q}_{00} + \frac{m}{2}\omega^2\delta Q_{00} - \delta\mathcal{K}_{00} &= 0, \\ \delta\dot{\mathcal{K}}_{00} + \frac{m}{2}\omega^2\delta\dot{Q}_{00} &= 0 \end{aligned} \quad (73)$$

and the remaining ones are the systems for the components of the quadrupole tensors with  $\mu = 0, 1, 2$ . All three have the same structure. For example:

$$\begin{aligned} \frac{m}{4}\delta\ddot{Q}_{20} + \left(\frac{m}{2}\omega^2 + 2\kappa Q_{00}^{eq}\right)\delta Q_{20} - \delta\mathcal{K}_{20} &= 0, \\ \delta\dot{\mathcal{K}}_{20} + \frac{m}{2}\omega^2\delta\dot{Q}_{20} &= 0. \end{aligned} \quad (74)$$

Imposing the time evolution via  $e^{i\Omega t}$  for all variables, one can easily find the corresponding collective eigenfrequencies. The system (73) supplies the frequency of monopole vibrations:

$$\Omega_0 = 2\omega. \quad (75)$$

which is just the unperturbed shell model value, since there is no residual monopole interaction in our model. The system (74) for the quadrupole vibrations gives

$$\Omega_2 = 2\sqrt{\omega^2 + \frac{2\kappa}{m}Q_{00}^{eq}}. \quad (76)$$

Using here expression (67) for the strength constant, one obtains the well-known [33, 36] result for the quadrupole eigenfrequency:

$$\Omega_2 = \sqrt{2}\omega. \quad (77)$$

It is seen from the second equation of (74) that the amplitudes of  $\delta Q_{2\mu}$  and  $\delta \mathcal{K}_{2\mu}$  have opposite signs, i.e. the nuclear density and the Fermi surface vibrate out of phase. Furthermore, the order of magnitude of the factor  $m\omega^2/2$  is close to unity, therefore the FS deformation and the density deformation are of the same order.

#### 4.2.3 Numerical solution and Fourier analysis

The systems of equations (55, 57, 59) are solved numerically by means of the Runge-Kutta procedure. Most of the calculations are performed for  $^{208}\text{Pb}$  and  $^{40}\text{Ca}$ . The solutions depend strongly on the Initial Conditions (IC). We take  $Q_{2\mu}(0) = \mathcal{K}_{2\mu}(0) = 0$ ,  $Q_{00}(0) = Q_{00}^{eq}$ ,  $\mathcal{K}_{00}(0) = \mathcal{K}_{00}^{eq}$ . The equilibrium value of the monopole moment is chosen to be  $Q_{00}^{eq} = \frac{3}{5}R_0^2A$  with  $R_0 = 1.18A^{1/3}$ . The equilibrium value of  $\mathcal{K}_{00}^{eq}$  is fixed by relation (61). The time derivatives  $\dot{Q}_{\lambda\mu}(0)$  are varied arbitrarily.

The most detailed analysis has been performed for the  $\beta$ -mode. The typical time dependence of the function  $Q_{20}(t)$  is displayed in fig.3. As one can see,  $Q_{20}(t)$  oscillates quite irregularly. The behaviour is quite similar to the one found by W.Bauer et al [38]. The period of oscillations  $\tau_2$  (when the curve begins to repeat itself) depends crucially on IC. For this figure  $\tau_2 = 457.4 \text{ MeV}^{-1}$  ( $\tau = t/\hbar$ ). Knowing the basic period of the oscillations one can produce the Fourier analysis corresponding to the curves and represent

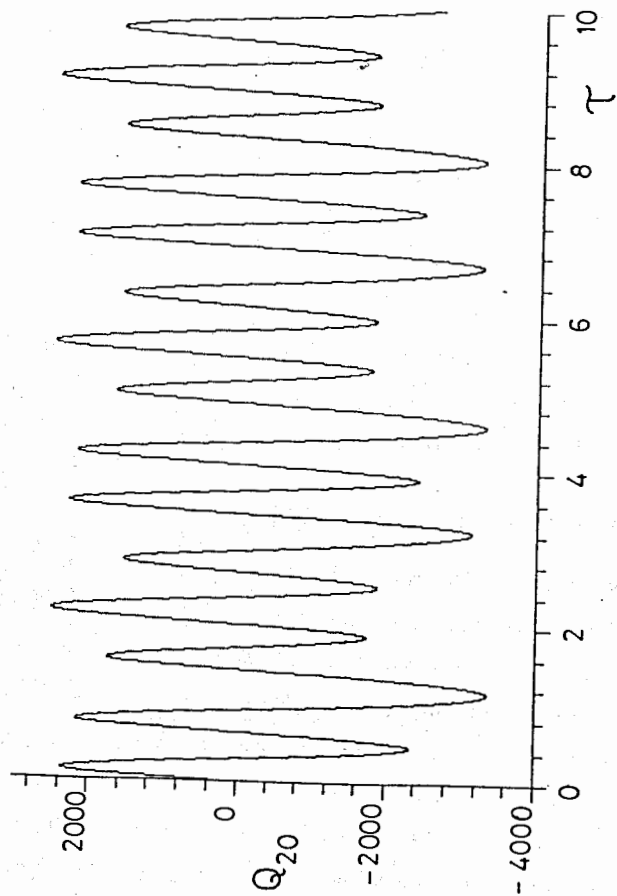


Fig.3. The time dependence ( $\tau = t/\hbar$ ) of the quadrupole moment of  $^{208}\text{Pb}$  for the initial conditions  $\dot{Q}_{00}(0) = 0$ ,  $\dot{Q}_{20}(0) = 25000$ .

all the functions by a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i \cos \omega_i t + b_i \sin \omega_i t). \quad (78)$$

The results of such calculations are shown in tables 3,4, where the frequencies  $\hbar\omega_i$  and the corresponding Fourier coefficients  $a_i$  and  $b_i$  of the functions  $Q_{20}$  and  $Q_{00}$  are presented for two versions of IC. Let us analyze Table 3 in detail. As one can see, there are about 30 frequencies whose amplitudes  $a_i, b_i$  range from  $10^{-2}$  to  $10^3$ , half of them range from 1 to  $10^3$ . How to interpret this variety of frequencies? As we have seen, the Fourier analysis of the one-dimensional model describing the dynamics of the monopole moment gave one basic frequency and an infinite number of satellites. In the case of a two-dimensional problem of coupled monopole and quadrupole motion ( $\beta$ -mode) it is quite natural to

Table 3 Fourier coefficients and energies for  $\dot{Q}_{00}(0) = 0$ ,  $\dot{Q}_{20}(0) = 18000$

i	$\hbar\omega_i$ MeV		$Q_{00}$		$Q_{20}$	
			$b_i$	$a_i$	$b_i$	$a_i$
0	0.00	.	0.00	202.47	0.00	-481.45
1	1.69	3M-4Q	-0.00	0.02	0.24	-0.84
2	2.05	3Q-2M	3.10	-0.39	-15.32	1.72
3	3.74	M-Q	-6.68	-0.91	196.11	32.15
4	5.80	2Q-M	-0.28	27.83	3.46	-68.38
5	7.48	2M-2Q	1.04	-3.68	-7.16	21.24
6	7.85	5Q-3M	-0.16	0.03	0.09	-0.00
7	9.17	5M-6Q	0.05	0.02	-0.01	0.02
8	9.54	Q	-83.83	-12.37	1727.16	373.68
9	9.90	8Q-5M	-0.06	0.01	0.00	0.03
10	11.22	3M-3Q	0.05	0.06	2.37	1.25
11	11.59	4Q-2M	-0.12	4.83	-0.68	6.70
12	13.28	M	64.72	-222.54	25.33	-64.26
13	13.64	7Q-4M	0.00	-0.20	0.01	-0.00
14	14.97	4M-4Q	-0.04	-0.02	0.01	-0.02
15	15.33	3Q-M	27.42	4.29	-9.05	-2.44
16	17.02	2M-Q	-2.48	-1.15	4.27	2.55
17	17.39	6Q-3M	0.01	0.02	0.01	-0.05
18	19.07	2Q	-30.12	99.72	29.42	-64.80
19	20.76	3M-2Q	-0.19	0.32	0.05	-0.06
20	21.13	5Q-2M	-0.60	-0.09	0.28	0.09
21	22.81	Q+M	4.52	2.08	6.07	4.05
22	24.87	4Q-M	0.16	-0.50	-0.86	1.67
23	26.56	2M	-0.11	0.18	0.42	-0.45
24	28.61	3Q	2.08	0.98	-4.33	-3.22
25	30.66	6Q-2M	-0.01	0.02	0.02	-0.04
26	32.35	2Q+M	-0.05	0.08	0.04	-0.04

Table 4 Fourier coefficients and energies for  $\dot{Q}_{00}(0) = 0$ ,  $\dot{Q}_{20}(0) = 25000$

i	$\hbar\omega_i$ MeV		Q <sub>00</sub>		Q <sub>20</sub>	
			b <sub>i</sub>	a <sub>i</sub>	b <sub>i</sub>	a <sub>i</sub>
0	0.00		0.00	378.83	0.00	-787.69
1	0.22	7Q-5M	1.32	-0.08	-21.26	1.31
2	1.70	3M-4Q	-0.17	2.14	7.09	-88.25
3	1.92	3Q-2M	59.95	1.12	-318.53	-5.95
4	3.39	6M-8Q	-0.26	1.62	3.92	-24.25
5	3.62	M-Q	-67.69	-6.72	785.15	77.94
6	5.53	2Q-M	-16.15	136.67	43.90	-371.48
7	5.75	9Q-6M	-11.29	-0.64	34.09	1.92
8	7.01	7M-9Q	-1.31	-0.35	12.65	3.35
9	7.23	2M-2Q	14.97	-74.64	-74.15	369.79
10	7.45	5Q-3M	-18.08	-2.48	31.82	4.37
11	8.93	5M-6Q	5.38	-18.85	-37.80	132.43
12	9.15	Q	-157.77	-34.71	2092.70	460.45
13	9.37	8Q-5M	0.88	-5.62	-17.96	114.87
14	10.85	3M-3Q	-20.76	-6.35	-120.68	36.92
15	11.07	4Q-2M	-21.13	88.14	-43.78	182.65
16	12.76	M	177.67	-544.27	113.80	-348.64
17	12.98	7Q-4M	15.84	4.11	14.76	3.83
18	14.46	4M-4Q	-13.89	33.26	4.01	-9.60
19	14.68	3Q-M	202.16	70.20	-62.13	-21.58
20	16.38	2M-Q	-59.05	-25.98	66.71	29.35
21	16.60	6Q-3M	-5.03	13.65	3.05	-8.28
22	18.08	5M-5Q	-24.21	-13.06	14.38	7.76
23	18.30	2Q	-67.25	145.42	44.50	-96.22
24	18.52	9Q-5M	-15.51	-6.05	10.67	4.16
25	19.99	3M-2Q	-11.69	20.73	5.06	-8.98
26	20.21	5Q-2M	-21.15	-10.26	13.53	6.57
27	21.91	Q+M	31.04	18.28	2.67	1.57
28	23.83	4Q-M	1.21	-1.97	-9.65	15.71
29	25.53	2M	-1.77	2.43	6.18	-8.45

expect the appearance of two basic frequencies  $\Omega_1, \Omega_2$  and their satellites. Due to coupling there exist many linear combinations of these frequencies:  $n_1\Omega_1 \pm n_2\Omega_2$ . For the  $\gamma$ -mode there must be three basic frequencies with corresponding satellites. One can recognize immediately the energies of the quadrupole (GQR) and monopole (GMR) giant resonances. They are very close to their values in a small amplitude approximation:  $E_2$  ( $\hbar\omega_8$ ) becomes 9.54 MeV instead of 9.78 MeV, and  $E_0$  ( $\hbar\omega_{12}$ ) becomes 13.28 MeV instead of 13.84 MeV. It is elementary to show that all remaining energies are just the combinations of the two basic energies:  $E_2$  and  $E_0$ . The results of combinatorial analysis are shown in the third columns of the tables. The comparison of tables 3 and 4 shows that the results are quite sensitive to IC. As to mathematics, this fact is absolutely correct - we have

already observed it in the case of the one dimensional model. Furthermore, any textbook illustrates such a dependence by the example of a nonlinear pendulum [35, 39] (see also section 4 of the paper). In this context it is interesting to know the order of magnitude of the deformation, attained during the vibrations. To this purpose one needs to express the quadrupole moment  $Q_{20}$  in terms of the deformation parameter  $\beta$ . We derive it assuming a sharp nuclear edge. By definition

$$Q_{20}(\beta) = n_0(\beta) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^{R(\theta,\phi)} dr r^2 (x_1^2 + x_2^2 - 2x_3^2). \quad (79)$$

Here  $R(\theta, \phi) = R_0(1 + \beta Y_{20}(\theta, \phi))$ ,  $x_1^2 + x_2^2 - 2x_3^2 = -4\sqrt{\frac{\pi}{5}}r^2 Y_{20}(\theta, \phi)$ , and the density  $n_0(\beta)$  is defined as

$$n_0(\beta) = A \left( \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_0^{R(\theta,\phi)} dr r^2 \right)^{-1}$$

Performing simple but tedious calculations, we get:

$$Q_{20}(\beta) = -4\sqrt{\frac{\pi}{5}}n_0(\beta)R_0^5 \left( \beta + \frac{4}{7}\sqrt{\frac{5}{4\pi}}\beta^2 + \frac{15}{14\pi}\beta^3 + \frac{100}{77}\sqrt{\frac{5}{(4\pi)^3}}\beta^4 + \frac{25 \cdot 53}{77 \cdot 13(4\pi)^2}\beta^5 \right), \quad (80)$$

$$n_0(\beta) = 3\frac{A}{R_0^3} \left( 4\pi + 3\beta^2 + \frac{2}{7}\sqrt{\frac{5}{4\pi}}\beta^3 \right)^{-1}$$

The calculations show that the limit of maximum possible amplitudes is achieved at  $\dot{Q}_{00}(0) = 0$ ,  $\dot{Q}_{20}(0) \simeq 28000$  (in Mev fm<sup>2</sup>/h units): the maximum positive value of  $Q_{20}$  is  $\sim 2700$ , and the maximum negative value is  $\sim -3800$ . By using (80), we find that the vibrations with the maximum amplitude correspond to  $\beta$  ranging from  $\sim 0.42$  to  $\sim -0.42$ . So, the nuclear shape changes during the vibrations from oblate to prolate. Further increase of  $\dot{Q}_{20}(0)$  leads to instability: the amplitudes begin to grow indefinitely. The amplitudes presented in fig.3 correspond to  $\beta \simeq 0.25$ .

As it was expected, the solution of system (57) yields three basic energies. For example, with IC  $\dot{Q}_{00} = 6100$ ,  $\dot{Q}_{20} = 10^4$ ,  $\dot{J}_- = 10^3$  we get:  $E_M = 13.66$  Mev (GMR),  $E_B = 9.5$  Mev ( $\beta$ -mode of GQR) and  $E_G = 9.75$  Mev ( $\gamma$ -mode). The difference  $\Delta E = E_G - E_B = 0.25$  Mev represents the splitting of the GQR due to large amplitude vibrations.



#### 4.2.4 Radiation probability

The radiation probability can be calculated by two methods. The first one uses the classical formula for an intensity of the quadrupole radiation [40]:

$$\text{Int} = \frac{1}{180 c^5} \sum_{k,l=1}^3 \ddot{D}_{k,l}^2 \quad (81)$$

where  $D_{k,l} = eZ/A(3J_{kl} - \delta_{kl} \sum_{s=1}^3 J_{ss})$ . Using the relations  $6J_{11} = 2Q_{00} + Q_{20} + 3J_-$ ,  $6J_{22} = 2Q_{00} + Q_{20} - 3J_-$ ,  $3J_{33} = Q_{00} - Q_{20}$ , one can represent the expression (81) in terms of our variables. So, for the  $\alpha$ -mode we obtain:

$$\text{Int} = \left(\frac{eZ}{A}\right)^2 \frac{1}{120 c^5} (\ddot{Q}_{20}^2 + 3 \ddot{J}_-^2 + 12 \ddot{J}_{13}^2). \quad (82)$$

The formulae for  $\gamma$ - and  $\beta$ -modes are obtained from this expression by neglecting  $J_{13}$  and  $J_{13}$ ,  $J_-$  respectively. For the sake of simplicity the following formulae will be proposed for the  $\beta$ -mode only. Assuming  $J_{13} = J_- = 0$  and inserting into (82) the Fourier expansion for  $Q_{20}$ , we get after averaging over the greatest period of oscillations:

$$\overline{\text{Int}} = \left(\frac{eZ}{A}\right)^2 \frac{1}{120 c^5} \sum_{i=1}^{\infty} \omega_i^6 \frac{a_i^2 + b_i^2}{2} \equiv \sum_{i=1}^{\infty} \overline{\text{Int}}_i. \quad (83)$$

Dividing  $\overline{\text{Int}}_i$  by  $\hbar\omega_i$ , we obtain the radiation probability  $W_i$ . Taking into account the relation between  $W_i$  and the reduced probability [11], we find:

$$B(E2)_i = \left(\frac{eZ}{A}\right)^2 \frac{5}{64\pi} (a_i^2 + b_i^2) = \left(\frac{Z}{A}\right)^2 \frac{125}{144} \frac{a_i^2 + b_i^2}{R^4} B_W, \quad (84)$$

where  $B_W$  is the Weisskopf unit. The generalization for  $\gamma$ - and  $\alpha$ -modes is elementary.

The second method is based on a classical limit relation between the average value  $\langle \hat{f} \rangle$  of some operator  $\hat{f}$  and its classical counterpart  $f(t)$ . It is known [41] that in the classical limit the matrix elements  $\langle m | \hat{f} | n \rangle$  are equal to coefficients  $f_{m-n}$  of the Fourier expansion of the function  $f(t)$  (see also [33]):

$$f(t) = \sum_{s=-\infty}^{+\infty} f_s \exp i\omega_s t. \quad (85)$$

Hence, in a classical limit the formula

$$B(E2)_{m-n} = |\langle m | \hat{f} | n \rangle|^2 \quad (86)$$

is converted into

$$B(E2)_s = |f_s|^2 \quad (87)$$

Remembering that  $\hat{f} = \frac{eZ}{A} r^2 Y_{2\mu}(\theta, \phi) = \frac{eZ}{A} \sqrt{5/16\pi} q_{2\mu}(\mathbf{r})$  and taking into account the relation between the coefficients of different representations (85) and (78) of the Fourier expansion  $f_{\pm s} = (a'_s \pm ib'_s)/2$ , we find:

$$f_{\pm s} = \frac{eZ}{A} \sqrt{5/16\pi} (a_s \pm ib_s)/2 \quad (88)$$

Substituting this expression into (87), we reproduce immediately the formula (84). The second method is also applicable for the derivation of the formula for  $B(E0)$ :

$$B(E0)_i = \left(\frac{eZ}{A}\right)^2 \frac{1}{16\pi} (a_i^2 + b_i^2) = \left(\frac{Z}{A}\right)^2 \frac{1}{4} (a_i^2 + b_i^2) B_W, \quad (89)$$

Some numerical results will be presented in the next section.

## 5 Quantization of $\beta, \gamma$ -modes

The classical Hamiltonians for  $\beta, \gamma$  modes are found easily with the help of the energy integral (51) and the integrals of motion (54). The energy integral for these modes takes the form:

$$\sum_{i=1}^3 \left[ \Pi_i + \frac{m}{2} \omega^2 J_i + \frac{\kappa}{2} \sum_{s=1}^3 (J_i - J_s)^2 \right] = c_0. \quad (90)$$

where  $J_i \equiv J_{ii}$ ,  $\Pi_i \equiv \Pi_{ii}$ . Using the integrals (54), one can write the kinetic energy term  $\sum_{i=1}^3 \Pi_i$  as:

$$\sum_{i=1}^3 \Pi_i = \sum_{i=1}^3 \left( \frac{m}{8} J_i^2 + c_i \right) / J_i. \quad (91)$$

Introducing now the momenta  $P_i = \frac{m}{4J_i} \dot{J}_i$  instead of velocities  $\dot{J}_i$ , one obtains the Hamiltonian in terms of canonical variables  $P_i$  and  $J_i$ :

$$H = \sum_{i=1}^3 \left[ \frac{2J_i}{m} P_i^2 + \frac{c_i}{J_i} + \frac{m}{2} \omega^2 J_i + \frac{\kappa}{2} \sum_{s=1}^3 (J_i - J_s)^2 \right] = c_0. \quad (92)$$

It is easy to see that the first of equations (50) coincides with the Hamilton equation  $\dot{P}_i = -\frac{\partial H}{\partial J_i}$  that justifies our choice of canonical variables. The other Hamilton equation  $\dot{J}_i = \frac{\partial H}{\partial P_i}$  is identical to our definition of the momentum  $P_i$ .

Let us consider in more details the  $\beta$ -mode. Assuming  $J_1 = J_2$ ,  $c_1 = c_2$  and introducing the momenta  $P_1 = \frac{m}{2J_1} \dot{J}_1$ ,  $P_3 = \frac{m}{4J_3} \dot{J}_3$ , we find from (90, 91) the corresponding Hamiltonian:

$$H_\beta = \frac{J_1}{m} P_1^2 + \frac{2J_3}{m} P_3^2 + \frac{2c_1}{J_1} + \frac{c_3}{J_3} + \frac{m}{2} \omega^2 (2J_1 + J_3) + 2\kappa (J_1 - J_3)^2. \quad (93)$$

We need the Hamiltonian in terms of irreducible tensors  $Q_{00}, Q_{20}$  and their conjugate variables. Changing the variables  $J_1, J_3$  to  $Q_2 \equiv Q_{20} = 2(J_1 - J_3)$ ,  $Q_0 \equiv Q_{00} = 2J_1 + J_3$ , we find:

$$H_\beta = \frac{2}{3m} (Q_+ P_+^2 + Q_- P_-^2) + \frac{12c_1}{Q_+} + \frac{3c_3}{Q_-} + \frac{m}{2} \omega^2 Q_0 + \frac{\kappa}{2} Q_2^2, \quad (94)$$

where  $Q_+ = 2Q_0 + Q_2$ ,  $Q_- = Q_0 - Q_2$ ,  $P_+ = P_0 + P_2$ ,  $P_- = P_0 - P_2$ . The momenta  $P_2, P_0$  are defined by the formulae:

$$P_2 = \frac{m}{12} \left( \frac{\dot{Q}_+}{Q_+} - \frac{\dot{Q}_-}{Q_-} \right), \quad P_0 = \frac{m}{12} \left( 2\frac{\dot{Q}_+}{Q_+} + \frac{\dot{Q}_-}{Q_-} \right).$$

It is seen that Hamiltonian (94) is strongly nonlinear and nonpolynomial. The mass coefficients depend on the variables  $Q_0, Q_2$ . The collective potential

$$V = \frac{12c_1}{2Q_0 + Q_2} + \frac{3c_3}{Q_0 - Q_2} + \frac{m}{2} \omega^2 Q_0 + \frac{\kappa}{2} Q_2^2 \quad (95)$$

which one may call the diabatic potential, contains the terms (proportional to  $c_1, c_3$ ) that come from the microscopic kinetic energy. In agreement with the considerations of section 2.4, we take for the constants  $c_1, c_3$  the equilibrium values. Substituting into (54) the relation  $2\Pi_{ii}^{eq} = m\omega^2 J_{ii}^{eq}$  following from (61), we find:

$$c_1^{eq} = c_3^{eq} = \frac{1}{2} m\omega^2 (J_{ii}^{eq})^2 = \frac{1}{18} m\omega^2 (Q_0^{eq})^2.$$

The general picture of the potential energy surface is demonstrated by fig.4 where the family of equipotential curves for  $^{40}\text{Ca}$  is shown. Obviously, physical sense has only the area above the lines  $Q_0 = Q_2$  and  $2Q_0 = -Q_2$ . The potential has infinitely high walls on these lines and grows linearly with increasing  $Q_0$ . Its minimum lies at  $Q_2 = 0, Q_0 = Q_0^{eq} = 390 fm^2$ . There are two saddle points: at negative  $Q_2$  ( $Q_2 \simeq -290 fm^2, Q_0 \simeq 480 fm^2$ ) where the barrier height is  $\sim 25$  Mev, and at positive  $Q_2$  ( $Q_2 \simeq 710 fm^2, Q_0 \simeq 940 fm^2$ ), where the barrier height is  $\sim 233$  MeV. So, the stable vibrations in  $^{40}\text{Ca}$  are possible only with energies less than 25 MeV. The vibrations with higher energies are unstable:

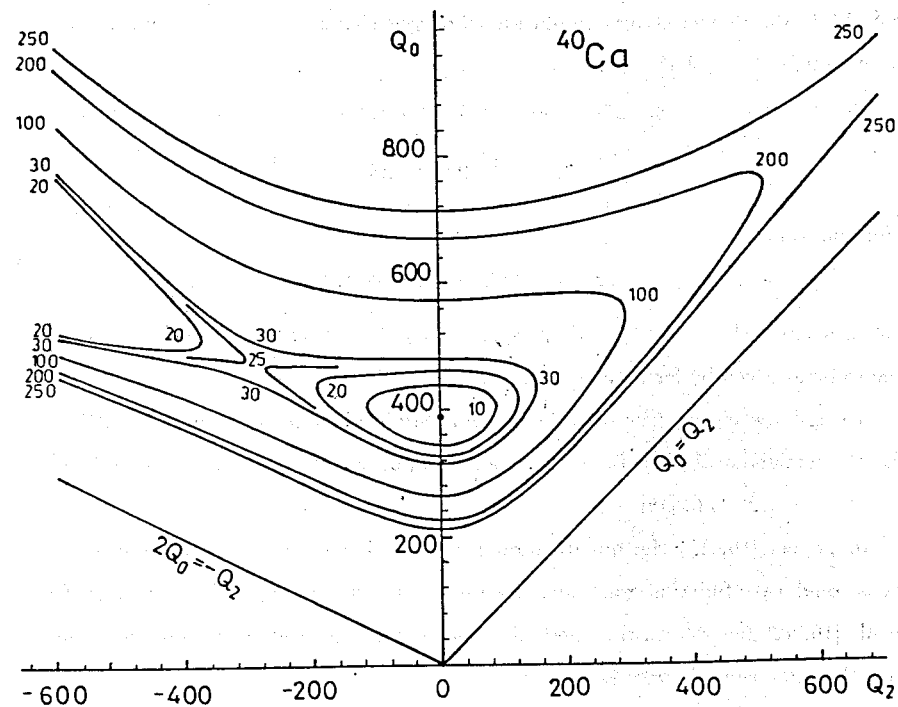


Fig.4. The equipotential curves of the collective potential for  $^{40}\text{Ca}$ . The scales for  $Q_0, Q_2$  are in  $fm^2$ . The numbers near the curves are energies in Mev.

after several oscillations (depending on the initial values of  $Q_2, Q_0$ ) the amplitudes begin to increase indefinitely. The equipotential curve at 25 MeV shows that the maximum amplitude vibrations change the nuclear deformation in the limits  $-0.32 \leq \beta \leq 0.49$ . The analogous results for  $^{208}\text{Pb}$  are: saddle points lie at  $Q_2 \simeq -4300 fm^2, Q_0 \simeq 7360 fm^2$  (the barrier height  $\simeq 130$  Mev) and at  $Q_2 \simeq 10600 fm^2, Q_0 \simeq 14121 fm^2$  (the barrier height  $\simeq 1212$  Mev). The maximum amplitude vibrations change the nuclear deformation in the limits  $-0.31 \leq \beta \leq 0.47$ .

In a small amplitude approximation the Hamiltonian splits into monopole

$$H_M = P_0^2/(2M) + 2M\omega^2 x^2$$

and quadrupole

$$H_Q = P_2^2/M + (M\omega^2 + \kappa/2)y^2$$

parts, where  $4M = m/Q_0^{eq}, x = \delta Q_0(t), y = \delta Q_2(t)$ . Naturally their eigenfrequencies are the same as given by formulae (75), (76).

To find the eigenvalues of the quantum Hamiltonian corresponding to (94) we will use the semiclassical procedures developed for applications in multidimensional problems [14, 17], [16, 42], [43], [18].

In papers [14, 17] the quantization rule is derived from the requirement that the variational wave function must be gauge invariant and periodic (GIPQ method). Caurier et al. [16, 42] generalized this method for weakly non-separable systems. Both methods give the same rule of quantization:

$$\oint_{c_i} \sum_k p_k dq_k = 2\pi\hbar n_i, \quad n_i = 0, 1, \dots, \quad (96)$$

where  $c_i$  is a closed trajectory in the space of conjugate variables  $p_k, q_k$ . This result practically coincides with the well known [43] Einstein-Brillouin-Keller quantization rule:

$$\oint_{c_i} \sum_k p_k dq_k = 2\pi\hbar(n_i + \alpha_i/4), \quad n_i = 0, 1, \dots, \quad (97)$$

The only difference is the Maslov index  $\alpha_i$ , showing the number of times the trajectory  $c_i$  touches the edges of the classically allowed region.

The described methods are the generalizations of the Bohr-Sommerfeld quantization rule. The Cambiaggio method [18] can also be generalized to be applied for multidimen-

sional systems. The generalization is trivial: the procedure described in section 3.4 is performed separately for each mode.

Poincare sections are usually used to find the periodic trajectories [16], [43]. However, in the paper [42] a method, very convenient for practical applications, was proposed. Using this method in our case, it is necessary to start with the initial conditions  $P_0(0) = 0, P_2(0) = 0$  at some energy  $E$  and to find the least coupled modes by varying  $Q_0(0)$  and  $Q_2(0)$ . The distinctive sign of such a mode is the periodicity (or quasiperiodicity) of the corresponding trajectory, i.e. after one oscillation the trajectory must return practically to the same point  $P_0(0), P_2(0), Q_0(0), Q_2(0)$  (the deviation from these values must be minimal). Changing the energy  $E$ , one finds the trajectory obeying the quantization condition (96).

The set of initial values of  $Q_0, Q_2$ , at which the quadrupole mode is least coupled with the monopole one for  $^{208}\text{Pb}$ , is shown in fig.5 by crosses. The example of the least-coupled trajectory for the energy  $E \simeq 24$  Mev is presented by the dashed curve. It is easy to see that for this trajectory the Maslov index is  $\alpha = 2$ . There exists a great variety of non-periodic trajectories and their pictures resemble very much the analogous ones of the other studies [42], [43], [44].

The results of quantization of the quadrupole mode in  $^{208}\text{Pb}$  are shown in table 5. The second, third and fourth columns present the results of calculations by formulae (96,97) and the Cambiaggio method, respectively, for  $P_0(0) = 0, P_2(0) = 0$  and  $Q_0(0), Q_2(0)$  corresponding to the least-coupled modes (crosses in fig.5). It is seen that all spectra are quite similar, the third and fourth columns (marked by II and III) being practically indistinguishable. So, one can conclude that formula (97) is preferable to quantize our equations of motion. The table demonstrates the weak anharmonicity of the quadrupole excitations spectrum. The distance between the levels is decreased by  $\sim 0.06$  Mev, when the excitation is increasing. The same order of magnitude has the difference  $E_{RPA} - (E_1 - E_0) = 0.04$  Mev (let us remind that  $E_{RPA} = 9.78$  Mev), which also can serve as a measure of the anharmonicity.

The Cambiaggio method allows one to quantize with arbitrary initial conditions. We have calculated the quadrupole spectrum with the initial values of  $Q_0, Q_2$  disposed approximately on the line OB of fig.5 (fifth column of table 5). The comparison of the fourth

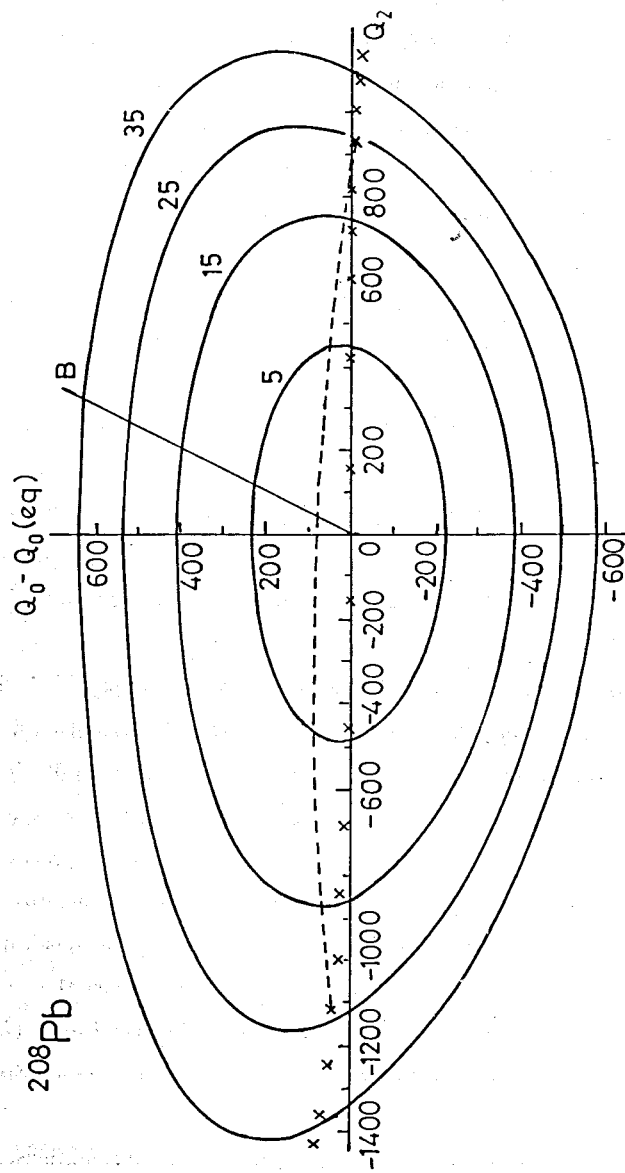


Fig.5. The equipotential curves of the collective potential for  $^{208}\text{Pb}$ . The scales for  $Q_0$ ,  $Q_2$  are in  $\text{fm}^2$ . The numbers near the curves are energies in Mev. The set of initial values of  $Q_0$ ,  $Q_2$ , at which the quadrupole mode is least coupled with the monopole one, is shown by crosses. The least-coupled trajectory for the energy  $E \simeq 24$  Mev is demonstrated by the dashed curve.

and fifth columns leads to the conclusion that the lower part of spectrum is not very sensitive to the choice of initial conditions, the difference being appreciable starting from the five-phonon state.

Table 5. Distances (in Mev) between the levels of the quadrupole spectrum of  $^{208}\text{Pb}$ . I, II, III- the energies are calculated for the least-coupled modes (see the text) using formulae (96, 97) and the Cambiaggio method [18] respectively. IV - the Cambiaggio method for initial conditions corresponding to the line 0B of fig.5.

$\Delta E$	I	II	III	IV
$E_1 - E_0$	9.7613	9.7354	9.7358	9.7349
$E_2 - E_1$	9.7081	9.6793	9.6799	9.6830
$E_3 - E_2$	9.6490	9.6172	9.6177	9.6314
$E_4 - E_3$	9.5831	9.5467	9.5484	9.5792
$E_5 - E_4$	9.5094	9.4691	9.4708	9.5263

The monopole mode is exactly decoupled from the quadrupole one for the initial condition  $Q_2(0) = 0$  that corresponds to the particular solution  $Q_2(t) = 0$  of the system (55). This result has a natural physical explanation. It is possible to change the root mean square radius (r.m.s.) of a spherical nucleus without disturbing its shape (breathing mode), but it is impossible to change the quadrupole moment of a nucleus (i.e. to excite quadrupole vibrations) without disturbing its r.m.s. (i.e. without exciting monopole vibrations). Due to the absence of nonlinear terms the system (55) exhibits in this particular case the lack of any anharmonicity and has only one monopole excitation that is shown in the appendix A.

The nonlinear effects increase remarkably when the mass number  $A$  is decreasing. Two factors are most important here. First: the height of the collective potential barrier is decreasing. Second: the giant resonance energy  $E_{GQR}$  is increasing. Their combined action leads to a rapid decrease of the number of bound states with decreasing mass.

For example the barrier height  $V_B$  in  $^{208}\text{Pb}$  is about 130 Mev,  $E_{GQR} \simeq 10$  Mev, so one can find about 12 bound states in its potential well (taking into account the zero oscillations energy  $\sim 5$  Mev). The nucleus with  $A=70$  has  $V_B \sim 40$  Mev allowing only two bound states for which  $E_1 - E_0 \equiv E_{GQR} = 13.69$  Mev and  $E_2 - E_1 = 13.08$  Mev. Here the difference  $(E_1 - E_0) - (E_2 - E_1) = 0.6$  Mev is an order of magnitude more than

in  $^{208}\text{Pb}$ . The same is true also for the difference  $E_{RPA} - E_{GQR} = 0.38\text{Mev}$ .

The  $^{40}\text{Ca}$  has only one bound state with  $E_{GQR} = 15.78\text{ Mev}$  (see the fig.4). This energy differs from  $E_{RPA} = 16.95\text{ Mev}$  by 1.17 Mev that exceeds 30 times the respective value for  $^{208}\text{Pb}$ , i.e. the anharmonicity in light nuclei is considerably stronger than in heavy ones.

And finally, the nuclei with  $A < 40$  have no bound states at all. After the appearance of multiphonon states in heavy nuclei it is the most remarkable effect of the nonlinearity.

Naturally an interpretation of this result is required because one knows that the GQR in light nuclei is observed experimentally. The interpretation could be twofold: 1) the theory is in line with the general trends of the experimental situation, 2) the model must be modified to improve the agreement with an experiment.

In connection with the first statement it will be useful to remind that the GQR systematic is usually given separately for  $A > 40$  and  $A < 40$  nuclei because of the principally different behaviour of GQR in heavy and light nuclei. A citation from the review of A. van der Woude [45] will be helpful here: "For  $A > 40$  nuclei 50-100% of the E2 EWSR has been localized in a peak at about  $65A^{-1/3}\text{ Mev}$  and with a width which is slowly increasing with decreasing mass from  $\sim 2.5\text{ Mev}$  for  $^{208}\text{Pb}$  to about 4.5 Mev around  $A \sim 90$ ." He continues further: "In sd-shell nuclei the GQR strength is fragmented..., around 40%, 30%, 25% and 60% of the E2 EWSR was localized in the region between 14 and 25 Mev for  $^{24}\text{Mg}$ ,  $^{26}\text{Mg}$ ,  $^{28}\text{Si}$  and  $^{40}\text{Ca}$  respectively." And else [46]: "... it should be noted that in still lighter nuclei like  $^{12}\text{C}$  only a small amount of the GQR strength has been located, less than 16%, which probably signifies the disappearance of this collective phenomenon in very light nuclei."

The exact bound of the area of the GQR existence depends naturally on the forces used in calculations and on the definition of GQR. Microscopically giant resonances are described as a coherent superposition of 1particle-1hole excitations. In heavy nuclei these excitations are coherent enough to create the collective state which exhausts most of the EWSR and can be treated as the vibration of the nucleus' quadrupole moment. The coherence is decreasing gradually with decreasing mass number, so the E2 strength becomes more and more distributed and less and less concentrated in the region of GQR and this region itself becomes too wide. At some critical value  $A_{cr}$  of the mass number

the contribution of the GQR region into EWSR becomes less than that of low (and maybe higher) lying states. Strictly speaking there is no GQR in nuclei with  $A < A_{cr}$ .

In connection with the second point of the interpretation it is necessary to note that our simple model gives only one  $2^+$  state that exhausts 100% of EWSR. So we are able to reproduce the experimental situation more or less adequately when most of the experimental E2 strength is concentrated in a small region. The more complicated situation of strongly distributed strength can not be described by the model - that is revealed by the lack of any bound states in the collective potential. The generalization of the model for the description of more than one collective  $2^+$  excitations can be done by taking into account additional degrees of freedom. The respective collective variables are derived by taking the phase space moments of Wigner function with the weights  $x_i x_j x_k x_l$ ,  $x_i x_j x_k p_l$ ,  $x_i x_j p_k p_l$ ,  $x_i p_j p_k p_l$ ,  $p_i p_j p_k p_l$  and reducing the obtained tensors. It was shown in [47] (in a small amplitude approximation) that low lying  $2^+$  states appear in this case.

One should not be confused by the small amplitude approximation result  $E = \sqrt{2}\hbar\omega$ , that has no formal limits of applicability, because this result does not say anything about the existence or nonexistence of GQR but only shows the scale of the quantum energy spectrum. So, if one accepts the definition, that GQR is the vibration of the nucleus' quadrupole moment, then  $A_{cr}$  can be determined very well with the help of our model. It is interesting to recall the A. van der Woude's [45] definition of giant resonances: "Giant resonances are small amplitude, high frequency, simple, collective modes of excitations of nuclei." We agree with all these attributes of GR except the first one. The latter does not work everywhere. For example the GQR in  $^{208}\text{Pb}$  corresponds to the vibrations of its quadrupole moment  $Q_{20}$  from  $750\text{fm}^2$  to  $-850\text{fm}^2$  what can be seen from fig.5 (the equipotential curve for 15 Mev:  $E_{GQR} + \text{energy of zero point oscillations}$ ). This range of the change of  $Q_{20}$  corresponds to a variation of the quadrupole deformation  $\beta$  from -0.1 to +0.1. In principle this is not such a small deformation, however its square value  $\beta^2$  is negligible, being an order of magnitude less, what explains the applicability of a small amplitude approximation in heavy nuclei. The situation in light nuclei is quite different. During the vibrations the quadrupole moment of  $^{40}\text{Ca}$  is changed from  $130\text{fm}^2$  to  $-250\text{fm}^2$  (see the equipotential curve of fig.4 for 24 Mev). The respective quadrupole

deformation  $\beta$  is changed from -0.30 to 0.43. This is already a rather large amplitude that does not allow the small amplitude approximation.

Let us also mention that the coincidence of the disappearances of the GQR around  $^{40}\text{Ca}$  in our model on the one hand and also on the experimental side may be a pure accident. Indeed it is difficult to imagine that a realistic force produces a barrier in quadrupole direction in a purely diabatic scenario neglecting on top of it the Coulomb force. On the other hand we have seen that for lighter nuclei the deformations attained during the GQR are much stronger and therefore also the deformations in momentum space. The latter ones probably can easily become so large that particles are spilled into the continuum what implies a strong damping from escape. This may also indicate the end of a well defined GQR. In this context it would be very interesting to study the potential of deformation energy in the purely diabatic case for a realistic situation.

Concerning the transition probabilities, having formulæ (84, 89) at hand, we are able to calculate them for the various levels. For example, substituting into (84) the values of  $a_i$  and  $b_i$ , calculated for  $^{208}\text{Pb}$  with the input energy  $E=9.735$  Mev, we find the  $B(E2)$ -factors for GQR in the case of  $\beta$ -excitation:

$$B(E2, \text{GQR}) = 23.5 B_W = 1609.9 e^2 fm^4.$$

This result is in qualitative agreement with other studies [46]. Repeating the calculations with the input energy  $E=9.735+9.679=19.414$  Mev and using  $a_i, b_i$  corresponding to twice the energy of GQR (satellite), we get:

$$B(E2, 2 \times \text{GQR}) = 0.02 B_W = 0.9 \cdot 10^{-3} B(E2, \text{GQR}).$$

So, the  $B(E2)$ -factor of the double GQR is three orders of magnitude less than that of the GQR. The  $B(E2)$ -factor for the three-phonon state is six orders of magnitude less than that of the GQR.

It is interesting to compare the dependence of  $B(E2)$  factors of the GQR and double GQR on the value of  $\kappa$  (the strength parameter of the quadrupole-quadrupole interaction). For example, taking  $\kappa = 0.5\kappa_{\text{Bohr}}$  we find that the  $B(E2)$  of the GQR is decreased by 1.23 times, whereas that of the double GQR is decreased by 13.9 times. Further decreasing of  $\kappa$  (to  $0.1\kappa_{\text{Bohr}}$ ) leads to the decreasing of the  $B(E2)$  of the GQR by 1.37 times (in comparison with the case of  $\kappa = \kappa_{\text{Bohr}}$ ), whereas that of the double GQR is decreased by 735.6 times

that means practically the disappearance of this excitation. This is the natural result because the multi-phonon resonances existence itself is only due to the anharmonic term of the Hamiltonian. The similar conclusion was done in the paper [48]: "the inclusion of small anharmonicities and non-linearities strongly enhance the excitation cross section of the two phonon states without modifying much the population of the one phonon state".

The estimate of the  $B(E0)$  factor for GMR can be found in the Appendix A.

The authors of Ref. [49] found the deexcitation rates of the one- and two-phonon GQR by using a microscopic approach with the Skyrme forces. The calculations were done for  $^{40}\text{Ca}$  only. Their results are:

$$W(\text{GQR}) = 0.6 \cdot 10^{16} s^{-1},$$

$$W(2 \times \text{GQR}) = 0.26 \cdot 10^{15} s^{-1} \simeq \frac{1}{23} W(\text{GQR}).$$

In our calculations the GQR energy of  $^{40}\text{Ca}$  is  $E(\text{GQR})=15.78$  Mev. Using  $a_i, b_i$ , calculated with the input energy  $E=15.78$  Mev, we find

$$W(\text{GQR}) = 1.12 \cdot 10^{17} s^{-1},$$

that is approximately twenty times more than that of [49]. Using the Fourier coefficients  $a_i, b_i$  of the  $2E(\text{GQR})$  satellite, calculated with the same input energy  $E=15.78$  Mev, we get

$$W(2 \times \text{GQR}) = 8.92 \cdot 10^{15} s^{-1} \simeq \frac{2}{25} W(\text{GQR}),$$

that is approximately forty times more than that of [49] though the ratio  $W(2)/W(1)$  is only off by a factor of 2. However, this number has not much physical meaning because the double GQR energy  $2E(\text{GQR})=31.56$  Mev exceeds the barrier energy  $E_B \simeq 25$  Mev, and a stable two-phonon GQR in  $^{40}\text{Ca}$  does not exist in our model (see the discussion after fig.4) contradicting the experimental prediction [3]. This is quite similar to the situation with the disappearance of GQR in  $A < 40$  nuclei and we suppose that it is resolved by the same considerations (see above).

Of course, the barrier height depends on the forces used in the calculations, so it would be interesting to repeat the calculations of the paper [49] for the harmonic oscillator with the Q-Q residual interaction. In any case one should not expect the absolute coincidence of the results because working in a coordinate space we treat all interactions exactly,

whereas the authors of [49] are compelled to do approximations in their approach (boson representations of fermion operators).

## 6 Conclusions

Let us list the main results of the paper.

Two models with separable forces are considered. Sets of nonlinear dynamical equations for monopole  $Q_{00}$  and quadrupole  $Q_{2\mu}$  moments of nuclei are derived from the TDHF equation by using the method of the Wigner function moments. The collective Hamiltonians, which generate these equations, are constructed. In accordance with general theorems they are just the mean values of the respective microscopic Hamiltonians. The new element of our approach is the division of the standard variational procedure, described in the introduction, in two steps. One derives at first the equation of motion for the density matrix (or Wigner function). After it, taking the phase space moments of this equation, one introduces the collective variables (the variational parameters  $\rho_i, \pi_i$ , mentioned in the introduction), deriving simultaneously the dynamical equations for them.

The classical and quantum aspects of the analytically solvable one-dimensional monopole model of Suzuki are revisited. It is shown that the anharmonicity of the collective spectrum, being the specific property of quantum systems, cannot be observed in classical ones. The choice of the initial conditions, necessary for a quantization of the models, is established.

Large amplitude vibrations of  $Q_{2\mu}$  and  $Q_{00}$  are described in the model of the harmonic oscillator with the Q-Q residual interaction. The nonlinear equations of motion are derived exactly, without any approximations. They are solved numerically. It is found that the functions  $Q_{2\mu}(t)$  and  $Q_{00}(t)$  oscillate irregularly. The maximum amplitudes correspond to the deformation parameter  $\beta \simeq 0.50$ . Their Fourier analysis exhibits giant quadrupole and monopole resonances and multiphonon states built on them. The essential features of the large amplitude motion manifest themselves by the coupling of GMR and GQR and by the GQR splitting in spherical nuclei. The radiation probability of the two-phonon giant quadrupole resonance turns out three orders of magnitude less than that of the one-phonon GQR. The quantization of the model allows one to observe the spectrum's anharmonicity that increases with the mass number decreasing. The analysis of the

potential energy surface shows that the number of bound states in the collective potential (or multiphonon excitations of the model) is decreased quickly with decreasing mass, reaching zero at  $A \sim 40$ , i.e. the model predicts the disappearance of the GQR in these nuclei. The experimental situation is more complicated, demonstrating the large decrease of the collectivity of GQR in light nuclei and the real disappearance of this phenomenon in very light ones. So, an improvement of the model is required.

The theory can be modified to take into account spin and isospin degrees of freedom. Then it will be possible to study spin and isovector collective modes for the case of large amplitude motion. Considering higher rank tensors, one is able to describe low-lying modes. The extension to the description of excitations of higher multiplicities is straightforward.

It should also be mentioned that the separable multipole-multipole forces considered here lead to a bounded (time-dependent) mean field. This entails that the Fourier spectrum only reveals discrete states. In a realistic TDHF calculation, such as it was performed by Flocard et al [50], there is very likely a strong coupling to the continuum and the Fourier analysis may only yield a spectrum of strongly overlapping resonances of various multiplicities. Though such a spectrum may bear some resemblance with an experimental one, it certainly will lack quantization. On the other hand quantisation of damped TDHF motion is to our knowledge an unresolved problem.

Acknowledgements:

We would like to thank C.Volpe for valuable discussions.

## Appendix A

In the case  $Q_{20}(t) = 0$  the system of equations (55) is reduced to more simple system

$$\begin{aligned} \frac{m}{4} \ddot{Q}_{00} + \frac{m}{2} \omega^2 Q_{00} - \mathcal{K}_{00} &= 0, \\ \dot{\mathcal{K}}_{00} + \frac{m}{2} \omega^2 \dot{Q}_{00} &= 0 \end{aligned} \quad (98)$$

that describes the giant monopole resonance. It can be solved analytically. Second equation gives the energy integral  $\mathcal{K}_{00}(t) + \frac{m}{2} \omega^2 Q_{00}(t) = E$ , that allows to rewrite the first

equation as

$$\frac{m}{4}\ddot{Q}_{00} + m\omega^2 Q_{00} = E.$$

It is evident that its solution has the following form:  $Q_{00}(t) = a_0 + a_1 \cos \Omega t + b_1 \sin \Omega t$ .

Using it together with the initial condition  $\dot{Q}_{00}(0) = 0$  one gets:

$$m\omega^2 a_0 = E, \quad b_1 = 0, \quad (m\omega^2 - m\Omega^2/4)a_1 = 0.$$

The last relation determines the frequency:  $\Omega = 2\omega$ . The coefficient  $a_1$  is determined with the help of the initial condition  $Q_{00}(0) = a_0 + a_1 = E/(m\omega^2) + a_1$  and the integral of motion (54), that in our case takes the form  $Q_{00}(0)\mathcal{K}_{00}(0) = 9c_1$ . We have chosen  $c_1 = m\omega^2(Q_{00}^{eq})^2/18$ . So, one has:  $2\mathcal{K}_{00}(0) = m\omega^2(Q_{00}^{eq})^2/Q_{00}(0)$ . Substituting this expression into the energy integral  $\mathcal{K}_{00}(0) + m\omega^2 Q_{00}(0)/2 = E$  one obtains the quadratic equation which allows one to determine  $Q_{00}(0)$  as a function of  $E$ , i.e. to determine the classical turning points:

$$Q_{00}(0)_{1,2} = \frac{E}{m\omega^2} \pm \sqrt{\frac{E^2}{m^2\omega^4} - (Q_{00}^{eq})^2}.$$

One finds from here, that  $a_1 = \pm \sqrt{E^2/(m^2\omega^4) - (Q_{00}^{eq})^2}$ . Using this expression in the formula (89) and taking into account the relation  $E_{eq} = m\omega^2 Q_{00}^{eq}$  one gets:

$$\frac{B(E0)}{B_W} = \frac{Z^2}{A^2} \frac{E^2 - E_{eq}^2}{4m^2\omega^4} = \frac{Z^2}{A^2} \frac{[Q_{00}^2(0) - (Q_{00}^{eq})^2]^2}{16Q_{00}^2(0)}.$$

To estimate the numerical value of  $B(E0)$  one can use the Cambiaggio [18] self-consistency condition  $E - E_{eq} = \hbar\Omega$ :

$$\frac{B(E0)}{B_W} = \frac{Z^2}{A^2} \frac{\hbar\Omega}{4m^2\omega^4} (2E_{eq} + \hbar\Omega) \simeq \frac{Z^2}{A^2} \frac{\hbar}{m\omega} Q_{00}^{eq} \simeq \frac{Z^2}{A^2} \frac{\hbar^2}{m} \frac{A^{1/3}}{41} \frac{3}{5} A r_0^2 A^{2/3} = Z^2 \frac{\hbar^2}{m} \frac{3}{5} \frac{r_0^2}{541}.$$

Taking  $r_0 = 1.2$  one gets for  $^{208}\text{Pb}$  the value  $B(E0) = 5923 B_W = 471 e^2 f m^4$  that is in qualitative agreement with other [46] calculations of GMR.

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