СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

98-170

E4-98-170

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APPROXIMATE METHODS OF FINITE TEMPERATURE MANY-BODY THEORY WITH THE LIPKIN—MESHKOV—GLICK MODEL

1998

1. Introduction

Theoretical tools to deal with many-body systems at finite temperatures were developed long ago. But the observation in the early eighties of collective dipole oscillations in hot nuclei up to temperatures of an order of several MeV [1] gave a strong impact on the renewed interest in the field. Shortly after the discovery of a giant dipole resonance in hot nuclei several theoretical investigations were performed where a linear response theory at finite temperature was explored [2] [1]. Later on more elaborated approaches were suggested and applied to the problem of temperature dependence of the GDR width in hot nuclei [4]. A standard technique of treating quantum many-body systems at finite temperature T is the thermal Green function (Matsubara) method. But in the early seventies an alternative approach – the thermo-field dynamics (TFD) [5, 6] – was formulated. In the present context TFD has at least two appealing features: a) temperature effects arise explicitly as T-dependent vertices, providing a good starting point for various approximations; b) generalization to the time-dependent situation is easy since temperature and time are independent variables in TFD. Both the features allow for straightforward extensions of well-established zero-temperature approximations, as it was already demonstrated in [7, 8]. Recently, by the use of the TFD formalism a new approximate method going beyond the thermal RPA (TRPA) has been proposed [9] to describe collective excitations in hot finite Fermi systems. This method called the thermal renormalized RPA (TRRPA) is an extension to finite temperature of the so-called extended RPA of Ken-ji Hara [10].

New approximate methods of a nuclear structure theory are usually examined by applying them to simple exactly soluble models in order to gain some insights into a range of their validity. One of the widely used models is the two level schematic shell model, which possesses the SU(2) symmetry and is often called the SU(2) or Lipkin – Meshkov – Glick (LMG) model [11]. This well-known model has been used many times to justify approximate methods of the many-body theory at finite temperature as well. For example, the works [7, 12] have focused on boson expansion methods and symmetry breaking in hot LMG-systems. The so-called mixed state representation has been formulated and then applied to the LMG- model in refs. [13-16]. The thermal Hartree - Fock approximation (THFA) [15] as well as TRPA [16] have been studied within the approach. The THFA and the static path approximation were also analyzed within the LMG- model in ref. [17].

In the present paper, we investigate the accuracy and the range of validity of TRRPA by comparing it with exact calculations for the grand canonical ensemble with the LMG-model. Moreover, a comparison with THFA and TRPA is also made.

The organization of the paper is as follows. Basic elements of the formalism of thermo-

[1] It is worthwhile mentioning that some aspects of collective motion in hot nuclei have also been studied earlier [3]

field dynamics are given in Section 2. In Section 3 a derivation of the TRRPA equations for the LMG- model based on the TFD formalism is presented. The necessary formulae for the exact numerical grand canonical calculations with the LMG-model are given in Section 4. The results of approximate methods and their comparison with the exact ones are discussed in Section 5. Section 6 contains the summary and brief concluding remarks.

2. Thermo field dynamics: basic elements

To be more understandable while describing approximate methods, we briefly recapitulate the formalism of thermo-field dynamics (see, refs.[5-7, 18, 19]).

The extension of quantum field theory to finite temperature requires the field degrees of freedom to be doubled. In TFD, the doubling is achieved by introducing an additional tilde space. A tilde conjugate operator $\tilde{A}$ is assigned to an operator A (acting in ordinary space) through the tilde conjugation rules

$$\widetilde{(AB)} = \tilde{A}\tilde{B}; \quad \widetilde{(aA+bB)} = a^*\tilde{A} + b^*\tilde{B}\,,$$

where A and B represent ordinary operators and a and b are c-numbers. The asterisk denotes the complex conjugation. The tilde operation commutes with hermitian conjugation and any tilde and non-tilde operators are assumed to commute or anticommute with each other. A double application of tilde operation changes a sign of a fermionic operator and saves it for a bosonic one. The whole Hilbert space of a heated system is a direct product of ordinary and tilde spaces. A formal quantity playing a central role in the present discussion is the so-called thermal Hamiltonian

$$\mathcal{H} = H - \tilde{H}$$

The operator $\mathcal{H}$ serves to translate temperature dependent wave functions along the time axis. It means that an "excitation spectrum" of a hot system (or, in other words, a set of energies corresponding to the thermal equilibrium states) should be obtained by the diagonalization of $\mathcal{H}$.

The temperature-dependent vacuum $|\Psi_0(T)\rangle$ is the eigenvector of $\mathcal{H}$ with eigenvalue 0

$$\mathcal{H}|\Psi_0(T)\rangle = 0\,.$$

If one determines the thermal vacuum state as

$$|\Psi_0(T)\rangle = \frac{1}{\sqrt{Tr(\exp(-H/T))}} \sum_n \exp(-\frac{E_n}{2T})|n\rangle \otimes |\tilde{n}\rangle\,,$$

where $E_n, |n\rangle$ and $|\tilde{n}\rangle$ are eigenvalues, eigenvectors and their tilde counterparts of the Hamiltonian H, respectively, the expectation value $\langle\Psi_0(T)|O|\Psi_0(T)\rangle$ will exactly correspond to the grand canonical ensemble average $\ll O \gg$ of a given observable O.

In practice, it is impossible to find an exact thermal vacuum for a full Hamiltonian of a many-body system. In setting up approximate schemes, the usual starting point is the thermal mean-field approximation. In this case, the thermal vacuum $|\Psi_0(T)\rangle$ is an eigenvector of the uncorrelated thermal Hamiltonian

$$\mathcal{H}_{MF}|0(T)\rangle = (H_{MF} - \tilde{H}_{MF})|0(T)\rangle = \sum_i \varepsilon_i(a_i^+a_i - \tilde{a}_i^+\tilde{a}_i)|0(T)\rangle = 0\,. \qquad (1)$$

The solutions of eq. 1 define the vacuum $|0(T)\rangle$ for so-called thermal quasiparticles $\beta, \tilde{\beta}$

$$\begin{aligned} \beta_i &= x_i a_i - y_i\tilde{a}_i^+ \\ \tilde{\beta}_i &= x_i\tilde{a}_i + y_i a_i^+ \end{aligned} \qquad (2)$$

$$\beta_i|0(T)\rangle = \tilde{\beta}_i|0(T)\rangle = 0\,.$$

The coefficients x_i, y_i are dependent on the thermal Fermi occupation probabilities of the states $a_i^+|0\rangle$ ($|0\rangle$ is a vacuum for a_i)

$$x_i = \sqrt{1-f_i}\,,\; y_i = \sqrt{f_i}\,,\; f_i = \frac{1}{1+\exp(\varepsilon_i/T)} \qquad (3)$$

Transformation (2) is a unitary transformation and thus conserves the commutation relations. It is often called the thermal Bogoliubov transformation.

3. Thermal renormalized RPA with the LMG- model

Now we apply the TFD formalism to evaluate the TRRPA equations for the LMG- model. A more general consideration of the approximation can be found in refs. [9, 18].

We use the version of the LMG- model with an interaction acting between a pair of particles with parallel spins only. The model system consists of N fermions distributed over two levels with degeneracy Ω ($\Omega = N$). The energy of the lower and upper level is $-\varepsilon/2$ and $+\varepsilon/2$, respectively. Thus, the Hamiltonian has the form

$$H_{LMG} = \varepsilon J_z - \frac{1}{2}V\,(J_+J_+ + J_-J_-)\,, \qquad (4)$$

where the operators of quasispin J and its components J_+, J_-, J_z are defined as follows:

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2\,,$$

$$J_z = \frac{1}{2}\sum_{p=1}^{\Omega}\left(a_{2p}^+a_{2p} - a_{1p}^+a_{1p}\right)\,, \quad J_+ = \sum_{p=1}^{\Omega} a_{2p}^+a_{1p}\,, \quad J_- = \left(\hat{J}_+\right)^+\,.$$

Indices "1" and "2" label the lower and upper levels, respectively, index p enumerates the sublevels.

The thermal model Hamiltonian $\mathcal{H}_{LMG}$ is equal to $H_{LMG} - \tilde{H}_{LMG}$. At the first step, we like to formulate the thermal Hartree – Fock approximation. To this aim, we make a unitary transformation D from the initial particle operators a^+_{ip}, a_{ip} to the Hartree - Fock quasiparticle operators $\alpha^+_{ip}, \alpha_{ip}$ [20] [2]

$$a^+_{ip} = D_{i1}\alpha^+_{1p} + D_{i2}\alpha^+_{2p} \qquad (5)$$
$$a_{ip} = D^*_{i1}\alpha_{1p} + D^*_{i2}\alpha_{2p}.$$

and then the thermal Bogoliubov transformation (2) from $\alpha, \tilde{\alpha}$ to thermal quasiparticles. At $T = 0$ one determines the matrix D (5) in such a way that the ground state energy of the system (i.e. the average of H_{LMG} over the HF vacuum state) is minimized.

Now we determine both the D and $\{x, y\}$ transformations together under the condition for the system to be in the thermal equilibrium at $T = \text{const}$. It means that we have to find a minimum of the free energy $F = E - TS - \lambda N$ (see, e.g. [14]), where E is the intrinsic energy of the heated system and S is the entropy.

In accordance with the TFD prescriptions, the energy E is equal to the expectation value of $H_{LMG}(\beta^+, \beta, \tilde{\beta}^+, \tilde{\beta})$ over the thermal vacuum state $|0(T)\rangle$ (see Sect. 2) If the coefficients $D_{ii'}$ are parametrized as in [20]

$$D_{11} = D_{22} = \cos\theta, \qquad D_{12} = -D_{21} = \exp(i\varphi)\sin\theta$$

one gets the following expression for $E \equiv \langle 0(T)|H|0(T)\rangle$:

$$E = \frac{\varepsilon\Omega}{2}(y_2^2 - y_1^2)\left[\cos 2\theta - \frac{\chi_0(y_2^2 - y_1^2)}{2}\sin^2 2\theta \cos 2\varphi\right]$$

where χ_0 is the effective coupling constant

$$\chi_0 = \frac{V(\Omega - 1)}{\varepsilon}.$$

The entropy of the system is

$$S = -2\Omega \sum_{i=1,2}(y_i^2 \ln y_i + x_i^2 \ln x_i). \qquad (6)$$

After variation of F over $\theta, \varphi, x_i, y_i$ and λ taking into account the constraints $x_i^2 + y_i^2 = 1$, one finds two different solutions depending on the value of the effective coupling constant χ_0 and temperature T.

The first solution (the normal phase) exists if $\chi(T) \equiv \chi_0(f_1 - f_2) \leq 1$. It corresponds to the following values of variables:

$$\theta = 0,\ \varphi = 0,\ \varepsilon(T) = \varepsilon,\ \lambda = 0,$$

[2] The tilde a and α operators are connected by the complex conjugate transformation D^*

$$f_{1,2} = y^2_{1,2} = \frac{1}{1 + \exp(\mp\varepsilon/2T)}.$$

In the normal phase the thermal ground state energy is

$$E = \frac{\varepsilon\Omega(f_2 - f_1)}{2}$$

and the Hartree - Fock part of the whole thermal Hamiltonian $\mathcal{H}$ does not depend on temperature

$$\mathcal{H}_{HF} = \varepsilon\left(B - \tilde{B}\right),$$

where

$$B = \frac{1}{2}\sum_{p=1}^{\Omega}\left(\beta^+_{2p}\beta_{2p} - \beta^+_{1p}\beta_{1p}\right).$$

The second solution (the deformed phase) exists when $\chi(T) \equiv \chi_0(f_1 - f_2) > 1$. It corresponds to the values of variables

$$\cos^{-1} 2\theta = \chi(T),\ \varphi = 0,\ \varepsilon(T) = \varepsilon\chi(T),\ \lambda = 0$$

and

$$f_{1,2} = y^2_{1,2} = \frac{1}{1 + \exp(\mp\varepsilon(T)/2T)}.$$

The energy of the thermal "deformed" ground state is

$$E = \frac{\varepsilon\Omega(f_2 - f_1)}{4}\left(\chi(T) + \frac{1}{\chi(T)}\right)$$

In this regime the thermal Hartree - Fock Hamiltonian appears to be temperature - dependent

$$\mathcal{H}_{HF} = \varepsilon(T)\left(B - \tilde{B}\right).$$

The value of the chemical potential λ is always equal to zero due to the symmetry of the LMG-system (two levels only).

Thus, we have derived the thermal Hartree - Fock Hamiltonian for the LMG- model in different regimes. It is worthwhile noting that expression (6) for the entropy already implies that we deal with a heated system of independent HF quasiparticles. So, it is not a big surprise that we get the traditional Fermi - Dirac formulae (3) for x and y. By the way, one met the same situation while evaluating the formulae of the thermal BCS approximation within the TFD formalism [7, 21, 22].

After extracting the Hartree – Fock part of the Hamiltonian (4) we take into account the interaction of thermal quasiparticles. For further studies we need only that part of

$\mathcal{H}_{LMG}$ which consists of the terms with an even numbers of both creation and annihilation thermal quasiparticle operators (and the thermal HF part, of course). Namely,

$$\mathcal{H}_{TRPA} = \varepsilon(T)\left(B - \tilde{B}\right) - \\ - \frac{V(f_1 - f_2)(1+\cos^2 2\theta)}{4}\left[(A^+A^+ + AA) - \left(\tilde{A}^+\tilde{A}^+ + \tilde{A}\tilde{A}\right)\right] + \\ + \frac{V(f_1 - f_2)\sin^2 2\theta}{2}\left[A^+A - \tilde{A}^+\tilde{A}\right], \quad (7)$$

where

$$A^+ = \sum_{p=1}^{\Omega} \beta_{2p}^+ \tilde{\beta}_{1p}^+, \qquad \tilde{A}^+ = \sum_{p=1}^{\Omega} \beta_{1p}^+ \tilde{\beta}_{2p}^+.$$

The following exact commutation rules are valid for the thermal biquasiparticle operators A, A^+, $\tilde{A}$ and $\tilde{A}^+$:

$$\left[A, A^+\right] = N - \sum_{p=1}^{\Omega} \tilde{\beta}_{1p}^+ \tilde{\beta}_{1p} - \sum_{p=1}^{\Omega} \beta_{2p}^+ \beta_{2p}, \quad \left[\tilde{A}, \tilde{A}^+\right] = N - \sum_{p=1}^{\Omega} \beta_{1p}^+ \beta_{1p} - \sum_{p=1}^{\Omega} \tilde{\beta}_{2p}^+ \tilde{\beta}_{2p}.$$

In TRPA the biquasiparticle operators A and A^+ are supposed to be boson operators. With the present definition it means that in TRPA the corresponding commutator relation is $[A, A^+] = N$. Now in accordance with the idea of ref.[10] (see, also [9, 18]) we suppose that

$$\left[A, A^+\right] = \left[\tilde{A}, \tilde{A}^+\right] = N\left(1 - \rho_1 - \rho_2\right) \equiv N\left(1 - 2\rho\right). \quad (8)$$

The ρ_i are c-numbers and can be treated as numbers of thermal quasiparticles in the temperature - dependent ground state $|\Psi_0(T)\rangle$. This state is not any more the thermal Hartree - Fock vacuum state, since we involve into consideration the quasiparticle interaction, and will be defined later. Thus,

$$\rho_i = \frac{1}{N}\langle\Psi_0(T)|N_i^\beta|\Psi_0(T)\rangle = \frac{1}{N}\langle\Psi_0(T)|\tilde{N}_i^\beta|\Psi_0(T)\rangle,$$

where N_i^β is the operator of the number of thermal quasiparticles $N_i^\beta = \sum_{p=1}^{\Omega} \beta_{ip}^+ \beta_{ip}$.

The thermal Hamiltonian (7) can be diagonalized in the space of two one-phonon states constructed as bilinear forms of the thermal quasiparticle operators

$$Q_1^+|\Psi_0(T)\rangle = \left(\psi_1 A^+ - \phi_1 A\right)|\Psi_0(T)\rangle \\ Q_2^+|\Psi_0(T)\rangle = \left(\psi_2 \tilde{A}^+ - \phi_2 \tilde{A}\right)|\Psi_0(T)\rangle. \quad (9)$$

We define the wave function $|\Psi_0(T)\rangle$ as the thermal phonon vacuum, i.e. $Q_{1,2}|\Psi_0(T)\rangle = 0$. In contrast with the thermal Hartree - Fock vacuum state this new thermal ground state

allows for some kinds of thermal quasiparticle correlations. Also, the role of the Pauli principle in its structure is taken into account in a better way than in the TRPA vacuum state.

The states (9) have to be orthonormal, and taking account of eq. (8) the following constraints on the amplitudes ψ and ϕ are derived:

$$\psi_i^2 - \phi_i^2 = [N(1-2\rho)]^{-1}, \quad i = 1, 2.$$

The system of equations for ψ_i, ϕ_i and the phonon frequencies ω_i is derived by the equation of motion method. It appears that only a positive value of ω_1 and a negative value of ω_2 are allowed under a requirement that the wave functions $Q_1^+|\Psi_0(T)\rangle$ and $Q_2^+|\Psi_0(T)\rangle$ are vectors of the Hilbert space. The eigenvalue - eigenvector problem has the following solution:

$$\omega_1 = \omega \equiv \sqrt{\mathcal{E}^2 - \left[\varepsilon\chi(T)\left(1-2\rho\right)\frac{(1+\cos^2 2\theta)}{2}\right]^2}, \\ \psi_1^2 = \frac{\mathcal{E}+\omega}{2N\omega(1-2\rho)}, \quad \phi_1^2 = \frac{\mathcal{E}-\omega}{2N\omega(1-2\rho)}, \\ \omega_2 = -\omega, \quad \psi_2^2 = \psi_1^2, \quad \phi_2^2 = \phi_1^2,$$

where

$$\mathcal{E} = \varepsilon(T) + \frac{\varepsilon\chi(T)\left(1-2\rho\right)\sin^2 2\theta}{2}$$

To evaluate the equation for ρ we need an expression for the thermal phonon vacuum state. The latter can be derived from the thermal quasiparticle vacuum state $|0(T)\rangle$ by a unitary transformation

$$|\Psi_0(T)\rangle = \sqrt{R}e^S|0(T)\rangle, \quad S = \frac{1}{2(1-2\rho)}\frac{\phi_1}{\psi_1}\left(A^+A^+ + \tilde{A}^+\tilde{A}^+\right).$$

By the use of standard techniques of the operator calculus [10] we get

$$\rho = \frac{1}{2}\frac{\mathcal{E}-\omega}{N\omega} \quad (10)$$

It is interesting to note that in the thermodynamic limit, i.e. as $N \to \infty$, ρ vanishes and the TRRPA equations are reduced to the TRPA ones.

4. The grand canonical ensemble calculations with the LMG- model

Our goal is to compare the results of the approximations described in Section 3 with the exact calculations for the grand canonical ensemble. In this section, the procedure for exact evaluation of the grand canonical partition function of the LMG- model is described.

The operators of a quasispin and its projections $J_\pm$ and J_z form the SU(2) algebra, and the quasispin operator commutes with H_{LMG}. So the Hamiltonian matrix breaks up into submatrices Θ_J of dimension $2J+1$. The LMG- Hamiltonian can be diagonalized in each of these subspaces independently. The corresponding eigenvalues are denoted by $E_1^J, E_2^J, ...E_{2J+1}^J$. They can easily be calculated analytically (for small N) or numerically (see, e.g. [23]).

To calculate the grand canonical partition function besides the eigenvalues E_k^J, one needs degeneracies of the irreducible quasispin representations Θ_J for different particle numbers from the range $0 < N \leq 2\Omega$. To determine the latter, we use the results of ref. [15]. The whole number of the ensemble states, i.e., the whole number of the eigenstates of the LMG- systems formed by two Ω- degenerated levels with a number of particles varying from 1 to 2Ω is equal to $2^{2\Omega}$. A particular distribution of given number of particles over two degenerate levels can be characterized by numbers ν_1 and ν_2 where ν_1 is the number of sublevels occupied by particles on both the lower and upper levels and ν_2 is the number of sublevels occupied on neither the lower nor the upper level. The quasispin J of the state is determined by the distribution of the rest of particles over $2\tau = \Omega - \nu_1 - \nu_2$ sublevels. The number $2(\tau + \nu_1)$ is equal to the number of particles. The dimension of the subspace of states with ν_1 occupied and ν_2 empty sublevels is $2^{2\tau}$. There exist $\Omega!/(2\tau)!\nu_1!\nu_2!$ such distinct subspaces for fixed τ and ν_1. Each of them may be decomposed into irreducible subspaces with fixed quasispin values Θ_τ (appearing once), $\Theta_{\tau-1}$ (appearing g_1^τ times), $\Theta_{\tau-2}$ (appearing g_2^τ times), ... , $\Theta_{\tau-k}$ (appearing g_k^τ times),... , $\Theta_{\tau-[\tau]}$ (appearing $g_{[\tau]}^\tau$ times). Here

$$g_k^\tau = \frac{(2\tau)!}{k!(2\tau-k)!} - \frac{(2\tau)!}{(k-1)!(2\tau-k+1)!},$$

and $[\tau] = \tau$, if τ is integer, $[\tau] = \tau - 1/2$ if τ is half-integer.

Thus, the exact grand partition function of our LMG ensemble is

$$Z(T) = \sum_{\tau\nu_1\nu_2} \frac{\Omega!}{(2\tau)!\nu_1!\nu_2!} \sum_k g_k^\tau \sum_m \exp\left[-\frac{E_m^{\tau-k} - 2(\tau+\nu_1)\lambda}{T}\right] \qquad (11)$$

The expressions for average energy, quasispin z-projection and the total fermion number with $Z(T)$ (11) can be found in refs. [15, 24].

5. Results and discussion

The numerical calculations are performed for the LMG- system with N =10 and $\varepsilon = 1$, i.e. we adopt ε as an energy unit.

Let us consider first the dependence of ω on T (Fig. 1). It seems appropriate to distinguish two cases: a) $\chi_0 < 1$; b) $\chi_0 > 1$. A key for understanding a displayed

behaviour of $\omega(T)$ is that in the present version of the LMG - model heating effectively weakens the interaction of particles since at $T \neq 0$ the effective coupling constant χ_0 is multiplied by a thermal factor $f_1 - f_2 < 1$; hence $\chi(T) < \chi_0$ and $\chi(T)$ vanishes when $T \to \infty$. In the case a) the LMG- system is in the normal phase at $T = 0$ and stays there when $T \to \infty$. Then $\omega \to \varepsilon$ with increasing T due to vanishing of the effective interaction. A picture is more complicated if $\chi_0 > 1$. Then, the LMG- system is in the deformed phase at $T = 0$. In this phase the distance between single-particle levels is proportional to $V(f_1 - f_2)$ and goes down when temperature increases. As a result, the energy ω of the excited state goes down as well. But near the point $T_{cr} = \frac{\varepsilon}{2} \ln^{-1} \frac{\chi_0+1}{\chi_0-1} \simeq 1.0$, where the temperature dependent effective coupling constant $\chi(T) = 1$, the rearrangement of the Hartree - Fock field (i.e. the phase transition) occurs and at $T > T_{cr}$ the LMG- system appears to be already in the normal phase. Note that within TRPA the energy ω vanishes at $T = T_{cr}$ whereas within TRRPA ω stays finite. With a further increase in T ω starts to increase and again goes to ε when $T \to \infty$. As one can see in Fig. 1, within TRRPA the phase transition appears at a slightly lower temperature than within TRPA. The reason for this will be discussed later on. There is a noticeable difference between the TRRPA and TRPA results only near the critical temperature, and at much lower or higher T both the approximations give close results. A temperature dependence of ω on T within TRPA was also studied in [12]. The results are in complete agreement with the present ones. The same is true for the THFA calculations of refs. [15, 17].

Now we discuss T-dependencies of the intrinsic energy $\langle H \rangle$ and the average value of quasispin z-projection $\langle J_z \rangle$ as well as a particle number variance ΔN.

The exact values $\langle H_{LMG} \rangle_{GCE}, \langle J_z \rangle_{GCE}$ and ΔN_{GCE} have been calculated with the grand canonical partition function (11). The expressions for $\langle H \rangle, \langle J_z \rangle$ and ΔN in TRRPA are obtained by evaluation of the expectation values of the corresponding operators over the thermal vacuum state $|\Psi_0(T)\rangle$. While evaluating $\langle H \rangle_{TRRPA}$ the thermal Hartree - Fock ground state energy E has to be taken into account as well

$$\langle H \rangle_{TRRPA} = \frac{\Omega(f_2 - f_1)(1-2\rho)}{2} \left[\mathcal{E} - \varepsilon\chi(T)\sin^2 2\theta\right] + \frac{(\mathcal{E}-\omega)(\varepsilon(T)+\omega)}{2\omega} \times \frac{(f_2-f_1)^2+1}{2(f_2-f_1)}. \qquad (12)$$

The expression for $\langle H \rangle_{TRPA}$ can be derived from (12) if one puts $\rho = 0$. The THFA ground state energy E of both the phases has been evaluated in Section 3.

The results of all three approximations together with the exact one are displayed in Fig. 2. Three typical cases are shown: a) a weak coupling case $\chi_0 = 0.5$; b) an intermediate coupling case $\chi_0 = 0.95$; c) a strong coupling case $\chi_0 = 1.0$. At $\chi_0 = 0.5$ the results of TRRPA, TRPA and THFA are very close to each other and to the exact one though formally the TRRPA curve is closer to the exact result. More interesting is

the case b) (it has already been discussed in [24]). At $\chi_0 = 0.95$ the system is close to the phase transition point. Here, the advantages of TRRPA appear to be most evident. The difference between the approximations is noticeable when $T < 0.3 - 0.5$ and again with the increase in T, results of different approximations approach the exact one. In the case c) the LMG- system is in the deformed phase at $T = 0$. Although the interaction is strong, the results of TRRPA, TRPA and THFA do not deviate far from each other at $T < T_{cr}$. It means that already THFA is good enough to allow for the main part of thermal quasiparticle correlations and the other approximations give only minor corrections to THFA. But in the vicinity of T_{cr} TRRPA is again the best approximation.

One could already notice that if $\chi_0 > 1$, all the three approximations predict the phase transition in the system with the increase in T. The transition between the two phases manifests itself as a break point of the intrinsic energy $\langle H \rangle$ in THFA and TRRPA and as a singular point of this function in TRPA. Obviously, the phase transition does not occur in reality, i.e. in exact calculations. This is quite a typical situation when approximate methods are applied to study finite many-body systems. THFA, TRPA and TRRPA predict phase transitions of different characters; also a quality of the description of a system evolution in the vicinity of a critical temperature is different. It is more clearly seen in Fig. 3, where a dependence of a heat capacity C on T for the three values of χ_0 is displayed. The heat capacity is calculated as a partial derivative with respect to T of the intrinsic energy (12): $C = \partial < H > /\partial T$. At any value of χ_0 the exact heat capacity as a function of T has quite a sharp maximum at $T \sim 0.5$. At the weak interaction case all the three approximations describe $C(T)$ well. But in the cases b) and c) all the approximations demonstrate much sharper behaviour of C in the region of the maximum. Moreover, in the case c) maxima of approximate functions are at noticeably higher temperatures than that of exact one. Note also that $C(T)$ calculated within TRPA has a singular discontinuity at $T = T_{cr}$ whereas the THFA and TRRPA heat capacities have jump discontinuities only.

As it is seen in Fig. 2c, the phase transition in TRRPA occurs at slightly lower T than in TRPA and THFA. In these two approximations the phase transition is at the same value T_{cr} because the rearrangement of the Hartree – Fock field and the collapse of the TRPA collective state are at the same value of $\chi(T)$. Within TRRPA a picture of the phase transition is the following. In the vicinity of T_{cr} one can compare the values of $\langle H \rangle_{TRRPA}$ calculated with the two different mean field configurations corresponding to two phases. It appears that the value of $\langle H \rangle_{TRRPA}$ corresponding to the normal phase remains lower than the value of $\langle H \rangle_{TRRPA}$ calculated with the deformed mean field at $T < T_{cr}$ (here T_{cr} is the critical temperature of the phase transition in THFA). In other words, within TRRPA the normal phase of the LMG- system survives in a larger temperature range than within THFA or TRPA. This fact is intimately connected with the behaviour

of the collective state energy as a function of the coupling constant. One cannot calculate $\langle H \rangle_{TRPA}$ at $\chi(T) > \chi_{cr}$ because the value of ω is imaginary there. But within TRRPA it is possible because ω remains real and finite at any value of $\chi(T)$.

The expectation value of the operator J_z is proportional to the difference of the numbers of particles on the lower and upper levels of the system. Hence, with the increase in T $J_z \to 0$. But the behaviour of J_z appears to be dependent on χ_0. The expressions for J_z for the different phases have the following forms:

$$\langle J_z \rangle_{TRRPA} = \begin{cases} \dfrac{\Omega(f_2 - f_1)(1 - 2\rho)}{2} & \text{(Normal phase)} \\ \dfrac{\varepsilon\Omega(1 - 2\rho)}{2V(\Omega - 1)} & \text{(Deformed phase)} \end{cases}$$

Note that within THFA and TRPA the expression for $\langle J_z \rangle$ appears to be the same. In the deformed phase $\langle J_z \rangle$ does not depend on temperature. This seems to be the result of the two opposite tendencies. With the increase in T the difference $f_1 - f_2$ (and hence the value $N_1 - N_2$) decreases but at the same time the difference of the energies of the single-particle levels decreases (see the corresponding expression for $\varepsilon(T)$ in Sect.3) and this compensates the first effect.

As one can see in Fig. 4, the results of different approximations in the normal phase (the cases b) and c) are very close to each other as well as to the exact one. The largest difference between them is $\sim$10% at $\chi_0 = 0.95$ and $T \leq 0.1$. Nevertheless, formally the results of TRRPA are in better agreement with the exact ones than those of TRPA (and THFA). The worst agreement with the exact result is for the strong coupling case (Fig. 4c). First, in the deformed phase the exact curve manifests quite a non-trivial dependence of $\langle J_z \rangle_{GCE}$ on T, which can hardly be approximated by the constant predicted by our approximate methods. Moreover, at $T > T_{cr}$, i.e. in the normal phase, the exact value $\langle J_z \rangle_{GCE}$ goes to zero faster than the approximate one $\langle J_z \rangle_{TRRPA}$. The absolute value of the difference between the approximate and exact results is also the largest for the strong coupling case. In spite of these discrepancies one can make a conclusion about a qualitative agreement between the exact and approximate results in this case too.

The expression for the particle number variance ΔN_{TRRPA} is

$$\Delta N_{TRRPA} = \sqrt{2N f_1 f_2 (1 - 2\rho)}\,.$$

This expression is valid both in the normal and deformed phase. While evaluating the above formula we face a difficulty to calculate the matrix element $\langle \Psi_0(T)|N^2|\Psi_0(T)\rangle$, where N is the particle number operator in the ordinary space. This matrix element was expanded on the TRRPA phonon (9) basis. Then, in this expansion only the phonon vacuum and two-phonon terms were taken into account [25]. The two-phonon terms give

a correction of an order of $\sim \Omega\rho$. A contribution of four-phonon and even more complex terms seems to be small.

The results of calculations of ΔN are presented in Fig. 5. In the cases of weak and intermediate couplings the difference between exact and approximated results is negligible (2-3%), though formally the TRRPA curve is closer to the exact one. In the strong coupling case the exact and approximat results differ noticeably only in the vicinity of T_{cr}. Here, TRRPA works evidently better than the other two approximations (their results coincide with each other). From the expression for ΔN one can see that in TRPA and THFA the particle number fluctuations have only the thermal origin (we deal with the grand canonical ensemble). At the same time, within TRRPA quantum fluctuations exist as well. Note that quantum fluctuations slightly damp thermal ones. The reason for this destructive interference of the two types of fluctuations seems to be the Pauli principle. The nonvanishing ρ values mean that the single-particle levels are already partially occupied and this is an obstacle for their thermal feeding with the increase in temperature. But the link between the quantum and thermal fluctuations appears to be quite intimate because, when $T \to 0$ the particle variance vanishes, i.e. the quantum fluctuations disappear together with the thermal ones.

6. Concluding remarks

The thermodynamic properties of the Hamiltonian of the two-level model of Lipkin, Meshkov and Glick have been calculated with the approximate methods of a many-body theory at finite temperatures and compared with the exact grand canonical calculations. The equations of the approximate methods – TRRPA, TRPA and THFA – have been evaluated with the formalism of the thermo field dynamics.

On the whole, the TRRPA results are in better agreement with the exact ones than the results of TRPA and THFA. This is most evident when the system is in the vicinity of the phase transition point. Actually, the exact calculations do not demonstrate the phase transition and its appearance is a result of the approximations. But among the approximations studied in the present work the TRRPA curves agree with the exact calculations near the point of rearrangement of the thermal Hartree - Fock field better than the other ones. At the phase transition point the heat capacity has only a jump discontinuity in TRRPA whereas it has a singular point (as well as $\langle H \rangle$) in TRPA. Though in THFA $C(T)$ has also a jump discontinuity, TRRPA produces much better agreement for the absolute values of $\langle H \rangle$ above the phase transition.

The main reason for these TRRPA advantages is allowance for a nonvanishing number of the thermal quasiparticles in the TRRPA thermal ground state. Due to this, the role of the Pauli principle is taken into account more properly than in the standard TRRPA.

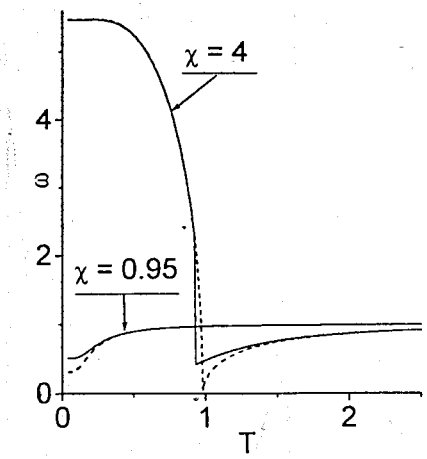

Fig. 1 The energy ω of the lowest excited state as a function of T for two values of the effective coupling constant $\chi_0 = 0.95$ and $\chi_0 = 4.0$. Notation: the TRRPA results – solid lines; the TRPA results – dashed lines.

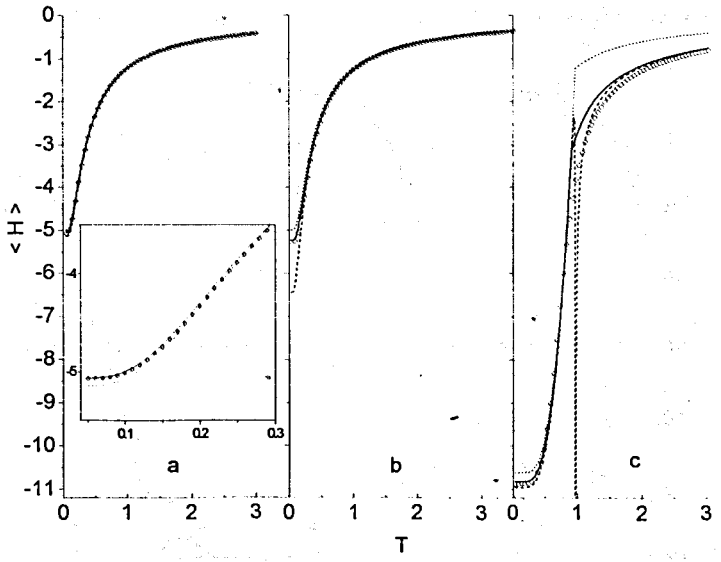

Fig. 2 The intrinsic energy $\langle H \rangle$ as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$. The exact results (the grand canonical ensemble calculations) – open circles; the THF results – short-dashed lines; the TRPA results – long-dashed lines; the TRRPA results – solid lines.

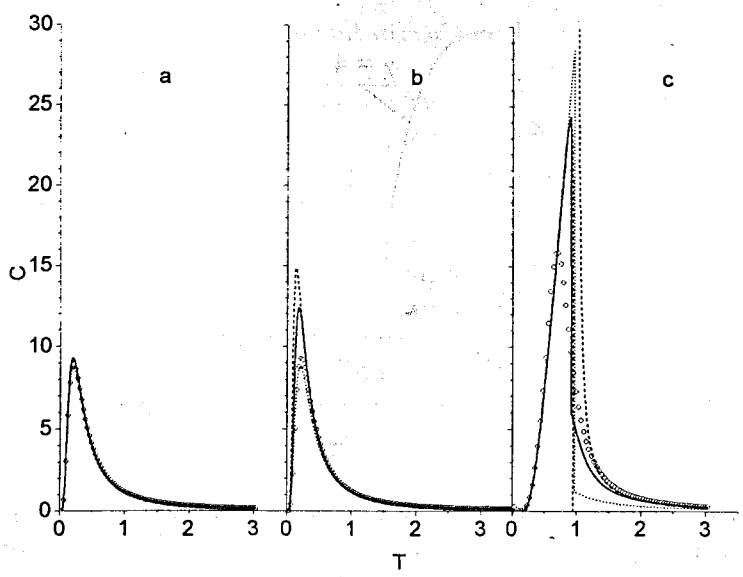

Fig. 3 The heat capacity C as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$. For notation, see Fig. 2.

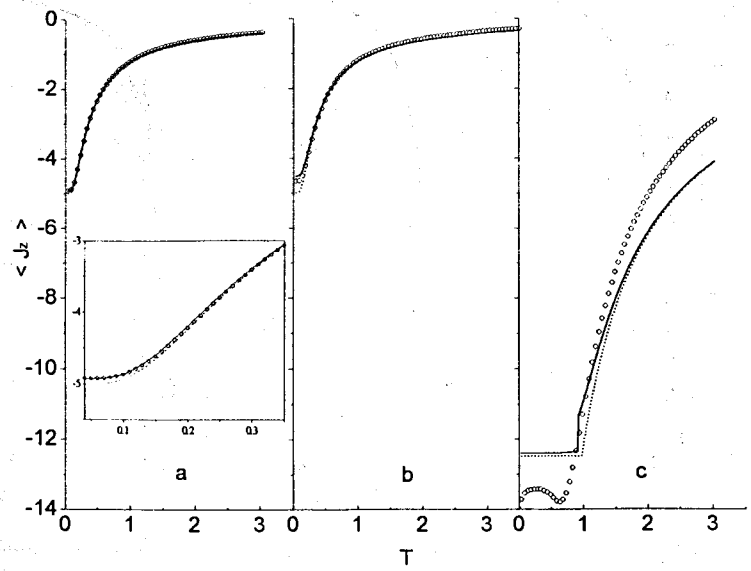

Fig. 4 The average value of the quasispin projection $\langle J_z \rangle$ as a function of temperature T for three values of the effective coupling constant a) $\chi_0 = 0.5$; b) $\chi_0 = 0.95$; c) $\chi_0 = 4.0$. For notation, see Fig. 2.

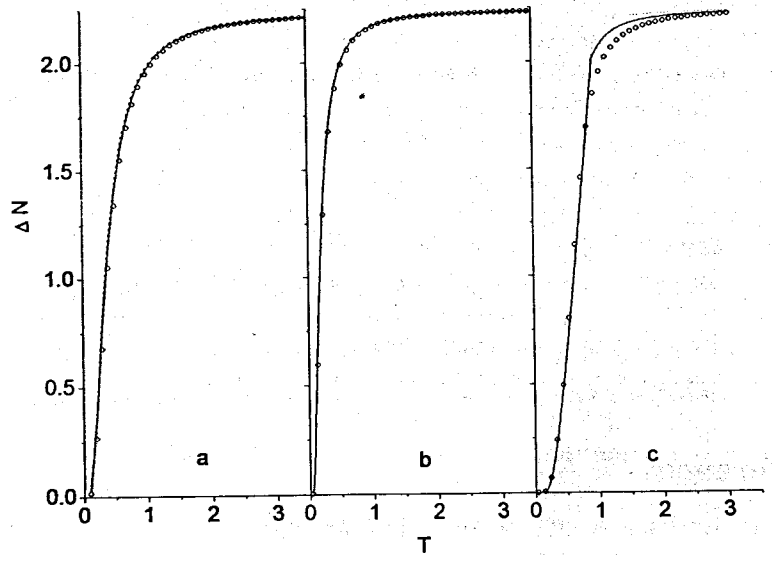

Fig. 5 The particle number variance ΔN as a function of temperature T. For notation, see Fig. 2.

As a rule, the ρ value is not large but it reaches the maximum near and at the phase transition point (see (10) and Fig. 1), and this explains our results. Also, the shift of the phase transition temperature in TRRPA as compared to TRPA (and THFA) seems to be a quite interesting result. This shift pushes the maximum of the heat capacity in the right direction (Fig. 4c). With increasing T and N, results of all the approximate methods improve rapidly and at $T \geq 3\varepsilon$ the difference between exact and approximate results is invisible.

The approach of Ken-ji Hara [10] that was extended to finite temperatures in the present work is one of the simplest and well-known approximations going beyond RPA. The RPA approach has known many other generalizations over the past decades. The renormalized RPA proposed by Rowe [26] was already more elaborated than that of ref [10]. Many other papers can be pointed out as well [27, 25, 28] and this list of references is obviously not complete. Most of the improvements suggested in the cited papers have not yet been considered for hot Fermi - systems with only a few exceptions [18, 19, 28]. We suppose to continue our efforts in this direction.

The authors acknowledge an important contribution of Dr. D.S. Kosov at the early stage of this work and are grateful to him for his interest and correspondence. The comments of Prof. P.Schuck to the first version of this work are appreciated. The work was partially supported by the Council of President of Russia on grants and support of leading scientific school and RFBR (grant of RFBR 96-15-96729).

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Received by Publishing Department
on June 15, 1998.