97-416

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E4-97-416

V.M.Dubovik, A.G.Galperin, V.S.Richvitsky, S.K.Slepnyov

# THE CONDITIONS OF EXISTENCE OF FIRST INTEGRALS AND HAMILTONIAN STRUCTURES OF THE LOTKA—VOLTERRA EQUATIONS. COMMENT ON SOME OF THE RECENT PAPERS

Submitted to «Journal of Mathematical Physics»

1997

# 1 Introduction.

The Lotka-Volterra equations (LVE) are systems of ordinary nonlinear differential equations of the form

$$\dot{x}_i = x_i y_i, \quad i = 1, \ldots, N; \text{ where } y = Ax + b. \tag{1}$$

The Volterra equations (VE) are a special case of the LVE [1], when there exist such $\beta_i \neq 0$, that

$$\beta_i a_{ij} = -\beta_j a_{ji}. \tag{2}$$

In the odd-dimensional case of the VE, the matrix $A$ is degenerate. Initially even-dimensional systems were studied with variables coupled in predator-prey pairs, admitting classical Hamiltonian approach (see [2]) with non-degenerate symplectic structure. The classical approach, however, cannot be applied in a straightforward way for odd-dimensional systems, and also in even dimensions, when the equations for the central equilibrium point $p$

$$Ap = -b \tag{3}$$

gives $p_i = 0$ for at least one $i$. In the biological implementations, the dependent variables $x_i$ are regarded as real positive numbers, representing populations of species $i$, the components of the vector $b$ are called linear growth rates, or malthusian terms, and $A$ is called the interactions matrix, the diagonal terms describing self-interactions of species, and the off-diagonal terms being responsible for interactions between different species. The terms of interactions matrix $A$ and the malthusian terms $b$ are arbitrary real numbers in the case of the general LVE. In the case of the VE, the terms of the interactions matrix are not completely arbitrary, for instance, the self-interaction (diagonal) terms are all equal to zero.

The Volterra lattice model, usually written as $\frac{d}{dt}N_i = N_i(N_{i+1} - N_{i-1})$, studied with the Hamiltonian methods in [3], known for close relations to the Toda lattice model and the Korteweg-de Vries equation (see [16, 18, 20, 19]), and cited by Gümral and Nutku [8] as Faddeev-Takhtajan system, turns into different subcases of (2) under different boundary conditions. For example, the 3D case of periodic boundary conditions (22) complies with the form of "ABC-matrix" (4), used in [5, 4, 8], while the conditions, used in [16], do not. Both are subcases of the general anti-symmetric interactions matrix studied in [21], which, in turn, is a subcase ($\beta_i = 1, \forall i; b = 0$) of the VE, and the latter are a subset (2) of the LVE

(1). We classify the systems with multiple pairwise interactions, called LVE in [17], as Volterra equations.

To make things clear, we use the definition of the LVE and the VE complying with that given in [7, 11], the works that we cite most extensively. But, in contradistinction with [7, 11], we do not assume the "natural" $x_i > 0$ conditions.

## 2 Bi-Hamiltonian structure.

The first example of a bi-Hamiltonian structure for an LVE system of a special form was given by Nutku [4]. For the system studied earlier by Grammatikos and others [5] with "ABC" interactions matrix:

$$A = \begin{pmatrix} 0 & C & 1 \\ 1 & 0 & A \\ B & 1 & 0 \end{pmatrix} ; b = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \tag{4}$$

on the conditions

$$ABC + 1 = 0 \ , \nu = \mu B - \lambda AB$$

for the constants of motion

$$H_1 = AB \ln x_1 - B \ln x_2 + \ln x_3 \ ; \ H_2 = ABx_1 + x_2 - Ax_3 + \nu \ln x_2 - \mu \ln x_3 \tag{5}$$

he has written Hamiltonian equations as

$$\dot{x}^i = J_1^{ik} \nabla_k H_2 = J_2^{ik} \nabla_k H_1 \tag{6}$$

with the antisymmetric Poisson structure matrices $J_1$ and $J_2$ which we represent here by corresponding vectors $j_1$ and $j_2$ so that $j_i = ((J_i)^{23}, -(J_i)^{13}, (J_i)^{12})^T$:

$$j_1 = (-x_2 x_3, -BCx_1 x_3, Cx_1 x_2)^T \ , \tag{7}$$

$$j_2 = (x_1 x_2 x_3, -Cx_1 x_3 (x_2 + \nu), Cx_1 x_2 (Ax_3 + \mu))^T \ , \tag{8}$$

satisfying the Jacobi identity

$$J^{k[m} \nabla_k J^{np]} = 0 \ . \tag{9}$$



In the 3D case, the Jacobi identity for the Poisson structure matrix $J$, represented by vector $j$, becomes

$$(j, \mathrm{rot}\ j) \equiv 0, \tag{10}$$

which is recognizably the condition of the theorem of Frobenius on the integrability of Pfaff's form, the fact being more than a mere coincidence, and subsequently substantially used in [8]. However, the "ABC-matrix", on the terms of the constraints used, is a very special case of the interactions matrix of a Volterra type.

## 3 The primary invariants of Cairó and Feix.

The invariants of motion for the LVE of the most general form were studied in [6, 7] by means of the generalized Carleman embedding method. These invariants come together with certain constraints and have been classified by Cairó and Feix [7] into the primary invariants of three types, the secondary invariants and those deduced by rescaling.

If and only if $det(A) = 0$, the primary invariant type I

$$\mathcal{I}_I = \prod_{i=1}^{N} x_i^{\alpha_i} e^{st} \tag{11}$$

exists with $\alpha_i$ and $s$ satisfying

$$A^T \alpha = 0, \ s = -(\alpha, b). \tag{12}$$

Defining the auxilliary matrix $D$ and the vectors $n$ and $\mathrm{diag}(A)$ according to

$$d_{ij} = a_{ij} - a_{jj}, \ n = (1, 1, \ldots, 1)^T, \ \mathrm{diag}(A) = (a_{11}, a_{22}, \ldots, a_{NN})^T, \tag{13}$$

the conditions for the existence of the primary invariant type II are $(N-1)(N-2)/2$ equations

$$\overline{R}_{ijk} = d_{ij} d_{jk} d_{ki} + d_{ji} d_{kj} d_{ik} = 0 \tag{14}$$

together with the conditions

$$b_1 = b_2 = \ldots = b_N = b_0, \ \text{that is, } b = b_0 n. \tag{15}$$

The form of the invariant type II is

$$\mathcal{I}_{II} = \prod_{i=1}^{N} x_i^{\alpha_i} \Big( -x_1 + \sum_{l=2}^{N} \frac{d_{1l}}{d_{l1}} x_l \Big) e^{st}, \tag{16}$$

where $\alpha_i$ and $s$ are found from the equations:

$$A^T \alpha = -\mathrm{diag}(A),\ s = -b_0(1 + (\alpha, n)). \tag{17}$$

Considering the time dependence, Cairó and Feix state, that $s = 0$ when $N$ is odd. This statement is based on the assumption $rank(A) = N$ which does not appear among the conditions, but is used in the proof of their theorem, and becomes not valid, when $det(A) = 0$. Here is the example that makes this clear:

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix};\ b = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix};\ \mathcal{I}_{II} = x_2^{-3} x_3^3 (-x_1 - x_2 - x_3) e^{-3t}. \tag{18}$$

The primary invariant type III of Cairó and Feix

$$\mathcal{I}_{III} = \prod_{i=1}^{N} x_i^{\alpha_i} \Big( 1 + \sum_{l=1}^{N} \frac{a_{ll}}{b_l} x_l \Big) e^{st} \tag{19}$$

exists on $N(N-1)/2$ conditions

$$R_{ij} = \frac{a_{ii}}{b_i} d_{ij} + \frac{a_{jj}}{b_j} d_{ji} = 0. \tag{20}$$

For this invariant, $\alpha$ and $s$ are defined from

$$A^T \alpha = -\mathrm{diag}(A),\ s = -(\alpha, b). \tag{21}$$

There is a certain correspondence between invariants II and III in the neighbouring odd and even dimensions that Cairó and Feix have discovered. However, for the primary invariant III their statement is that $s = 0$ for even $N$. The controversial example for the latter statement is the same as (18), with an additional equation

$$\dot{x}_4 = x_4(-3 - 3x_4);\ \mathcal{I}_{III} = x_2^{-3} x_3^2 x_4^{-1} (1 + x_1 + x_2 + x_3 + x_4) e^{-3t}.$$

We can see from the given examples, that the conditions $s = 0$ for the primary invariants type II in the odd number of dimensions and type III in the even number of dimensions to be explicitly independent of the time are not fulfilled automatically. Time-dependent cases exist for these invariants, as well as for the corresponding secondary invariants [7], containing only a subset of species in the linear polynomial part of the invariant expression.



Considering the classical Volterra invariant, Cairó and Feix use a procedure of obtaining a limit of invariant III when the diagonal terms of the interaction matrix tend to zero. They have managed to obtain it for $N = 2$, but in the case of the "ABC-matrix" (5) their result for the Volterra invariant is $H_1$, which is not correct, since the expression for the Volterra invariant should contain the coordinates of the central equilibrium point, or stable population levels, thus the correct expression should be $H_2$. This is a consequence of the fact, that the generalized Carleman ansatz does not contain logarithmic terms additively to the linear ones.

# 4 Bi-Hamiltonian technique versus rescaling.

Gümral and Nutku [8] studied the Poisson structures of dynamical systems with three degrees of freedom from the point of view of the theorem of Frobenius on the integrability of Pfaff's equation. Among the others, they used the same "ABC-matrix" example (4) and Faddeev-Takhtajan system closed modulo 3

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix};\ b = 0 \tag{22}$$

as its particular case. Although the bi-Hamiltonian structures for the LVE given in [8] are the same as in [4], the general considerations on the forms of the bi-Hamiltonian structures are important. Namely, the Poisson structures include, in general, the terms of the order from 0 to 3 in the powers of $x_i$. The Poisson 1-forms, corresponding to the Poisson structures, should be compatible, so a conformal factor should be used to add two of them. In a certain case, the equations of motion can be written in a manifestly bi-Hamiltonian form through the exterior product of the gradients of two Hamiltonians. It was also pointed out in [8], that a ratio of components of Poisson structure functions obeys a partial differential equation, which could be quite a manageable one. An analogous idea was used also in [9, 10].

In [9] a representative set of three-dimensional autonomous systems was studied, the LVE being the last and the most difficult case. The procedure implemented therein, included rescaling of the vector field and using the Jacobi identities for the Poisson structure matrix as partial differential equations to obtain one of its components. The idea was that every particular solution of these equations should identically satisfy both the Hamiltonian form of the rescaled equations and the Jacobi identities. However, to find a particular solution in the case of the LVE with primary invariant I as the Hamiltonian function, an additional constraint

$$d_{32}(a_{23}a_{11} - a_{13}a_{21}) = d_{31}(a_{23}a_{12} - a_{13}a_{22}) \tag{23}$$

was imposed. In [10] the same idea was used, but two constraints were imposed. The common feature of both the works [9, 10] is that no numerical examples are given, so the question arises, whether the solutions obtained are consistent with the initial systems. On our part, we have found that the formulae from [10] do not reproduce the malthusian terms $b$ for $s = 0$. The Poisson structure functions obtained in [9] are also not applicable if $s = 0$, though the constraint (23) and the two constraints imposed in [10] are satisfied with matrices (32) given in section 6. The correct Poisson structures in this case we give in section 7.

# 5 Hamiltonian structures by Plank.

Plank studied generalized Hamiltonian structures in the LVE [11] using time-independent constants of motion as Hamiltonian functions and quadratic Poisson structure functions

$$j_{il} = c_{il}x_i x_l, \tag{24}$$

where $c_{il}$ are the matrix elements of a constant skew-symmetric matrix $C$. To the usual items of the definition of the generalized Hamiltonian system: " (i) $\dot{x} = J\nabla H$ is the vector field with smooth real valued Poisson structure matrix $J$ and Hamiltonian function $H$ defined on an open subset $G$ of $R^N$, and (ii) the Jacobi identities for the skew-symmetric $J$ are satisfied", he added the third item "(iii) The matrix of linearization at every fixed point can be written as a product of a symmetric and a skew-symmetric matrices."



The forms for Hamiltonian functions Plank deduced from explicitly solved case N=2:

$$H(x) = \sum_{i=1}^{N} \beta_i(x_i - p_i \ln x_i); \tag{25}$$

$$H(x) = \prod_{i=1}^{N} x_i^{\alpha_i}\Big(1 + \sum_{l=1}^{N} B_l x_l\Big),\ B_l \neq 0; \tag{26}$$

$$H(x) = \prod_{i=1}^{N} x_i^{\alpha_i}\Big(\sum_{l=1}^{N} B_l x_l\Big),\ B_l \neq 0; \tag{27}$$

$$H(x) = \sum_{i=1}^{N} \alpha_i \ln x_i + \frac{B_0 + \sum_{l=1}^{N} B_l x_l}{x_k},\ B_k = 0. \tag{28}$$

All the forms (25-28) are explicitly independent of the time. The quantities $\beta_i$ and $p_i$ in the expression (25) for the Volterra invariant are the same that enter in its conditions of existence (2), (3). Since the expressions (26) and (27) are the time-independent versions of Cairó and Feix's invariants III and II, respectively, the coefficients $B_i$ are proportional to the coefficients in the expressions (19) and (16), while the $\alpha_i$ are obtained from the equations (21) and (17), respectively, in which $s$ is to be set to zero. Cairó and Feix [12] regard the constant of motion of the form (28) as a limiting case of their primary invariant type III. The following example shows that this limit is not so simple, if at all possible, to obtain:

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -3 & -1 & -2 \\ 0 & 2 & 1 \end{pmatrix}; \ b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; \ H(x) = \ln\frac{x_1 x_2}{x_3^2} + \frac{1 - x_2 - x_3}{x_1}, \tag{29}$$

because the equations for $\alpha$ (21) give $\alpha_2 = \alpha_3$ for $\mathcal{I}_{III}$ in the contrast with $\alpha_2 = 1$, $\alpha_3 = -2$ in the example. Although the conditions of existence of the invariant type III are satisfied automatically in this case, the coefficient $a_{11}/b_1$ goes to infinity in the limit, when $b_1$ tends to zero while $a_{11}$ tends to 3. So, it is impossible to obtain the algebraic expression of $H(x)$ in (29) as the limit of a type III invariant. It is more natural to derive the invariant expression (28) from the Volterra invariant of a related system, which is the result of the transformation $x_k \to 1/x_k, x_i \to x_i/x_k, \forall i \neq k$ and differs from certain VE by a common factor $x_k$ in the right-hand sides. When the invariants (26) and (28) are used as Hamiltonian functions, as we shall see in the following sections, they imply second Poisson structure matrices of different algebraic forms, as well as different measure preserving density functions, so they should be thought of as separate invariants.

However, all the Plank's theorems, with the exception of that on the Volterra invariant, in case $N > 2$ are not valid on the part of the proof that the first part of the definition of a Hamiltonian system is satisfied. Calculating the Hamiltonian vector field $\dot{x} = J\nabla H$, the author [11] gets the correct expressions

$$\dot{x}_i = g(x)x_i\Big(b_i + \sum_{i=1}^{N} a_{ij}x_j\Big), \tag{30}$$

where $g(x) = 1$ for (25), $g(x) = \prod_{i=1}^{N} x_i^{\alpha_i}$ for (26, 27), $g(x) = x_k^{-1}$ for (28). The proofs of the mentioned Plank's theorems end with the following similar words: "Since the factor $g(x)$ is positive in the first orthant, it can be dropped without altering the phase portrait of the differential equation. Q.E.D." All these words are true except for the last 3 letters "Q.E.D.", because the definition point (i) demands *the differential equation itself* to be written in the Hamiltonian form, *not* the phase portrait. So, Plank has discovered, or, rather, constructed Hamiltonian systems with quadratic Poisson structure matrices, having the same phase portrait as certain LVE in the first orthant. For the genuine LVE another form of Poisson structure matrices should be used with Plank's Hamiltonian functions:

$$j_{il} = \frac{1}{g(x)} c_{il} x_i x_l. \tag{31}$$

In three dimensions, the Jacobi identities with this form of Poisson structure matrices are satisfied. When $N = 4$, additional constraints arise from the closure of the Jacobi identities: $a = b = 0$ when $det(C) \neq 0$, or $det(C) = det(A) = 0$. Of course, when $N > 4$, still more additional constraints will appear. However, the open subset $G$ in which the Hamiltonian system should be defined, may be extended now, in certain cases, to the entire of $R^N$, excluding the subspaces $x_i = 0$.

## 6 Degeneracies with 3D Plank's structures.

The puzzling absence of the analogue of Cairó and Feix's primary invariant type I among Plank's Hamiltonian functions can be explained comparing Nutku's example (8), with cubic terms, and Plank's ansatz (24), without



cubic terms in the Poisson structure matrices. But, in fact, all the Plank's Hamiltonian functions (26), (27), (28) imply the degeneracy of interactions matrices in three dimensions, which is easily proved by straightforward calculations of the vector fields through $J\nabla H$. The following formulae (32), (33) are the results of such calculations. Defining $\gamma$ as the vector dual to the matrix $C$ and introducing $B_0 = 1$ for (26) and $B_0 = 0$ for (27), we have for the cases of Hamiltonian functions (26), (27):

$$A = \begin{pmatrix} \lambda_1 B_1 & (\lambda_1 + \gamma_3)B_2 & (\lambda_1 - \gamma_2)B_3 \\ (\lambda_2 - \gamma_3)B_1 & \lambda_2 B_2 & (\lambda_2 + \gamma_1)B_3 \\ (\lambda_3 + \gamma_2)B_1 & (\lambda_3 - \gamma_1)B_2 & \lambda_3 B_3 \end{pmatrix}; \; b = B_0 \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \tag{32}$$

where $\lambda = C\alpha \equiv [\alpha, \gamma]$. In the same notations, for the case of Hamiltonian function (28), with $k = 1$ we have:

$$A = \begin{pmatrix} \lambda_1 & \gamma_3 B_2 & -\gamma_2 B_3 \\ \lambda_2 & \gamma_3 B_2 & (\gamma_1 + \gamma_3)B_3 \\ \lambda_3 & -(\gamma_1 + \gamma_2)B_2 & -\gamma_2 B_3 \end{pmatrix}; \; b = \begin{pmatrix} 0 \\ \gamma_3 B_0 \\ -\gamma_2 B_0 \end{pmatrix}. \tag{33}$$

The determinants of these matrices are

$$det(A) = B_1 B_2 B_3 (n, \gamma)(\lambda, \gamma) \equiv 0 \tag{34}$$

for (32) and

$$det(A) = B_2 B_3 (n, \gamma)(\lambda, \gamma) \equiv 0 \tag{35}$$

for (33). They are identical to zero because $(\lambda, \gamma) \equiv ([\alpha, \gamma], \gamma) \equiv 0$. This means, that primary invariants type I should exist in both cases. Moreover, $\gamma$ is a solution of equations for this invariant with $s = 0$. The corresponding equations in Plank's form are $A^T\gamma = 0$; $(\gamma, b) = 0$ for the constant of motion

$$K(x) = \prod_{i=1}^{3} x_i^{\gamma_i}. \tag{36}$$

Any solution $\gamma$ of these equations may be added to $\alpha$ in the equations (12), (17), when $s = 0$, so, $\alpha$ and $\gamma$ are not uniquely defined. However, $\lambda$ is defined uniquely. The equations (12), when $s = 0$, coincide with Plank's conditions for the LVE to be volume preserving with density function $\prod_{i=1}^{N} x_i^{\alpha_i - 1}$. When $N = 3$, the degeneracy of the matrices (32), (33) implies, that these density functions are also defined not uniquely. So, some normalization may be used. When $(n, \gamma) \neq 0$, the normal form of

$\alpha_i$ can be defined as the limit of the solution of the corresponding linear nondegenerate system, when $(\lambda, \gamma)$ tends to zero. This normal form obeys the relation $\alpha = -(\gamma + [\lambda, n])/(n, \gamma)$ for (32), which is consistent with the form of the Morse function for (22). The corresponding relation for (33) is $\alpha = -[\lambda, n]/(n, \gamma)$. Note, that the exact expressions from [7] make use of a different normalization.

It should be noted, that the LVE of the form (33) are volume preserving with the density function $x_1^{-2}x_2^{-1}x_3^{-1}$, owing to the fact that $(-1, 0, 0)^T$ is obviously a particular solution of (12) with $s = 0$. The existence of the corresponding measure preserving density can be established for Hamiltonian versions of the invariant (28) in higher dimensions also, which Plank has not mentioned.

# 7 The "manifestly bi-Hamiltonian" equations.

With the invariant (36), we can write, using Gümral and Nutku's expression, the "manifestly bi-Hamiltonian form" of the equations of motion of the system as

$$\dot{x} = m(x)[\nabla H, \nabla K], \tag{37}$$

so that

$$j_H = m(x)\nabla K, \text{ and } j_K = -m(x)\nabla H, \tag{38}$$

where the scalar function $m(x)$ is defined from (37) using a component of the vector field:

$$\begin{array}{l} m(x) = \prod_{i=1}^3 x_i^{1-\gamma_i-\alpha_i} \text{ for (26), (27)} \\ \text{and} \\ m(x) = x_1 \prod_{i=1}^3 x_i^{1-\gamma_i} \text{ for (28).} \end{array} \tag{39}$$

Along this line, we get the correct form of Poisson structure matrices with Plank's Hamiltonian functions, in dual representation:

$$j_H = m(x) \prod_{i=1}^{3} x_i^{\gamma_i - 1}(\gamma_1 x_2 x_3, \gamma_2 x_1 x_3, \gamma_3 x_1 x_2)^T \tag{40}$$

which is equivalent to (31), containing quadratic terms due to their origin from $\nabla K$. The Poisson structure matrices for time-independent case of the invariant $\mathcal{I}_I$ of Cairó and Feix contain only cubic terms when this invariant exists together with Plank's Hamiltonian function (27), and with Hamiltonian functions (26), (28) the quadratic terms are included also. In the dual representation, the expressions for the Poisson structure matrices corresponding to the invariants (26) and (27) are:

$$j_K = -\prod_{i=1}^{3} x_i^{-\gamma_i} \begin{pmatrix} (\alpha_1 L + B_1 x_1)x_2 x_3 \\ (\alpha_2 L + B_2 x_2)x_1 x_3 \\ (\alpha_3 L + B_3 x_3)x_1 x_2 \end{pmatrix}, \text{ with } L = B_0 + \sum_{l=1}^{3} B_l x_l, \tag{41}$$

and for the invariant (28)

$$j_K = -\prod_{i=1}^{3} x_i^{-\gamma_i} \begin{pmatrix} (\alpha_1 x_1 - B_0 - B_2 x_2 - B_3 x_3)x_2 x_3 \\ (\alpha_2 x_1 + B_2 x_2)x_1 x_3 \\ (\alpha_3 x_1 + B_3 x_3)x_1 x_2 \end{pmatrix}. \tag{42}$$

Reminding of the statements we have made at the end of the section 4, we note here, that the constraint (23) used in [9] is satisfied identically, while the additional constraints imposed in [10] are satisfied for interactions matrices (32) when $B_1 = B_2 = B_3$. While checking the validity of the structure functions from these two papers, one should take care of the proper normalization, then the result is, that they reproduce the appropriate vector fields with the invariant (11) as the Hamiltonian function, when $s = 0$, only if $b = 0$. Our formulae include the case $b \neq 0$ for $s = 0$. In general, coordinates and time rescaling procedure used in [9, 10], alters the structure of the phase space, so it can destroy the correspondence between the Hamiltonian and the Poisson brackets, in other words, it is not always a valid transformation, analogous of canonical transformations in the classical Hamiltonian method. It should be noted, that the logarithmic form of the invariant $K(x)$ could be used in the bi-Hamiltonian formulation (37) instead of (36). If this is done, then the functions $1/m(x)$ (39) become equal to density functions, with which the equations are volume preserving.

# 8 Non-degenerate 3D interactions matrices.

The degeneracy of interactions matrices (32) in the case of Plank's Hamiltonian function (26) is implied in three-dimensional case by the conditions of the corresponding theorem $C\alpha = b$; $a_{il} = B_l(b_i + c_{il})$. But in the case of Plank's theorem for the Hamiltonian function (27) the conditions $b=0$; $A^T\alpha=-\mathrm{diag}(A)$; $B_id_{ik}=-B_kd_{ki}$ do not imply the degeneracy of $A$. In the latter case, if a row $\epsilon(B_1,B_2,B_3)$ is added to each row of a degenerate interactions matrix $\tilde{A}$, the corresponding matrix $D$ (13) remains the same. The determinant of the new interactions matrix $A$ is nonzero, thus, the invariant $K(x)$ ceases to exist for the new system. The phenomenon appearing in such a case is clear from the following example with $\epsilon=1, b=0$:

$$\tilde{A}=\begin{pmatrix}-2 & 1 & 0\\ 1 & 0 & -1\\ 0 & -1 & 2\end{pmatrix}\rightarrow A=\begin{pmatrix}1 & 0 & 3\\ 4 & -1 & 2\\ 3 & -2 & 5\end{pmatrix}; \tag{43}$$

with the constant of motion

$$\tilde{H}(x)=x_1^{-3/4}x_2^{1/2}x_3^{-3/4}(3x_1-x_2+3x_3) \tag{44}$$

in common with both the systems. It is a Hamiltonian function of the type (26) for $\tilde{A}$, but it is not for the new system with the interactions matrix $A$, since the values of the components of the vector field coincide for these two matrices only in the invariant plane $\overline{P}: 3x_1-x_2+3x_3=0$ of the invariant $\mathcal{I}_{II}$, defined in [7]. Thus, the conditions of Plank's theorem for this case are not sufficient to reproduce even the phase portrait of the differential equation.

However, all the known examples of Hamiltonian structures for LVE up to this moment, have been of degenerate interactions matrices in the case $N=3$. Here follows an example with a non-degenerate interactions matrix with bi-Hamiltonian structure of Lie-Poisson form:

$$A=\begin{pmatrix}-\alpha & \beta & \gamma\\ \alpha & -\beta & \gamma\\ \alpha & \beta & -\gamma\end{pmatrix}; b=0 \tag{45}$$

$$H_1=x_1(\beta x_2-\gamma x_3);\ H_2=x_2(\alpha x_1-\gamma x_3) \tag{46}$$

$$j_{H_1}=-\frac{1}{\gamma}\begin{pmatrix}\alpha x_2\\ \alpha x_1-\gamma x_3\\ -\gamma x_2\end{pmatrix};\ j_{H_2}=\frac{1}{\gamma}\begin{pmatrix}\beta x_2-\gamma x_3\\ \beta x_1\\ -\gamma x_1\end{pmatrix}. \tag{47}$$

The equations (45) were derived by Brenig [13] from the equations of asymmetric top and resonant three-wave interaction system. The Hamiltonian



functions (46) could be thought of as secondary invariants of Cairó and Feix, since Plank's Hamiltonian function of the type (27) must have all the coefficients $B_l\neq 0$ in the linear polynomial expression. The Poisson structures (47) are not equal to those given in [8] for Euler's top, but are of the same linear type.

# 9 Conclusions.

The conditions of Plank's theorems on the Hamiltonian systems for the LVE with Hamiltonian functions of the types (26), (27), (28) are not sufficient to reproduce the vector field of the LVE with the quadratic Poisson structure matrix (24). For (27), the conditions are not sufficient to reproduce even the phase portrait of the LVE (43). A modified Poisson structure matrix (31) should be used, an additional constraint $det(A)=0$ is implied in the 3D case of the Hamiltonian function (27).

In the 3D case, the interactions matrices (32), (33) are identically degenerate, implying the existence of the second Hamiltonian function (36) and allowing for the Poisson structure matrices to be obtained using the gradients of the two Hamiltonians. The parametric representations (32), (33) served us as the source for controversial examples, which show that Plank's Hamiltonian function (28) is not a limiting case of the primary invariant III. This helps to realize, that the generalized Carleman ansatz, used in [6, 7], is not sufficient to obtain the constants of motion in which both the linear and the logarithmic terms appear.

In three dimensions, Lie-Poisson type structures may appear in the cases when secondary linear polynomial invariants of Cairó and Feix exist. In the given example (45), the interactions matrix is nondegenerate.

Plank's conditions for the absence of time dependence of the constants of motion are more exact, than that of Cairó and Feix. The existence of a sufficient number of time independent constant s of motion is important, as it makes possible to apply directly bi-Hamiltonian [8] or, more generally, multi-Hamiltonian [14, 15] formulation, thus the complete integrability might be proved more easily. Up to this moment, the maximum number of analytically established functionally independent constants of motion, has been $N+1$ symmetric functions of odd powers for $2N+1$ dimensions, given by Itoh [22] for the VE without the malthusian terms. Damianou's integrals of motion for the Volterra lattice model [16], containing both odd and even powers, are not functionally independent. The most useful feature of Cairó and Feix's secondary invariants, appearing due to some special symmetries of the terms of interactions matrices, is, that there may exist several invariats of the same order in the powers of variables, which is demonstrated in the example (46). The methods used in [22, 16] do not show such a possibility, because the symmetries of the systems studied in this papers are of another special type.

On the other hand, for the more comlicated case of the general LVE, the subcase $s > 0$ leading Cairó and Feix to the asymptotic reduction of the dimension of the phase space by one for the cases of the invariant type II from even to odd number of dimensions and of the invariant type III from odd to even number of dimensions, remains valid, but the examples of $s \neq 0$, that we have given, make us suppose, that such reductions can take place in other cases also. However, since the asymptotic reduction can occur only after the infinite time interval, it could not be accompanied in reality with a time rescaling, so the reduction of the dimensions by two, which has been suggested in [7], cannot take place without additional model assumptions which should allow to perform the transition across the barrier of the infinite time.

With the exception of the case of the Volterra invariant, the correct forms of Poisson structure matrices include the product of certain powers of dependent variables, which give their contributions to the left-hand sides of the Jacobi identities, implying some additional constraints when $N > 3$. For instance, in the case of $N = 4$, either both the Poisson structure matrix and the interactions matrix are degenerate, or the Hamiltonian function is linear.

The authors thank V.P. Gerdt, O.M. Khudaverdian, I.V. Komarov, A.N. Leznov, and P. Winternitz for discussion, and A.B. Shabat for a useful remark.

# References

[1] V.Volterra, Leçons sur la Théorie Mathematique de la Lutte pour la Vie. Gauthier Villars, Paris, 1931.

[2] E.H.Kerner, "Comment on Hamiltonian structures for the n-dimensional Lotka-Volterra equations," J.Math.Phys. **38**(2), 1218-1223(1997).



[3] L.Fadeev, L.Takhtajan, "Hamiltonian method in the theory of solitons." (Springer, Berlin, 1986).

[4] Y.Nutku, "Hamiltonian structure of the Lotka-Volterra equations, " Phys.Lett.A **145**,245(1990).

[5] B.Grammatikos, J.Moulin-Ollagnier, A.Ramani, J.M.Strelcyn, and S.Wojciechowski, "Integrals of quadratic ordinary differential equations in $R^3$: the Lotka-Volterra system," Physica A **163**,683(1990).

[6] L.Cairó, M.R.Feix, and J.Goedert, "Invariants for models of interacting populations," Phys.Lett.A **140**,421(1989).

[7] L.Cairó, and M.R.Feix, "Families of invariants of the motion for the Lotka-Volterra equations: The linear polynomial family," J.Math.Phys. **33**,2440-2455(1992).

[8] H.Gümral, and Y.Nutku, "Poisson structures of dynamical systems with three degrees of freedom," J.Math.Phys. **34**,5691(1993).

[9] J.Goedert, F.Haas, D.Hua, M.R.Feix, and L.Cairó, "Hamiltonian structures for systems in three dimensions with a rescalable constant of motion," J.Phys.A:Math.Gen. **27**,6495-6507(1994).

[10] F.Haas, and J.Goedert, "On the generalized Hamiltonian structure for 3D dynamical systems, "Phys.Lett.A **199**,173-179(1995).

[11] Manfred Plank, "Hamiltonian structures for the n-dimensional Lotka-Volterra equations," J.Math.Phys. **36**,3520-3534(1995).

[12] L.Cairó, and M.R.Feix "Comments on Hamiltonian structures for the n-dimensional Lotka-Volterra equations," J.Math.Phys. **37**,3644(1996).

[13] L.Brenig, "Complete factorization and analytic solutions of generalized Lotka-Volterra equations," Phys.Lett.A **133**,378(1988).

[14] J.Hiertarianta, "Nambu tensor and commuting vector fields, "J.Phys.A: Math.Gen. **30**,L27-L33(1997).

[15] R.Chatterjee, "Dynamical symmetries and Nambu mechanics. " Letter in Math. Phys. **36**,117-126(1996).

[16] P.Damianou, "The Volterra model and its relation to the Toda lattice," Phys.Lett.A **155**,126(1991).

[17] A.Nagai and J.Satsuma, "The Lotka-Volterra equations and the QR algorithm,"J.Phys.Soc.Jpn. **64**,3669(1995).

[18] A.Leznov, M.Saveliev, "Group Theoretical Methods for Integration of Nonlinear Dynamical Systems", (in Russian): Moscow, Nauka, 1985; (in English): Progress in Physics, #15, Basel, Birkhauser-Verlag, 1992.

[19] A.Razumov, M.Saveliev, "Lie Algebras, Geometry, and Toda-Type Systems", Cambridge, 1997.

[20] O.Bogoyavlensky, "Integrable discretizations of the KdV equation," Phys.Lett.A **134**,34(1988).

[21] A.G.Galperin, V.M.Dubovik, V.S.Richvitsky, "Nonlinear dynamical model of evolution of a 3-component system conserving total number of objects," in: CMCP96 Book of abstracts, JINR,D5,11-96-300,P.69, Dubna, 1996; А. Г. Гальперин, В. М. Дубовик. В. С. Рихвицкий, Preprint of JINR, P4-97340, Dubna,1997, (in Russian, more complite), "Нелинейная динамическая модель эволюции трехкомпонентной системы с сохранением суммарного числа объектов".

[22] Y.Itoh, "Integrals of a Lotka-Volterra System of Odd Number of Variables," Prog.Theor.Phys. **78**,507(1987).

Received by Publishing Department
on December 31, 1997.



Дубовик В.М., Гальперин А.Г., Рихвицкий В.С., Слепнев С.К. E4-97-416

Условия существования первых интегралов
и гамильтоновы структуры уравнений Лотка—Вольтерра.
Комментарий к недавно опубликованным работам

Уточняются условия существования первых интегралов и гамильтоновых структур для уравнений Лотка—Вольтерра (ЛВ), сформулированные недавно рядом авторов. В частности, в матрице пуассоновой структуры, предложенной Планком для не зависящих от времени гамильтонианов, отсутствует важный конформный множитель, в силу чего условия ряда теорем не являются достаточными для записи уравнений движения в гамильтоновой форме. В случае 3D представление Планка для матрицы пуассоновой структуры приводит к вырождению матрицы взаимодействий и к существованию времянезависимого инварианта Кэро—Фэкса типа I, что делает возможной бигамильтонову формулировку. С другой стороны, пуассоновы структуры, построенные для вырожденной матрицы взаимодействий при наличии лишь одной константы движения, не позволяют воспроизвести линейные (мальтусовы) члены уравнений ЛВ, когда эта константа не зависит явно от времени. Результаты работы основаны на параметризации 3D уравнений ЛВ, вытекающей из гамильтоновой формулировки с использованием модифицированного представления Планка.

Работа выполнена в Лаборатории высоких энергий, в Лаборатории теоретической физики им.Н.Н.Боголюбова, в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1997

Dubovik V.M., Galperin A.G., Richvitsky V.S., Slepnyov S.K. E4-97-416

The Conditions of Existence of First Integrals
and Hamiltonian Structures of the Lotka—Volterra Equations.
Comment on Some of the Recent Papers

The conditions of existence of first integrals (partial integrability) and Hamiltonian structures (possible complete integrability) of the Lotka—Volterra equations have been analyzed recently by many authors. In some cases, these conditions should be stated more correctly. In particular, an important conformal factor is not present in the Poisson structure matrix, suggested by Plank for time-independent Hamiltonian functions, for which reason the conditions of some of the theorems formulated by the mentioned author are not sufficient to write the equations of motion in the Hamiltonian form. In 3D case, Plank's ansatz for the Poisson structure matrix implies the degeneracy of the interactions matrix and the existence of the time-independent version of the invariant of Cairo and Feix's type I, thus the bi-Hamiltonian formulation becomes possible. On the other hand, the attempts to construct Poisson structures for degenerate interactions matrices, when only one constant of motion is present, which were made in two of the papers, do not give the possibility to reproduce the linear (malthusian) terms in the Lotka—Volterra equations, when this constant of motion is explicitly independent of the time. Our statements are based on the parametrization of 3D Lotka—Volterra equations, implied by Hamiltonian formulation with improved Plank's ansatz.

The investigation has been performed at the Laboratory of High Energies, at the Bogoliubov Laboratory of Theoretical Physics, at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1997