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G.N.Afanasiev*, V.G.Kartavenko

RADIATION OF CHARGE UNIFORMLY MOVING IN MEDIUM

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*E-mail: afanasev@thsun1.jinr.dubna.su

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Афанасьев Г.Н., Картавенко В.Г. E4-97-393
Излучение заряда, равномерно движущегося в среде

Проанализировано, как зависимость диэлектрической проницаемости от частоты влияет на электромагнитное поле, излучаемое заряженной частицей, равномерно движущейся в среде. Оказывается, что точечная частица излучает при любой скорости. Изучается пространственное распределение поля частицы. При этом осцилляции электромагнитного поля объясняются меняющейся во времени поляризацией среды, производимой движущимся зарядом. Вычислены спектральные распределения излучаемой энергии и числа квантов. Обсуждаются следствия, возникающие из-за применения поляризации, отличающейся от стандартной. Проанализированы соотношения Крамерса—Кронига применительно к рассматриваемой задаче.

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Radiation of Charge Uniformly Moving in Medium

We analyze how the frequency dependence of the dielectric permittivity affects the electromagnetic field radiated by a point charge uniformly moving in medium. It turns out that a moving charge radiates at every velocity. We study the space distribution of the electromagnetic field and show that its oscillations are due to the time-dependent medium polarization induced by the moving charge. Spectral distributions of the radiated energy and the photon number are given. Consequences arising from the choice of polarization different from the usual one are discussed. The analysis of the Kramers—Kronig dispersion relations for the treated problems is given.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

The goal of this consideration is to evaluate the electromagnetic field (EMF) arising from the uniform motion of a charge in the nonmagnetic medium described by the frequency-dependent one-pole electric permittivity

$$\epsilon(\omega) = 1 + \frac{\omega_L^2}{\omega_0^2 - \omega^2}. \tag{1.1}$$

This parametrization is a suitable parametrization between the static case $\omega = 0$, $\epsilon(\omega) = \epsilon_0 = 1 + \omega_L^2/\omega_0^2$ and the high-frequency limit $\omega = \infty$, $\epsilon(\omega) = 0$ when medium oscillators do not have enough time to be excited.

The radiation produced by fast electrons moving in medium was observed by P.A. Cherenkov in 1934 [1]. Tamm and Frank [2] considered the motion of a point charge in medium with a constant electric permittivity. They showed that the charge should radiate when its velocity exceeds the light velocity in medium. For the frequency independent electric permittivity the electromagnetic strengths have δ-type singularities on the surface of the so-called Cherenkov (or Mach) cone [3-6]. This leads to the divergence of the quantities involving the product of electromagnetic strengths. In particular, this is true for the flux of EMF. To avoid this difficulty Tamm and Frank made the Fourier transformation of the EMF and integrated the energy flux up to some maximal frequency ω_0.

However, Eq.(1.1) is a standard parametrization describing a lot of optical phenomena [7]. It is valid when the wavelength of the electromagnetic field is much larger than the distance between the particles of medium on which the light scatters. The typical atomic dimensions are of the order $a \approx \hbar/mc\alpha$, $\alpha = e^2/\hbar c$. This gives $\lambda = c/\omega >> a$ or $\omega << mc^2\alpha/\hbar \approx 5 \cdot 10^{18} sec^{-1}$. The typical atomic frequencies are of the order $\omega_0 \approx mc^2/\hbar\alpha^2 \approx 10^{16} sec^{-1}$. Thus, the integration region extends well beyond ω_0. For $\omega >> \omega_0$, $\epsilon(\omega) \approx 1$, that is, atomic electrons have no enough time to be excited. Following the book [8] and review [9] we extrapolate parametrization (1.1) to all ω. This means that we disregard the excitation of nuclear levels and discrete structure of scatterers.

So, we intend to consider the effects arising from the charge motion in medium with $\epsilon(\omega)$ given by (1.1). This was done by E. Fermi in 1940 [10]. He showed that a charged particle moving uniformly in medium with permittivity (1.1) should radiate at every velocity. He also showed that energy losses as a function of the charge velocity are less than those predicted by the Bohr theory [11]. However, Fermi did not evaluate the electromagnetic strengths for various charge velocities and did not show how the transition takes place from the subluminal regime to the superluminal one.

The Fermi theory was extended to the case of many poles case by Sternheimer [12] who obtained satisfactory agreement with experimental data. Another development of the Fermi theory is its quantum generalization [13-15].

In this consideration we restrict ourselves to the classical theory of the Vavilov-Cherenkov radiation with electric permittivity given by (1.1). It is suggested that uniform motion of a particle is maintained by some external force the origin of which is not of interest for us.

The plan of our exposition is as follows.

In section 2, the necessary mathematical formulas are presented.

In section 3, we evaluate electromagnetic potentials and field strengths for a charge moving uniformly in a dielectric with $\epsilon(\omega)$ given by (1.1). We observe the appearance of oscillations of the electromagnetic field inside the Cherenkov cone for the charge velocity above some critical value v_c.

In section 4, we evaluate the energy flux and the number of radiated photons as a function of the charge velocity. Their spectral distributions are also given. It turns out that for $v > v_c$ all frequencies contribute to the energy and photon number spectra, while for $v < v_c$ the range of available frequencies diminishes. In the same section, we demonstrate how the energy flux is distributed over the surface of a cylinder coaxial with the charge trajectory. Again, oscillations inside the Cherenkov cone are observed for $v > v_c$.

In section 5, we formulate the results obtained in the polarization language. It turns out that it is the medium polarization induced by the electromagnetic field of a moving charge that gives rise to the above-mentioned oscillations of EMF.

Another choice of polarization and its physical consequences are discussed in section 6.

The analysis of the Kramers-Kronig dispersion relations for the treated problem and short resume of the results obtained are given in sections 7 and 8.

2 Mathematical preliminaries

Consider a point charge e uniformly moving in a non-magnetic medium with a velocity v directed along the z axis. Its charge and current densities are given by

$$\rho(\vec{r},t) = e\delta(x)\delta(y)\delta(z-vt), \quad j_z = v\rho.$$

Their Fourier transforms are

$$\rho(\vec{k},\omega) = \int \rho(\vec{r},t)\exp[i(\vec{k}\vec{r}-\omega t)]d^3\vec{r}dt = 2\pi e\delta(\omega-\vec{k}\vec{v}), \quad j_z(\vec{k},\omega) = v\rho(\vec{k},\omega).$$

In the $(\vec{k},\omega)$ space the electromagnetic potentials are given by (see, e.g., [16])

$$\Phi(\vec{k},\omega) = \frac{4\pi}{\epsilon}\frac{\rho(\vec{k},\omega)}{k^2-\frac{\omega^2}{c^2}\epsilon}, \quad A_z(\vec{k},\omega) = 4\pi\beta\frac{\rho(\vec{k},\omega)}{k^2-\frac{\omega^2}{c^2}\epsilon}, \quad \beta = v/c. \tag{2.1}$$

Here $\epsilon(\omega)$ is the electric permittivity of medium. Its frequency dependence is chosen in a standard form (1.1). In the usual interpretation ω_L and ω_0 are the plasma frequency

$\omega_L^2 = 4\pi N_e e^2/m$ (N_e is the number of electrons per unit of volume, m is the electron mass) and some resonance frequency. Quantum-mechanically, it can be associated with the energy excitation of the lowest atomic level. Our subsequent exposition does not depend on this particular interpretation of ω_L and ω_0. The static limit of $\epsilon(\omega)$ is

$$\epsilon_0 = \epsilon(\omega=0) = 1 + \frac{\omega_L^2}{\omega_0^2}$$

$\epsilon(\omega)$ has poles at $\omega = \pm\omega_0$. Being positive for $\omega^2 < \omega_0^2$ it jumps from $+\infty$ to $-\infty$ when one passes the point $\omega^2 = \omega_0^2$. $\epsilon(\omega)$ has zero at $\omega^2 = \omega_0^2 + \omega_L^2$ and tends to unity for $\omega \to \infty$. It is seen that

$$\epsilon^{-1}(\omega) = 1 - \frac{1}{\omega_0^2+\omega_L^2-\omega^2}\,\omega_L^2 \tag{2.2}$$

has zero at $\omega^2 = \omega_0^2$ and a pole at $\omega^2 = \omega_3^2 = \omega_0^2 + \omega_L^2$. In the $\vec{r},t$ representation $\Phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ are given by

$$\Phi(\vec{r},t) = \frac{e}{\pi v}\int\frac{d\omega}{\epsilon}e^{i\omega(t-z/v)}\frac{kdk}{k^2+\frac{\omega^2}{v^2}(1-\beta^2\epsilon)}J_0(k\rho).$$

$$A_z(\vec{r},t) = \frac{e}{\pi c}\int d\omega e^{i\omega(t-z/v)}\frac{kdk}{k^2+\frac{\omega^2}{v^2}(1-\beta^2\epsilon)}J_0(k\rho). \tag{2.3}$$

First, we take integral over k. For this we use the Table integral (see, e.g., [17])

$$\int_0^\infty \frac{kdk}{k^2+q^2}J_0(k\rho) = K_0(\rho q), \tag{2.4}$$

where in the right-hand side the value of square root $\sqrt{q^2}$ corresponding to its positive real part should be taken.

3 Electromagnetic potentials and field strengths

We now define domains where $1-\beta^2\epsilon > 0$ and $1-\beta^2\epsilon < 0$.

For $\beta < \beta_c$ one has:

$1-\beta^2\epsilon > 0$ for $\omega^2 < \omega_c^2$ and $\omega^2 > \omega_0^2$ and $1-\beta^2\epsilon < 0$ for $\omega_c^2 < \omega^2 < \omega_0^2$.

For $\beta > \beta_c$ one gets: $1-\beta^2\epsilon > 0$ for $\omega^2 > \omega_0^2$ and $1-\beta^2\epsilon < 0$ for $0 < \omega^2 < \omega_0^2$.

Here $\beta_c = \epsilon_0^{-1/2}$, $\omega_c = \sqrt{\omega_0^2 - \beta^2\gamma^2\omega_L^2}$.

Sometimes in physical literature another representation of the dielectric permittivity is used (known as the Lorentz-Lorenz or Clausius-Mossotti formula, see, e.g., [18]):

$$\epsilon' = \frac{1+2\alpha(\omega)/3}{1-\alpha(\omega)/3}, \quad \alpha(\omega) = \frac{\omega_L^2}{\omega_0^2-\omega^2}.$$

It is generally believed that $\epsilon(\omega)$ describes optical properties of media for which $\epsilon(\omega)$ only slightly differs from unity (e.g., gases), whereas $\epsilon'(\omega)$ desribes more general media (liquids, solids, etc.).

For $\omega_0^2 < \omega_L^2/3$ one always has $1-\beta^2\epsilon' > 0$, which means the absence of radiation by the uniformly moving charge (see below).

Let now $\omega_0^2 > \omega_L^2/3$.

Then, for $\beta < \beta_c'$, $(\beta_c'^2 = 1-\omega_L^2/(\omega_0^2+2\omega_L^2/3)$ one has :

$1-\beta^2\epsilon' > 0$ for $\omega^2 < \omega_0^2 - \frac{1}{3}\omega_L^2 - \beta^2\gamma^2\omega_L^2$ and for $\omega^2 > \omega_0^2 - \frac{1}{3}\omega_L^2$;

$1-\beta^2\epsilon' < 0$ for $\omega_0^2 - \frac{1}{3}\omega_L^2 - \beta^2\gamma^2\omega_L^2 < \omega^2 < \omega_0^2 - \frac{1}{3}\omega_L^2$.

On the other hand, for $\beta > \beta_c'$:

$1-\beta^2\epsilon' > 0$ for $\omega^2 > \omega_0^2 - \frac{1}{3}\omega_L^2$ and

$1-\beta^2\epsilon' < 0$ for $0 < \omega^2 < \omega_0^2 - \frac{1}{3}\omega_L^2$.

We see that qualitative behaviour of ϵ and ϵ' is almost the same. The sole exception is that for $\omega_0^2 < \omega_L^2/3$ there is no solution corresponding to $1-\beta^2\epsilon' < 0$. This permits us to limit ourselves to the ϵ representation in form (1.1).

As it was admitted in [9], the inclusion of ω dependences in ϵ and ϵ' makes unnecessary the consideration of retardation effects. The very fact that the light velocity in medium c_n is less than the light velocity in vacuum c means that oscillators of medium react on the initial electromagnetic field with some delay (see section 5, for details). The deviation of c_n from c is due to the deviation of ϵ from unity. For the incoming plane wave and frequency-independent ω this was clearly demonstrated in refs. [19,20]. At first glance it seems that c_n will be greater than c for $\epsilon < 1$. However, a more accurate analysis shows [8] that the group velocity of light in medium is always less than c.

Now we satisfy the condition $Re\sqrt{1-\beta^2\epsilon} > 0$. It is fulfilled automatically if $1-\beta^2\epsilon > 0$. In this case the argument of the K_0 function is $\frac{|\omega|\rho}{v}\sqrt{1-\beta^2\epsilon}$ where there square root means its arithmetic value.

Now let $1-\beta^2\epsilon < 0$.

First, we consider the case when ω has the imaginary part:

$$\epsilon(\omega) = 1 + \frac{\omega_L^2}{\omega_0^2-\omega^2+ip\omega}, \quad p>0.$$

The positivity of p leads to poles of $\epsilon(\omega)$ lying only in the upper complex ω half-plane. This is needed to satisfy the causality condition (for details see [21]). Sometimes in physical literature [22] it is stated that the causality condition is fulfilled if the poles of $\epsilon(\omega)$ lie in the lower ω half-plane. This is due to a different definition of the Fourier transforms corresponding to different signs of ω of the exponentials occurring in (2.3). We write out explicit expressions for electromagnetic potentials and field strengths:

$$\Phi = \frac{e}{\pi v}\int_{-\infty}^{\infty}\frac{d\omega}{\epsilon}e^{i\alpha}K_0(k\rho), \quad A_z = \frac{e}{\pi c}\int_{-\infty}^{\infty}d\omega e^{i\alpha}K_0(k\rho),$$

$$H_\phi = \beta D_\rho = \frac{e}{\pi c}\int_{-\infty}^{\infty}d\omega e^{i\alpha}kK_1(k\rho), \quad E_\rho = \frac{e}{\pi v}\int_{-\infty}^{\infty}\frac{d\omega}{\epsilon}e^{i\alpha}kK_1(k\rho),$$

$$E_z = -\frac{ie}{\pi c^2}\int_{-\infty}^{\infty}d\omega\omega(1-\frac{1}{\beta^2\epsilon})e^{i\alpha}K_0(k\rho), \quad D_z = \frac{ie}{\pi v^2}\int_{-\infty}^{\infty}d\omega\omega(1-\beta^2\epsilon)e^{i\alpha}K_0(k\rho)\,. \quad (3.1).$$

Here $\alpha = \omega(t-z/v)$, $k^2 = (1-\beta^2\epsilon)\omega^2/v^2$. Again, k in Eq.(3.1) means the value of $\sqrt{k^2}$ corresponding to $Rek > 0$.

These expressions were obtained by Fermi [10]. Their drawback is that modified Bessel functions K are complex even for real ϵ (when $1-\beta^2\epsilon < 0$). We intend now to present Eqs. (3.1) in a manifestly real form. This greatly simplifies calculations.

We present $1-\beta^2\epsilon$ in the form

$$1-\beta^2\epsilon = a+ib = \sqrt{a^2+b^2}(\cos\phi + i\sin\phi) \quad (3.2)$$

where

$$a = 1-\beta^2-\beta^2\omega_L^2\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+p^2\omega^2}, \quad b = \beta^2\omega_L^2\frac{\omega p}{(\omega_0^2-\omega^2)^2+p^2\omega^2},$$

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\phi = \frac{b}{\sqrt{a^2+b^2}}\,.$$

Now we take square root of $1-\beta^2\epsilon$. The positivity of $Re\sqrt{1-\beta^2\epsilon}$ defines it uniquely:

$$\sqrt{1-\beta^2\epsilon} = (a^2+b^2)^{1/4}(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}),$$

$$\cos\frac{\phi}{2} = \frac{1}{\sqrt{2}}(1+\frac{a}{\sqrt{a^2+b^2}})^{1/2}, \quad \sin\frac{\phi}{2} = \frac{1}{\sqrt{2}}\frac{b}{|b|}(1-\frac{a}{\sqrt{a^2+b^2}})^{1/2}. \quad (3.3).$$

Thus, the argument of K functions entering into (3.1) is

$$\rho\frac{|\omega|}{v}(a^2+b^2)^{1/4}(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2})\,. \quad (3.4)$$

Although the integrands in (3.1) are complex, the integrals defining electromagnetic potentials and strengths are real (see Appendix). This is due to the fact that $\epsilon(-\omega) = \epsilon^*(\omega)$.

Now we take the limit $p \to 0+$. Let in this limit $1-\beta^2\epsilon > 0$. Then, $a > 0$, $b \to 0$, $\cos\frac{\phi}{2} \to 1$, $\sin\frac{\phi}{2} \to 0$ and $\sqrt{1-\beta^2\epsilon}$ coincides with its arithmetic value. Now let $1-\beta^2\epsilon < 0$. Then, $a < 0$, $b \to 0$, $\cos\frac{\phi}{2} \to 0$, $\sin\frac{\phi}{2} \to b/|b|$ and $\sqrt{1-\beta^2\epsilon} = i\sqrt{|1-\beta^2\epsilon|}\,sign(\omega)$. (it was taken into account that $p > 0$). This shows that K functions entering into the right-hand side of Eq. (3.1) reduce to

$$K_0(i\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}) = -\frac{i\pi}{2}H_0^{(2)}(\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}),$$

$$K_1(i\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}) = -\frac{\pi}{2}H_1^{(2)}(\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|})$$

for $\omega > 0$ and

$$K_0(-i\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}) = \frac{i\pi}{2}H_0^{(1)}(\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}),$$

$$K_1(-i\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}) = -\frac{\pi}{2}H_1^{(1)}(\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|})$$

for $\omega < 0$.

Now we are able to write out electromagnetic potentials and field strengths in a manifestly real form. For $\beta < \beta_c$ one finds

$$\Phi(\vec{r},t) = \frac{2e}{\pi v}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})\frac{d\omega}{\epsilon}\cos\alpha K_0 + \frac{e}{v}\int_{\omega_c}^{\omega_0}\frac{d\omega}{\epsilon}(\sin\alpha J_0 - \cos\alpha N_0),$$

$$A_z(\vec{r},t) = \frac{2e}{\pi c}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})d\omega\cos\alpha K_0 + \frac{e}{c}\int_{\omega_c}^{\omega_0}d\omega(\sin\alpha J_0 - \cos\alpha N_0) \qquad (3.5).$$

$$H_\phi(\vec{r},t) = \frac{2e}{\pi cv}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})\omega d\omega\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{cv}\int_{\omega_c}^{\omega_0}\omega d\omega\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1),$$

$$E_z = \frac{2e}{\pi c^2}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})(1-\frac{1}{\epsilon\beta^2})\omega d\omega\sin\alpha K_0 - \frac{e}{c^2}\int_{\omega_c}^{\omega_0}(1-\frac{1}{\epsilon\beta^2})\omega d\omega(N_0\sin\alpha + J_0\cos\alpha),$$

$$E_\rho = \frac{2e}{\pi v^2}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})d\omega\frac{\omega}{\epsilon}\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{v^2}\int_{\omega_c}^{\omega_0}d\omega\frac{\omega}{\epsilon}\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1).$$

On the other hand, for $\beta > \beta_c$

$$\Phi(\vec{r},t) = \frac{2e}{\pi v}\int_{\omega_0}^{\infty}\frac{d\omega}{\epsilon}\cos\alpha K_0 + \frac{e}{v}\int_0^{\omega_0}\frac{d\omega}{\epsilon}(\sin\alpha J_0 - \cos\alpha N_0),$$

$$A_z(\vec{r},t) = \frac{2e}{\pi c}\int_{\omega_0}^{\infty}d\omega\cos\alpha K_0 + \frac{e}{c}\int_0^{\omega_0}d\omega(\sin\alpha J_0 - \cos\alpha N_0). \qquad (3.6)$$

$$H_\phi(\vec{r},t) = \frac{2e}{\pi cv}\int_{\omega_0}^{\infty}\omega d\omega\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{cv}\int_0^{\omega_0}\omega d\omega\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1),$$

$$E_z = \frac{2e}{\pi c^2}\int_{\omega_0}^{\infty}(1-\frac{1}{\epsilon\beta^2})\omega d\omega\sin\alpha K_0 - \frac{e}{c^2}\int_0^{\omega_0}(1-\frac{1}{\epsilon\beta^2})\omega d\omega(N_0\sin\alpha + J_0\cos\alpha),$$

$$E_\rho = \frac{2e}{\pi v^2}\int_{\omega_0}^{\infty}d\omega\frac{\omega}{\epsilon}\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{v^2}\int_0^{\omega_0}d\omega\frac{\omega}{\epsilon}\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1).$$

Here $\alpha = \omega(t - z/v)$. The argument of all the Bessel functions is $\sqrt{|1-\beta^2\epsilon|}\rho\omega/v$. We observe that integrals containing usual (J, N) and modified (K) Bessel functions are taken over space regions where $1-\beta^2\epsilon < 0$ and $1-\beta^2\epsilon > 0$, resp.

Consider the limit cases of these expessions.

For $\omega_L \to 0$ we obtain: $\epsilon \to 1, \beta_c \to 1, \omega_c \to \omega_0$,

$$\Phi = \frac{2e}{\pi v}\int_0^{\infty}d\omega\cos\alpha K_0(\frac{\rho\omega}{v\gamma}) = \frac{1}{[(z-vt)^2+\rho^2/\gamma^2]^{1/2}}, \quad A_z = \beta\Phi; \quad \gamma = 1/\sqrt{1-\beta^2}$$

i.e. we get the field of a charge uniformly moving in vacuum.

Let $v \to 0$. Then, $\omega_c = \omega_0$ and

$$\Phi = \frac{2e}{\pi\epsilon_0}\int_0^{\infty}d\omega\cos(\frac{\omega z}{c})K_0(\frac{\rho\omega}{c}) = \frac{e}{\epsilon_0}\frac{1}{\sqrt{\rho^2+z^2}}, \quad A_z = 0$$

i.e., we obtain the field of a charge resting in medium.

Let $\omega_0 \to \infty$, $\omega_L \to \infty$,, but ω_L/ω_0 is finite. Then, $\omega_c = \omega_0\sqrt{1-\beta^2\gamma^2\omega_L^2/\omega_0^2} \to \infty$, $\epsilon(\omega) \to \epsilon_0$ and

$$\Phi = \frac{2e}{\pi v\epsilon_0}\int_0^{\infty}d\omega\cos\alpha K_0(\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon_0}) = \frac{e}{\epsilon_0}\frac{1}{[(z-vt)^2+\rho^2/\gamma_n^2]^{1/2}}, \quad A_z = \beta\epsilon_0\Phi$$

for $\beta < \beta_c$ and

$$\Phi = \frac{e}{v\epsilon_0}\int_0^{\infty}d\omega(\sin\alpha J_0 - \cos\alpha N_0) = \frac{2e}{\epsilon_0}\frac{1}{[(z-vt)^2-\rho^2/\gamma_n^2]^{1/2}}\Theta(vt-z-\rho/\gamma_n), \quad A_z = \beta\epsilon_0\Phi$$

for $\beta > \beta_c$. Here $\gamma_n = 1/\sqrt{|1-\beta_n^2|}$, $\beta_n = v/c_n$, $c_n = c/\sqrt{\epsilon_0}$.

Thus, we arrive at the charge motion in medium with a constant electric permittivity $\epsilon = \epsilon_0$.

It should be stressed that the integration over the whole range of ω is absolutely needed to obtain correct limit expressions and to guarantee the reversibility of the Fourier transformation.

The distributions of the magnetic vector potential A_z and field strengths as a function of z on the surface of a cylinder C_ρ of the radius ρ (Fig. 1) are shown in Figs. 2 - 7. If the dependence ϵ of ω were neglected ($\epsilon(\omega) = \epsilon_0$), then for $\beta > \beta_c$ the electromagnetic field would be confined to the interior of the Cherenkov cone with the solution angle $2\theta_c$ $\sin\theta_c = \beta_c/\beta$ (Fig. 1). This means that on the surface of C_ρ the electromagnetic field should be zero for $-z_c < z < \infty$, $z_c = \rho\cot\theta_c = \sqrt{\frac{\beta^2}{\beta_c^2} - 1}$. What can we learn from figures 2-7 ? For a small charge velocity ($\beta \le 0.4$) the magnetic field coincides with that of the charge moving inside medium with the constant $\epsilon = \epsilon_0$. For β slightly less than β_c ($\beta \approx 0.6$) oscillations appear for negative values of z. Their amplitude grows as β

increases. For $\beta \approx \beta_c$ we see a large peak at $z = 0$ and smaller ones in the region $z < 0$. For $\beta > \beta_c$ there is a large maximum at $z = z_c$ and smaller ones in the region $z < z_c$. The period of these oscillations approximately coincides with that of the medium polarization $T_z \approx 2\pi v\beta_c/\omega_0$ (see section 5). Figures 2-7 demonstrate how the EMF is distributed over the surface of the cylinder C_ρ at a fixed moment of time t. As all electromagnetic strengths depend on z and t via $z - vt$, the periodic dependence on time (with the period $2\pi\beta_c/\omega_0$) should be observed at a fixed spatial point.
It is seen that despite the ω dependence of ϵ, the critical velocity $\beta_c = 1/\sqrt{\epsilon_0}$ still has a physical meaning. Indeed, for $\beta > \beta_c$ the magnetic vector potential and field strength are very small outside the Mach cone ($z > z_c$) exhibiting oscillations inside it ($z < z_c$). For $\beta < \beta_c$ the Mach cone disappears. The EMF being relatively small differs from zero everywhere.

The magnetic vector potential presented in Figs. 2-4 can be compared with its non-oscillating behaviour for the the frequency-independent $\epsilon = \epsilon_0$:

$$A_z = \frac{\beta}{[(z - vt)^2 - \rho^2(\beta^2\epsilon_0 - 1)]^{1/2}}\Theta(vt - z - \frac{\rho}{\gamma_n}).$$

We turn again to Eqs. (3.5) and (3.6). The Fourier components of Φ and $\vec{E}$ have a pole at $\omega = \omega_3 = \sqrt{\omega_0^2 + \omega_L^2}$. This leads to the divergence of integrals defining Φ and $\vec{E}$. It would be tempting to approximate these integrals by their principal values. We illustrate this using Φ as an example (see Eq.(3.1)). Consider a closed contour C consisting of three real intervals $((-\infty, -\omega_0 - \delta),\ \ (-\omega_0 + \delta, \omega_0 - \delta),\ \ (\omega_0 + \delta, \infty))$, of two semi-circles C_1 and C_2 of the radius δ with their centers at $z = -\omega_0$ and $z = \omega_0$, resp. and of a semi-circle C_R of the infinite radius. All semi-circles C_1, C_2 and C_R lie in the upper half-plane. The integral

$$\int \frac{d\omega}{\epsilon} e^{i\alpha} K_0(k\rho)$$

taken over the closed contour C equals zero if the function K_0 has no singularities inside C. The same integral taken over C_R is also 0 for $t - z/v > 0$ due to the exponential factor $e^{i\alpha}$. Therefore,

$$\left(\int_{-\infty}^{-\omega_0-\delta} + \int_{-\omega_0+\delta}^{\omega_0-\delta} + \int_{\omega_0+\delta}^{\infty} + \int_{C_1} + \int_{C_2}\right)\frac{d\omega}{\epsilon} e^{i\alpha} K_0(k\rho) = 0.$$

In the limit $\delta \to 0$ one gets

$$V.P.\int_{-\infty}^{\infty} \frac{d\omega}{\epsilon} e^{i\alpha} K_0(k\rho) = -\left(\int_{C_1} + \int_{C_2}\right)\frac{d\omega}{\epsilon} e^{i\alpha} K_0(k\rho) =$$

$$= -2\pi\frac{\omega_L^2}{\omega_3}\Theta(t - z/v)\sin\omega_3(t - z/v) K_0(\rho\frac{|\omega_3|}{v}).$$

Then, for the electric potential one gets

$$\Phi = -2\frac{e}{v}\frac{\omega_L^2}{\omega_3}\Theta(t - z/v)\sin\omega_3(t - z/v) K_0(\rho\frac{|\omega_3|}{v}). \tag{3.7}$$

We see that the principal value of the treated integral does not describe the Cherenkov cone. Probably, this is due to singularities (poles and branch points) of the modified Bessel function in the upper ω half-plane. When evaluating (3.7) we did not take them into account.

The radiation field (described by the integrals in (3.5) and (3.6) containing usual Bessel functions) can be handled by the WKB method. We follow closely Tamm's paper [23] (see also review [24]). For this we change J_ν and N_ν functions by their asymptotic values:

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\nu\pi}{2} - \frac{\pi}{4}), \quad N_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\sin(x - \frac{\nu\pi}{2} - \frac{\pi}{4}).$$

Then,

$$H_\phi = \frac{e}{c}\sqrt{\frac{2}{\pi v\rho}}\int d\omega\sqrt{\omega}(\beta^2\epsilon - 1)^{1/4}\cos(f + \frac{\pi}{4}),$$

$$E_\rho = \frac{e}{v}\sqrt{\frac{2}{\pi v\rho}}\int d\omega\frac{1}{\epsilon}\sqrt{\omega}(\beta^2\epsilon - 1)^{1/4}\cos(f + \frac{\pi}{4}),$$

$$E_z = -\frac{e}{v}\sqrt{\frac{2}{\pi v\rho}}\int d\omega\frac{1}{\epsilon}\sqrt{\omega}(\beta^2\epsilon - 1)^{3/4}\cos(f + \frac{\pi}{4})\,. \tag{3.8}$$

Here $f = \omega(t - z/v) - \sqrt{\beta^2\epsilon - 1}\rho\omega/v$. The argument of cosine is a rapidly oscillating function of ω. The main contribution to the integrals comes from stationary points at which $df/d\omega = 0$. Or, explicitly,

$$(vt - z)\sqrt{\beta^2\epsilon - 1} = \rho[\beta^2 - 1 + \frac{\omega_0^2\omega_L^2}{(\omega^2 - \omega_0^2)^2}]. \tag{3.9}$$

This equation defines ω as a function of ρ, z. Let this ω be $\omega_1(\rho, z)$. Then the WKB method gives

$$H_\phi = -\frac{2e}{c}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{1/4}\sin f_1,$$

$$E_\rho = -\frac{2e}{v\epsilon_1}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{1/4}\sin f_1,$$

$$E_z = \frac{2e}{v\epsilon_1}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{3/4}\sin f_1 \tag{3.10}$$

for $\ddot{f}_1 > 0$ and

$$H_\phi = \frac{2e}{c}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{1/4}\cos f_1,$$

$$E_\rho = \frac{2e}{v\epsilon_1}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{1/4}\cos f_1,$$
$$E_z = -\frac{2e}{v\epsilon_1}\sqrt{\frac{\omega_1}{v\rho|\ddot{f}_1|}}(\beta^2\epsilon_1 - 1)^{3/4}\cos f_1 \qquad (3.11)$$

for $\ddot{f}_1 < 0$. Here

$$f_1 = f(\omega_1), \quad \epsilon_1 = \epsilon(\omega_1), \quad \ddot{f}_1 = \frac{d^2 f}{d\omega^2}|_{\omega=\omega_1}.$$

The electromagnetic strengths are maximal if

$$\omega_1(vt - z) - \rho\omega_1\sqrt{\beta^2\epsilon_1 - 1} = (m + \frac{1}{2})\pi v \qquad (3.12)$$

for $\ddot{f}_1 > 0$ and

$$\omega_1(vt - z) - \rho\omega_1\sqrt{\beta^2\epsilon_1 - 1} = m\pi v \qquad (3.13)$$

for $\ddot{f}_1 < 0$. Here $m = 0, \pm1, \pm2$ etc.The combined solution of (3.9) and (3.12),(3.13) defines the set of trajectories where electromagnetic strengths are maximal. Equations (3.9)-(3.13) were obtained by Tamm [23]. We apply them to the particular $\epsilon(\omega)$ given by Eq.(1.1). The trajectories of field strength maxima for selected v and m are shown in Figs. 8-10. The number of a particular curve means m. We observe that inclination of curves increases as β approaches β_c.

4 The energy flux and the number of photons

We evaluate now the energy flux per unit length through the surface of a cylinder C_ρ (Fig.1) coaxial with the z axis for the total time of motion. It is given by

$$W = 2\pi\rho\int_{-\infty}^{+\infty} S_\rho dt, \quad S_\rho = \frac{c}{4\pi}(\vec{E}\times\vec{H})_\rho = -\frac{c}{4\pi}E_z H_\phi. \qquad (4.1)$$

Substituting E_z and H_ϕ from (3.5) and (3.6) and taking into account that

$$\int_{-\infty}^{\infty} dt \sin\omega t\cos\omega' t = 0, \quad \int_{-\infty}^{\infty} dt\sin\omega t\sin\omega' t = \pi[\delta(\omega-\omega') - \delta(\omega+\omega')],$$
$$\int_{-\infty}^{\infty} dt\cos\omega t\cos\omega' t = \pi[\delta(\omega-\omega') + \delta(\omega+\omega')],$$

we get for energy losses per unit length

$$W = \frac{e^2}{c^2}\int_{\beta^2\epsilon>1}\omega d\omega(1 - \frac{1}{\epsilon\beta^2}). \qquad (4.2)$$

Or, explicitly,

$$W = \frac{e^2}{c^2}\int_{\omega_c}^{\omega_0}\omega d\omega(1 - \frac{1}{\epsilon\beta^2}) = -\frac{e^2\omega_L^2}{2c^2}[1 + \frac{1}{\beta^2}\ln(1-\beta^2)] \qquad (4.3)$$

for $\beta < \beta_c$ and

$$W = \frac{e^2}{c^2}\int_0^{\omega_0}\omega d\omega(1 - \frac{1}{\epsilon\beta^2}) = \frac{e^2}{c^2}[-\frac{\omega_0^2}{2}(\frac{1}{\beta^2} - 1) + \frac{\omega_L^2}{2\beta^2}\ln(1 + \frac{\omega_0^2}{\omega_L^2})] \qquad (4.4)$$

for $\beta > \beta_c$.

We observe that only those terms in (3.5) and (3.6) which contain the usual Bessel functions (J_μ and N_μ) and which correspond to $1 - \beta^2\epsilon < 0$ contribute to the radial energy flux. This permits us to escape troubles with the above-mentioned pole of ϵ^{-1} (at $\omega_3 = \sqrt{\omega_L^2 + \omega_0^2}$) which appears only in terms with modified Bessel functions in the region where $1 - \beta^2\epsilon > 0$.

Another way to escape these troubles is to evaluate EMF for an arbitrary value of the parameter p defining the imaginary part $\epsilon(\omega)$ and then let p go to zero. It was shown in the Appendix that this procedure leads to the same Eqs. (3.5),(3.6),(4.2),(4.3) and (4.4).

Similar expressions were obtained by E. Fermi [10]. The validity of Eq.(4.2) is also confirmed by the results obtained by Sternheimer [12] (whose equations pass into (4.2) in the limit $p \to 0$) and Ginzburg [25].

For $\beta \to 0$ the energy losses W tend to 0, while for $\beta \to 1$ (it is just this limit that was considered by Tamm and Frank [2]) they tend to the finite value $\frac{e^2\omega_L^2}{2c^2}\ln(1 + \frac{\omega_0^2}{\omega_L^2})$.

In Fig. 11, we present the dimensionless quantity $F = W/(e^2\omega_0^2/c^2)$ as a function of the particle velocity β. The numbers at curves mean β_c. Vertical lines with arrows divide a curve by two parts corresponding to the energy losses with velocities $\beta < \beta_c$ and $\beta > \beta_c$ and lying to the left and right of vertical lines, resp. We see that the charge uniformly moving in medium radiates at every velocity.

The dimensionless spectral distributions $f(\omega) = w(\omega)/(e^2\omega_0/c^2)$ of the energy loss $W = \int_0^\infty w(\omega)d\omega$ are shown in Fig. 12. The numbers of particular curves mean β. It is seen that for $\beta > \beta_c$ all ω from the interval $0 < \omega < \omega_0$ contribute to the energy losses. For $\beta < \beta_c$ the interval of permissible ω diminishes : $\omega_c < \omega < \omega_0$.

The total number of photons emitted per unit length is given by

$$N = \frac{e^2}{\hbar c^2}\int_{\omega_c}^{\omega_0} d\omega(1 - \frac{1}{\epsilon\beta^2}) = \frac{e^2}{\hbar c^2}[\frac{\omega_c - \omega_0}{\beta^2\gamma^2} + \frac{\omega_L^2}{2\beta^2\omega_3}\ln(\frac{\omega_3+\omega_0}{\omega_3-\omega_0}\frac{\omega_3-\omega_c}{\omega_3+\omega_c})],$$

for $\beta < \beta_c$ and

$$N = \frac{e^2}{\hbar c^2}\int_0^{\omega_0} d\omega(1 - \frac{1}{\epsilon\beta^2}) = \frac{e^2}{\hbar c^2}[-\frac{\omega_0}{\beta^2\gamma^2} + \frac{\omega_L^2}{2\beta^2\omega_3}\ln(\frac{\omega_3+\omega_0}{\omega_3-\omega_0})]$$

for $\beta > \beta_c$. It is seen that N grows from 0 for $\beta = 0$ up to

$$N = \frac{e^2}{\hbar c^2}\frac{\omega_L^2}{2\beta^2\omega_3}\ln(\frac{\omega_3+\omega_0}{\omega_3-\omega_0})$$

for $\beta = 1$. In Fig. 13, we present the dimensionless quantity $N/(e^2\omega_0/\hbar c^2)$ as a function of the particle velocity β. The numbers of curves mean β_c. The vertical lines with arrows divide curve into two parts corresponding to the photon numbers emitted by the charge with velocities $\beta < \beta_c$ and $\beta > \beta_c$ and lying to the left and right of vertical lines, resp. We see that an uniformly moving charge emits photons at every velocity.

The spectral distribution $n(\omega)$ of the photon number emitted per unit of length defined as $N = \int_0^\infty n(\omega)d\omega$ is given by

$$n(\omega) = \frac{e^2}{\hbar c^2}(1 - \frac{1}{\epsilon\beta^2}).$$

For $\beta < \beta_c$, $n(\omega)$ changes from 0 at $\omega = \omega_c$ up to $n(\omega) = e^2/\hbar c^2$ at $\omega = \omega_0$. For $\beta > \beta_c$, $n(\omega)$ changes from $\frac{e^2}{\hbar c^2}(1 - \frac{1}{\epsilon_0\beta^2})$ at $\omega = \omega_c$ up to $e^2/\hbar c^2$ at $\omega = \omega_0$.

The dimensionless spectral distributions $n(\omega)/(e^2/\hbar c^2)$ of the photon number are shown in Fig. 14. The numbers of particular curve mean β. It is seen that for $\beta > \beta_c$ all ω from the interval $0 < \omega < \omega_0$ contribute to the number of emitted photons. For $\beta < \beta_c$ the interval of permissible ω diminishes : $\omega_c < \omega < \omega_0$, i.e., only high-energy photons contribute.

The distributions of the radial energy flux S_ρ as a function of z on the surface of the cylinder C_ρ of the radius ρ (Fig. 1) are shown in Figs. 15-17. It is seen that despite the ω dependence of ϵ the critical velocity $\beta_c = 1/\sqrt{\epsilon_0}$ still has a physical meaning. Indeed, for $\beta > \beta_c$ the electromagnetic energy flux is very small outside the Mach cone exhibiting oscillations inside it. For $\beta < \beta_c$ the radial flux diminishes and becomes negligible for $\beta \le 0.4$.

For the frequency-independent $\epsilon = \epsilon_0$ the energy flux is confined to the surface of the Mach cone. Electromagnetic strengths inside the Mach cone fall as r^{-2} at large distances and, therefore, do not conribute to the radial flux.

5 Digression on the polarization

Another, more physical way to obtain EMF of a charge uniformly moving in medium is to start with the Maxwell equations

$$div\vec{D} = 4\pi\rho, \quad div\vec{B} = 0, \quad curl\vec{E} = -\frac{1}{c}\dot{\vec{B}}, \quad curl\vec{H} = \frac{1}{c}\dot{\vec{D}} + \frac{4\pi}{c}\vec{j} \quad (5.1).$$

As the medium is non-magnetic, $\vec{B} = \vec{H}$. The second and third Maxwell equations are satisfied if we put

$$\vec{H} = \vec{\nabla}\times\vec{A}, \quad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\dot{\vec{A}}\ .$$

We rewrite Maxwell equations in the ω representation:

$$H_\phi^\omega = -\frac{\partial}{\partial\rho}A_z^\omega, \quad E_z^\omega = \frac{i\omega}{v}(\Phi^\omega - \beta A_z^\omega),$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho(E_\rho^\omega + 4\pi P_\rho^\omega) - \frac{i\omega}{v}(E_z^\omega + 4\pi P_z^\omega) = 4\pi\rho^\omega,$$

$$H_\pi^\omega = \beta(E_\rho^\omega + 4\pi P_\rho^\omega), \quad \frac{i\omega}{v}E_\rho^\omega + \frac{\partial E_z^\omega}{\partial\rho} = \frac{i\omega}{c}H_\phi^\omega. \quad (5.2)$$

The last equation is satisfied trivially if we express electromagnetic strengths through the electromagnetic potentials:

$$E_\rho^\omega = -\frac{\partial\Phi^\omega}{\partial\rho}, \quad E_z^\omega = \frac{i\omega}{v}\Phi^\omega - \frac{i\omega}{c}A_z^\omega, \quad H_\phi^\omega = -\frac{\partial A_z^\omega}{\partial\rho}.$$

In deriving these equations we have taken into account that the z and t dependence of all Fourier components of electromagnetic potentials, field strengths, polarization, charge and current densities is given by the factor $\exp[i\omega(t - z/v)]$.

The electric field $\vec{E}$ of a moving charge induces the polarization $\vec{P}(\vec{r}, t)$ which being added with $\vec{E}$ gives electric induction $\vec{D} = \vec{E} + 4\pi\vec{P}$. Usually, it is believed (see, e.g.,[21,22]) that the ω components of $\vec{P}$ and $\vec{E}$

$$\vec{P}_\omega = \int e^{-i\omega t}\vec{P}(\vec{r}, t)dt, \quad \vec{E}_\omega = \int e^{-i\omega t}\vec{E}(\vec{r}, t)dt$$

are related by the formula

$$4\pi\vec{P}_\omega = \frac{\omega_L^2}{\omega_0^2 - \omega^2 + ip\omega}\vec{E}_\omega. \quad (5.3)$$

Using this fact and expressing electromagnetic strengths in Eq.(5.2) through the potentials we get (remember that the last equation (5.2) is satisfied trivially):

$$\Delta_2\Phi^\omega - \frac{\omega^2}{v^2}\Phi^\omega + \frac{i\omega}{c}div\vec{A}^\omega = -\frac{1}{\epsilon}4\pi\rho^\omega$$

$$\Delta_2 A_z^\omega + \frac{\omega^2}{c^2}\epsilon A_z^\omega - \frac{\omega_2}{cv}\epsilon\Phi^\omega = -\frac{4\pi}{c}j_z^\omega$$

$$\frac{\partial A_z^\omega}{\partial\rho} = \beta\epsilon\frac{\partial\Phi^\omega}{\partial\rho}. \quad (5.4)$$

Here

$$\rho^\omega = \frac{e}{v}\delta(x)\delta(y)\exp(-i\omega z/v), \quad j_z^\omega = e\delta(x)\delta(y)\exp(-i\omega z/v), \quad \Delta_2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}).$$

The last equation (5.4) is satisfied if we choose

$$A_z^\omega = \beta\epsilon(\omega)\Phi^\omega \quad (5.5)$$

while two others coincide after this substitution. The solutions of these equations are

$$\Phi^\omega = \frac{2e}{v\epsilon}K_0\left(\frac{\rho|\omega|}{v}\sqrt{1-\beta^2\epsilon}\right), \quad A_z^\omega = \frac{2e}{c}K_0\left(\frac{\rho|\omega|}{v}\sqrt{1-\beta^2\epsilon}\right).$$

In the $(\vec{r},t)$ space they are given by Eqs. (3.5) and (3.6).

Now we rewrite Eq.(5.3) in the $(\vec{r},t)$ representation:

$$\vec{P}(t) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} G(t-t')\vec{E}(t'),$$

where

$$G(t-t') = \omega_L^2 \int_{-\infty}^{+\infty} \frac{d\omega}{\omega_0^2-\omega^2+ip\omega} e^{i\omega(t-t')}. \tag{5.6}$$

Taking into account the positivity of p one gets:

a) for $p<\omega_0$:

$G(t-t')=0$ for $t'>t$ and

$G(t-t') = \frac{2\pi\omega_L^2}{\sqrt{\omega_0^2-p^2/4}}\exp\left[-p(t-t')/2\right]\sin[\sqrt{\omega_0^2-p^2/4}(t-t')]$ for $t'<t$.

b) for $p>\omega_0$:

$G(t-t')=0$ for $t'>t$ and

$G(t-t') = \frac{2\pi\omega_L^2}{\sqrt{\omega_0^2-p^2/4}}\exp\left[-p(t-t')/2\right]\sinh[\sqrt{p^2/4-\omega_0^2}(t-t')]$ for $t'<t$.

As a result of positivity of p, the value of polarization $\vec{P}$ at the moment t is defined by the values of the electric field $\vec{E}$ in preceeding times (causality principle). The source of polarization is distributed along the z axis:

$$div\vec{P} = \frac{e}{v}\delta(x)\delta(y)\frac{\omega_L^2}{\sqrt{\omega_0^2+\omega_L^2-p^2/4}}\exp\left[-p(t-z/v)/2\right]\sin[\sqrt{\omega_0^2+\omega_L^2-p^2/4}(t-z/v)]$$

for $z<vt$ and $div\vec{P}=0$ for $z>vt$ (this equation is related to the $\omega_0^2+\omega_L^2-p^2/4>0$ case).

Now the origin of oscillations of the potentials and field strengths behind the Mach cone becomes understandable. A moving charge gives rise to a time-dependent polarization which, in the absence of damping, oscillates with the frequency $\sqrt{\omega_0^2+\omega_L^2}$. The oscillations of polarization being added lead to the appearance of the smoothed Mach cones enclosed in each other. On the surface of the cylindrical surface C_ρ they are manifested as maxima of the potentials, field strengths, and intensities. The position of the first maximum approximately coincides with the position of the singular Mach cone in the absence of dispersion. The latter case is obtained if we neglect the ω dependence in the denominator of the integral in (5.5):

$$G(t-t') = 2\pi\frac{\omega_L^2}{\omega_0^2}\delta(t-t').$$

. Obviously, this can be realized for large values of ω_0. The introduction of damping should lead to decreasing of secondary maxima. To verify this, we evaluated the magnetic vector potential for different values of the parameter p defining the imaginary part of $\epsilon(\omega)$ (see section 3). We see (Fig. 18) that for $p\geq 1$ the secondary oscillations disappear.

Although the polarization formalism leads to the same expressions (3.5),(3.6) for the electromagnetic potentials and field strengths, it presents another, more physical, point of view on the nature of the Vavilov-Cherenkov radiation.

6 Another choice of polarization

So far we have dealt with the gauge condition of the form $A_z^\omega = \beta\epsilon(\omega)\Phi^\omega$. It looks highly non-local in the $(\vec{r},t)$ representation. There is another interesting possibility. We substitute

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{H} = \vec{\nabla}\times\vec{A}$$

into the first and fourth Maxwell equations (5.1) and obtain

$$\Delta\Phi + \frac{1}{c}div\dot{\vec{A}} = -4\pi\rho + 4\pi div\vec{P},$$

$$\Delta\vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = \vec{\nabla}(div\vec{A} + \frac{1}{c}\dot{\Phi}) - \frac{4\pi}{c}(\vec{j}+\dot{\vec{P}}).$$

We try to separate equations for Φ and $\vec{A}$ by imposing on them the Lorentz condition

$$div\vec{A} + \frac{1}{c}\dot{\Phi} = 0\,. \tag{6.1}$$

This equation is satisfied automatically if we put

$$A_x = A_y = 0, \quad A_z = \beta\Phi \tag{6.2}$$

and take into account that for the treated problem all the electromagnetic quantities depend on z and t through the combination $z-vt$. Thus, we obtain

$$\Delta\Phi - \frac{1}{c^2}\ddot{\Phi} = -4\pi\rho + 4\pi div\vec{P},$$

$$\Delta\vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = -4\frac{\pi}{c}\vec{j} - 4\frac{\pi}{c}\dot{\vec{P}}.$$

It follows from this that only the z component of $\vec{P}$ differs from zero in the chosen gauge (as only the z components of $\vec{A}$ and $\vec{j}$ differ from zero). We rewrite these equations in the ω representation

$$\Delta_2\Phi^\omega + \omega^2\left(\frac{1}{c^2}-\frac{1}{v^2}\right)\Phi^\omega = -4\pi\rho^\omega - 4\pi\frac{i\omega}{v}P^\omega,$$

$$\Delta_2 A_z^\omega + \omega^2(\frac{1}{c^2} - \frac{1}{v^2})A_z^\omega = -\frac{4\pi}{c}j_z^\omega - 4\pi\frac{i\omega}{c}P^\omega. \qquad (6.3)$$

As the treated medium is non-magnetic, it is natural to require the coincidence of equations (5.4) and (6.3) for vector potentials satisfying different gauge conditions. This takes place if P^ω is chosen to be proportional to A_z^ω:

$$P_\omega = -\frac{i\omega}{4\pi c}(\epsilon - 1)A_z^\omega. \qquad (6.4)$$

Then, one gets

$$\Delta_2\Phi^\omega + \omega^2(\frac{\epsilon}{c^2} - \frac{1}{v^2})\Phi^\omega = -4\pi\rho^\omega,$$

$$\Delta_2 A_z^\omega + \omega^2(\frac{\epsilon}{c^2} - \frac{1}{v^2})A_z^\omega = -\frac{4\pi}{c}j_z^\omega.$$

The solutions of these equations are

$$\Phi^\omega = \frac{2e}{v}K_0(\frac{\rho|\omega|}{v}\sqrt{1-\beta^2\epsilon}), \quad A_z^\omega = \frac{2e}{c}K_0(\frac{\rho|\omega|}{v}\sqrt{1-\beta^2\epsilon}),$$

$$H_\phi^\omega = \frac{2e|\omega|}{cv}\sqrt{1-\beta^2\epsilon}K_1, \quad E_\rho^\omega = D_\rho^\omega = H_\phi^\omega/\beta,$$

$$E_z = \frac{2ie\omega}{v^2}(1-\beta^2)K_0, \quad D_z = \frac{2ie\omega}{v^2}(1-\beta^2\epsilon)K_0,$$

where all K functions depend on the argument $\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon}$ in which the value of $\sqrt{1-\beta^2\epsilon}$ corresponding to its positive real part should be taken. Obviously, there is no proportionality between $\vec{D}$ and $\vec{E}$ for the chosen polarization. In the $(\vec{r}, t)$ representations the magnetic vector potential and field strength are the same as Eqs.(3.5) and (3.6), while for Φ, E_z and E_ρ one gets

$$\Phi(\vec{r}, t) = \frac{2e}{\pi v}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})d\omega \cos\alpha K_0 + \frac{e}{v}\int_{\omega_c}^{\omega_0} d\omega(\sin\alpha J_0 - \cos\alpha N_0),$$

$$E_z = \frac{2e}{\pi c^2}(1 - \frac{1}{\beta^2})[(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})\omega d\omega \sin\alpha K_0 - \frac{e}{c^2}\int_{\omega_c}^{\omega_0}\omega d\omega(N_0\sin\alpha + J_0\cos\alpha)],$$

$$E_\rho = \frac{2e}{\pi v^2}(\int_0^{\omega_c} + \int_{\omega_0}^{\infty})d\omega\omega\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{v^2}\int_{\omega_c}^{\omega_0} d\omega\omega\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1).$$

for $\beta < \beta_c$ and

$$\Phi(\vec{r}, t) = \frac{2e}{\pi v}\int_{\omega_0}^{\infty} d\omega\cos\alpha K_0 + \frac{e}{v}\int_0^{\omega_0} d\omega(\sin\alpha J_0 - \cos\alpha N_0),$$

$$E_z = \frac{2e}{\pi c^2}(1 - \frac{1}{\beta^2})[\int_{\omega_0}^{\infty}\omega d\omega\sin\alpha K_0 - \frac{e}{c^2}\int_0^{\omega_0}\omega d\omega(N_0\sin\alpha + J_0\cos\alpha)],$$

$$E_\rho = \frac{2e}{\pi v^2}\int_{\omega_0}^{\infty} d\omega\omega\sqrt{|1-\beta^2\epsilon|}\cos\alpha K_1 + \frac{e}{v^2}\int_0^{\omega_0} d\omega\omega\sqrt{|1-\beta^2\epsilon|}(\sin\alpha J_1 - \cos\alpha N_1).$$

for $\beta > \beta_c$.

These expressions satisfy the Maxwell equations but with the polarization different from the one used earlier. We observe that electric induction $\vec{D}$ is the same as earlier, but the electric strength differs. As the integrands defining Φ and $\vec{E}$ are finite for any value of ω, the corresponding integrals are convergent and can be evaluated numerically. We observe that for $\beta \to 1$, $E_z \to 0$. This means that for this choice of polarization the energy flux in the transverse direction disappears, that is, all the energy is radiated in the direction of charge motion.

It is surprising that the choice of Lorentz condition (6.1) almost inevitably leads to the solution with vanishing ρ component of polarization. But the physics cannot depend on the gauge choice. A probable resolution of this controversy is that polarization $\vec{P}$ is not observable. But electric strengths $\vec{E}$ defining the Poynting vector are also different in both the gauges. So, the question remains to be answered.

7 On the Kronig-Kramers dispersion relations

Up to now we considered the case when the imaginary part of the dielectic penetrability was chosen to be zero.Can this be reconciled with the Kramers-Kronig dispersion relations? Using the fact that for the chosen form of the Fourier integrals, the poles of $\epsilon(\omega)$ lie in the upper ω half-plane, one has (see, e.g.,[22]):

$$\int_{-\infty}^{+\infty}\frac{\epsilon(x) - 1}{\omega - x}d\omega + i\pi[\epsilon(x) - 1] = 0.$$

Or, separating real and imaginary parts:

$$\int_{-\infty}^{\infty}\frac{\epsilon_r - 1}{\omega - x}d\omega = \pi\epsilon_i(x), \quad \int_{-\infty}^{\infty}\frac{\epsilon_i}{\omega - x}d\omega = -\pi[\epsilon_r(x) - 1] \qquad (7.1)$$

(by the integrals, we mean their principal values which obtained by closing the integration contour in the lower ω half-plane).

Here ϵ_r and ϵ_i are the real and imaginary parts of ω:

$$\epsilon_r = 1 + \frac{\omega_L^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + p^2\omega^2}, \quad \epsilon_i = -\frac{p\omega\omega_L^2}{(\omega_0^2 - \omega^2)^2 + p^2\omega^2}. \qquad (7.2)$$

At first glance it seems that relations (7.1) cannot be fulfilled. Take, e.g., the second of them. For $\epsilon_i = 0$ its left-hand side disappears, which is not valid for its right-hand side.

Consider the integral entering into its left-hand side

$$\int_{-\infty}^{\infty} \frac{\epsilon_i}{\omega - x} d\omega = -p\omega_L^2 \int \frac{\omega d\omega}{\omega - x} \frac{1}{(\omega_0^2 - \omega^2)^2 + p^2\omega^2}. \tag{7.3}$$

A detailed consideration shows that the integral in the right-hand side of this equation is equal to

$$-\frac{\pi}{p} \frac{x^2 - \omega_0^2}{(x^2 - \omega_0^2)^2 + p^2 x^2}. \tag{7.4}$$

The factor p of the integral in (7.3) cancels out with the factor $1/p$ in (7.4). Thus,

$$\int_{-\infty}^{\infty} \frac{\epsilon_i}{\omega - x} d\omega = \pi\omega_L^2 \frac{x^2 - \omega_0^2}{(x^2 - \omega_0^2)^2 + p^2 x^2},$$

that exactly coincides with the the right-hand side of the second relation (7.1). The same resoning proofs the validity of the first relation (7.1). Thus, Kramers-Kronig relations are valid for any small $p > 0$. The positivity of p defines how the integration contour should be closed, which in turn leads to the fulfillment of the causality condition.

8 Conclusion.

We briefly summarize the main resuls obtained:

1. The space-time distributions of the EMF produced by a uniformly moving charge in medium with frequency-dependent dielectric permittivity are studied. The oscillating EMF arises inside the Cherenkov cone. It is associated with the time-dependent polarization induced by the moving charge.
2. It is proved that a charge uniformly moving in medium with frequency-dependent dielectric permittivity radiates at each velocity. The spectral characteristics of this radiation are given. It turns out that for a small charge velocity the main contribution comes from high frequencies.
3. It is shown that there are possible different definitions of polarization which are due to different choices of the gauge condition imposed on the electromagnetic potentials. However, their physical consequences are different.

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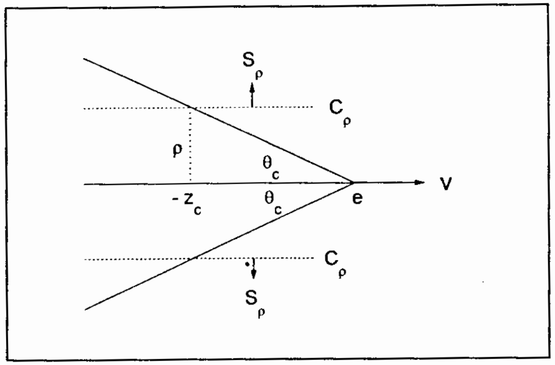

Figure 1: Schematic presentation of the Cherenkov cone for a constant electric permittivity. The radiation field is confined to the surface of the cone, the field inside the cone does not contribute to the radiation. On the surface of the cylinder C_ρ the electromagnetic field is zero for $z > -z_c$; S_ρ means the radial energy flux through the cylinder surface.

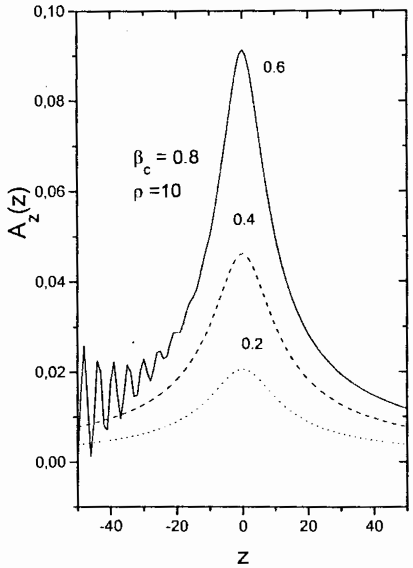

Figure 2: The distribution of the magnetic vector potential on the surface of cylinder C_ρ. The number of a particular curve means $\beta = v/c$; z and A_z are in units c/ω_0 and $e\omega_0/c$, resp.

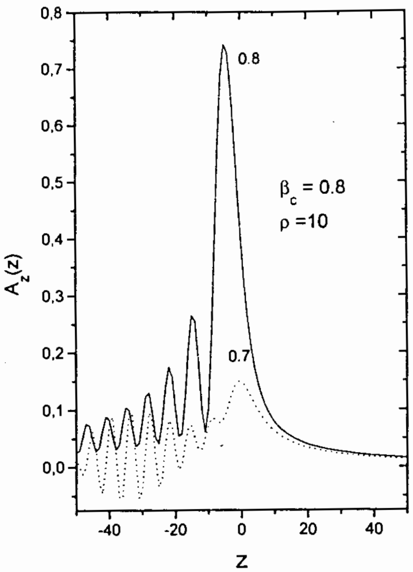

Figure 3: The same as in Fig.2, but for different values of β.

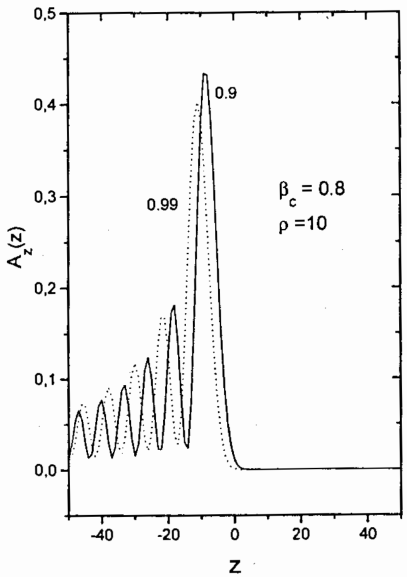

Figure 4: The same as in Fig.2, but for different values of β.

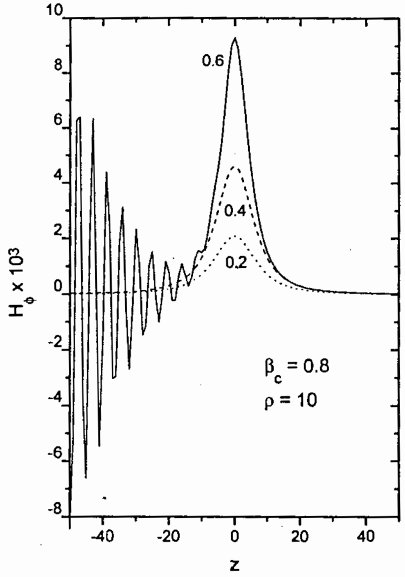

Figure 5: The distribution of the magnetic field strength on the surface of cylinder C_ρ. The number of a particular curve means $\beta = v/c$; z and H_ϕ are in units c/ω_0 and $e\omega_0^2/c^2$, resp.

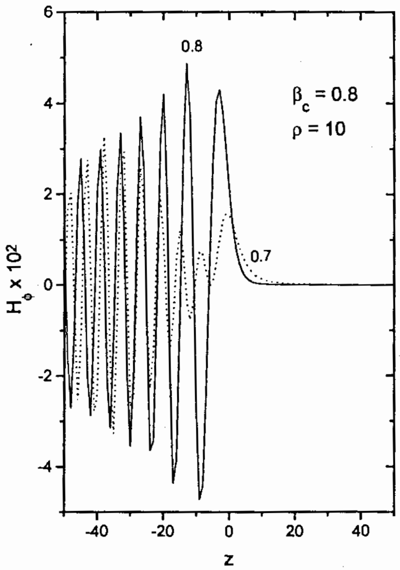

Figure 6: The same as in Fig.5, but for different values of β.

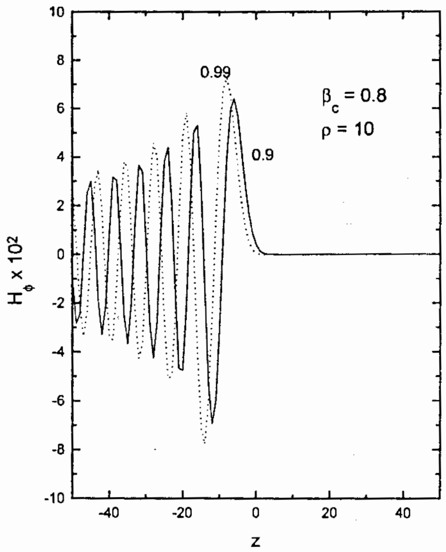

Figure 7: The same as in Fig.5, but for different values of β.

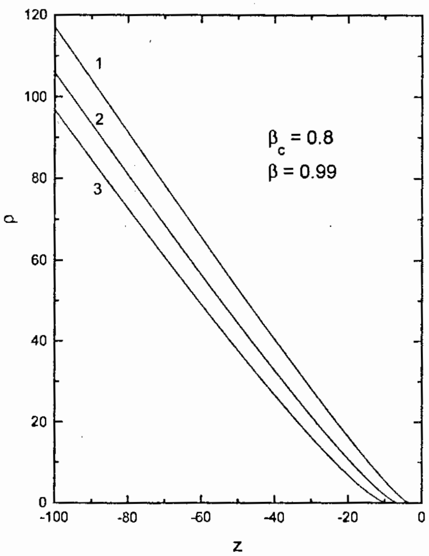

Figure 8: The positions of field strength maxima for $\beta = 0.99$. The number of a particular curve means the number m defining a particular trajectory (see the text); z and ρ are in units c/ω_0.

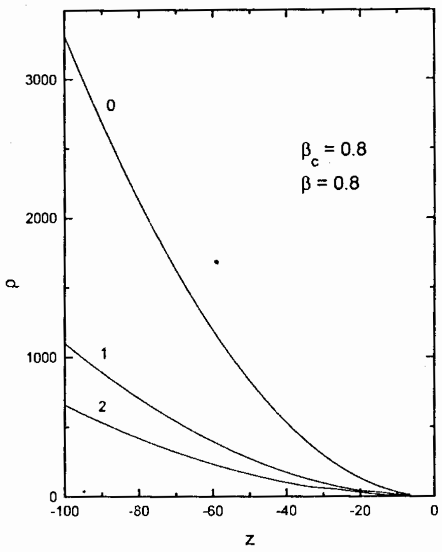

Figure 9: The same as in Fig.8, but for $\beta = 0.8$. It is seen that inclination of trajectories increases compared to the previous figure.

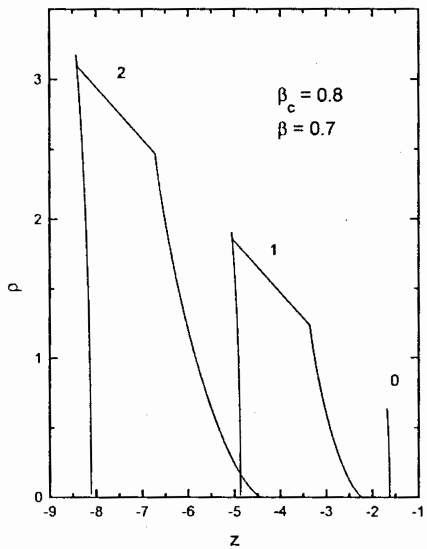

Figure 10: The same as in Fig.8, but for $\beta = 0.7$. It is seen that the trajectories are grouped near the z axis. The termination of the $m = 0$ trajectory is due to the vanishing of $|\tilde{f}_1|$. At this point the WKB approxoimation breaks.

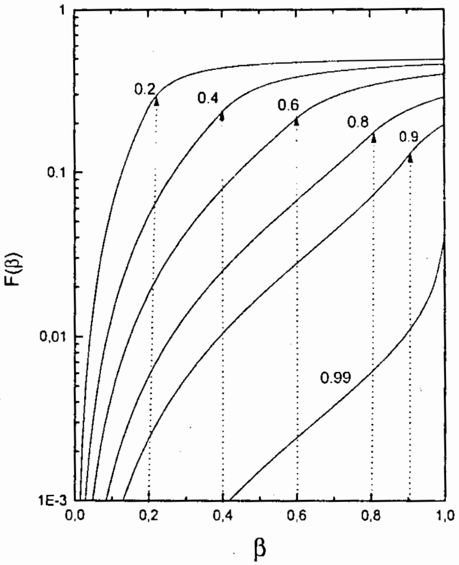

Figure 11: The radial energy losses per unit length (in units $e^2\omega_0^2/c^2$) as a function of $\beta = v/c$. The number of a particular curve means the critical velocity β_c.

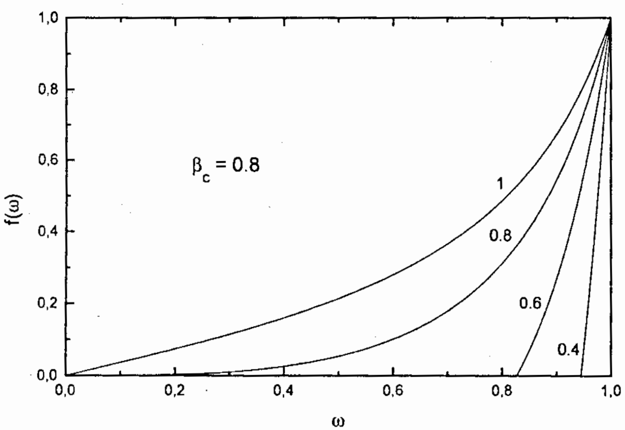

Figure 12: Spectral distribution of the energy losses (in units $e^2\omega_0/c^2$); ω is in units ω_0. The number of a particular curve means $\beta = v/c$.

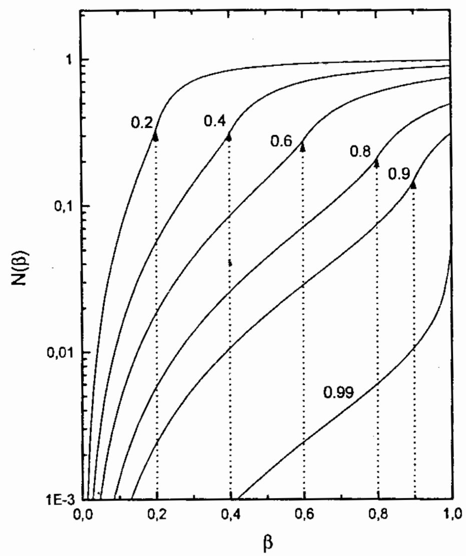

Figure 13: The number of emitted quanta in the radial direction per unit length (in units $e^2\omega_0/\hbar c^2$) as a function of $\beta = v/c$. The number of a particular curve means the critical velocity β_c.

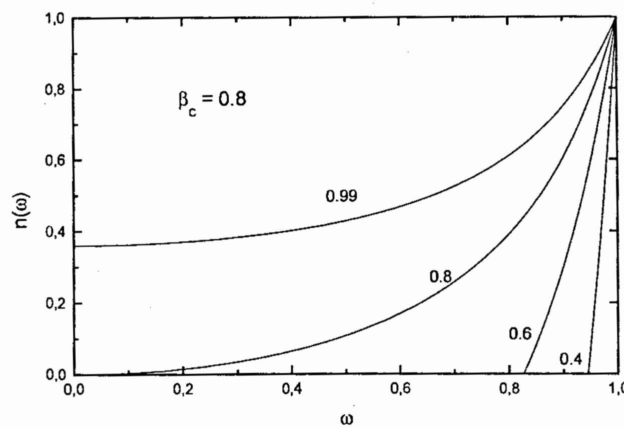

Figure 14: Spectral distribution of the emitted quanta (in units $e^2/\hbar c^2$); ω is in units ω_0. The number of a particular curve means $\beta = v/c$.

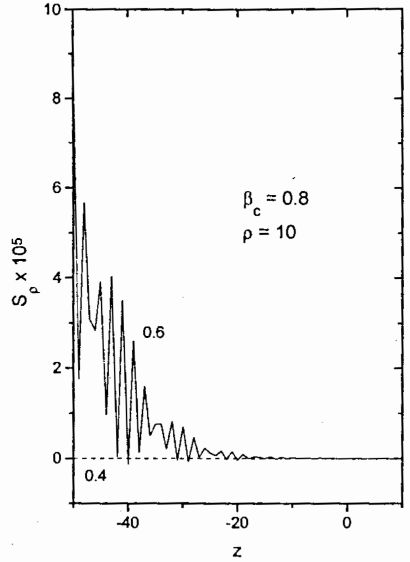

Figure 15: The distribution of the radial energy flux (in units $c^2\omega_0^3/c^2$) on the surface of cylinder C_ρ. z is in units c/ω_0. The number of a particular curve means $\beta = v/c$. For $\beta = 0.4$ the radial flux is negligible.

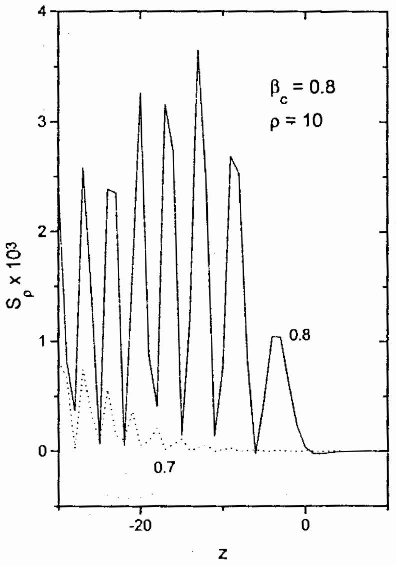

Figure 16: The same as in Fig.15, but for different values of β.

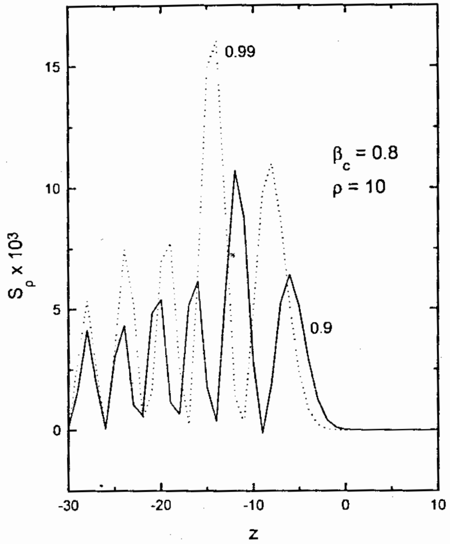

Figure 17: The same as in Fig.15, but for different values of β.

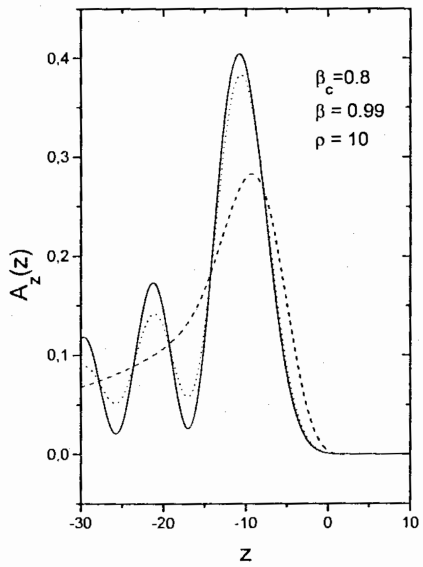

Figure 18: Shows how switching on the imaginary part p of dielectric permittivity affects the magnetic vector potential; z and A_z are in units c/ω_0 and $e\omega_0/c$, resp. The solid, point-like and short-dashed curves refer to $p = 0$, $p = 0.1$ and $p = 1$, resp. It is seen that secondary maxima are damped for $p = 1$ much stronger than the main one.

Appendix.

We write out electromagnetic potentials and field strengths for the finite value of a parameter p defining the imaginary part of ϵ. Since $\epsilon(-\omega) = \epsilon^*(\omega)$, the EMF can be written in a manifestly real form :

$$\Phi = \frac{2e}{\pi v}\int_0^\infty[(\epsilon_r^{-1}\cos\alpha - \epsilon_i^{-1}\sin\alpha)K_{0r} - (\epsilon_i^{-1}\cos\alpha + \epsilon_r^{-1}\sin\alpha)K_{0i}]d\omega,$$

$$A_z = \frac{2e}{\pi c}\int_0^\infty d\omega(\cos\alpha K_{0r} - \sin\alpha K_{0i}),$$

$$H_\phi = \frac{2e}{\pi vc}\int_0^\infty \omega d\omega(a^2+b^2)^{1/4}[\cos(\frac{\phi}{2}+\alpha)K_{1r} - \sin(\frac{\phi}{2}+\alpha)K_{1i}],$$

$$E_z = -\frac{2}{\pi v^2}\int_0^\infty \omega d\omega\{[\cos\alpha(\epsilon_r^{-1}-\beta^2) - \sin\alpha\epsilon_i^{-1}]K_{0i} + [\sin\alpha(\epsilon_r^{-1}-\beta^2) + \cos\alpha\epsilon_i^{-1}]K_{0r}\},$$

$$E_\rho = \frac{2}{\pi v^2}\int_0^\infty \omega d\omega(a^2+b^2)^{1/4}[(\epsilon_r^{-1}\cos\alpha - \epsilon_i^{-1}\sin\alpha)(\cos\frac{\phi}{2}K_{1r} - \sin\frac{\phi}{2}K_{1i}) -$$

$$-(\epsilon_i^{-1}\cos\alpha + \epsilon_r^{-1}\sin\alpha)(\sin\frac{\phi}{2}K_{1r} + \cos\frac{\phi}{2}K_{1i})]. \qquad (A.1)$$

Here we put

$$K_{0r} = ReK_0(\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon}), \quad K_{0i} = ImK_0(\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon}),$$

$$K_{1r} = ReK_1(\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon}), \quad K_{1i} = ImK_1(\frac{\rho\omega}{v}\sqrt{1-\beta^2\epsilon}),$$

ϵ_r and ϵ_i are the real and imaginary parts of $\epsilon(\omega)$ (see Eqs. (7.2)); $\epsilon_r^{-1} = \epsilon_r/(\epsilon_r^2+\epsilon_i^2)$, $\epsilon_i^{-1} = -\epsilon_i/(\epsilon_r^2+\epsilon_i^2)$; $\alpha = \omega(t - z/v)$; a, b and ϕ are defined by Eqs. (3.2) and (3.3). The energy flux per unit length through the surface of a cylinder of the radius ρ coaxial with the z axis for the whole time of charge motion is defined by Eq.(4.1). Substituting E_z and H_ϕ given by (A.1) into it one gets:

$$W = \int_0^\infty f(\omega)d\omega,$$

where

$$f(\omega) = -\frac{2e^2\rho}{\pi v^3}\omega^2(a^2+b^2)^{1/4}\{(K_{0r}K_{1r} + K_{0i}K_{1i})[(\epsilon_r^{-1}-\beta^2)\sin\frac{\phi}{2} - \epsilon_i^{-1}\cos\frac{\phi}{2}] -$$

$$-(K_{0i}K_{1r} - K_{0r}K_{1i})[(\epsilon_r^{-1}-\beta^2)\cos\frac{\phi}{2} + \epsilon_i^{-1}\sin\frac{\phi}{2}]\}. \qquad (A.2)$$

Consider now the limit $p \to 0$.
Let $1 - \beta^2\epsilon > 0$ in this limit, then (see section 3):

$$\sin\frac{\phi}{2} \to 0,\ \cos\frac{\phi}{2} \to 1,\ \epsilon_i \to 0,\ \epsilon_i^{-1} \to 0,\ K_{0i} \to 0,\ K_{1i} \to 0$$

and, therefore, $f(\omega) \to 0$ while electromagnetic potentials and field strengths coincide with those terms in (3.5) and (3.6) which contain modified Bessel functions.
On the other hand, if $1 - \beta^2\epsilon < 0$, then:

$$\sin\frac{\phi}{2} \to 1 \text{ (for } p > 0),\ \cos\frac{\phi}{2} \to 0,\ \epsilon_i \to 0,\ \epsilon_i^{-1} \to 0,$$

$$K_{0r} \to -\frac{\pi}{2}N_0,\ K_{0i} \to -\frac{\pi}{2}J_0,\ K_{1r} \to -\frac{\pi}{2}J_1,\ K_{1i} \to \frac{\pi}{2}N_1,$$

where the argument of the Bessel functions is $\rho\frac{|\omega|}{v}\sqrt{|1-\beta^2\epsilon|}$. Substituting this into (A.2) and using the relation

$$J_\nu(x)N_{\nu+1}(x) - N_\nu(x)J_{\nu+1}(x) = -\frac{2}{\pi x}$$

one arrives at

$$f(\omega) = \frac{e^2\omega}{c^2}(1 - \frac{1}{\epsilon\beta^2}).$$

This in turn leads to W exactly coinciding with (4.2),(4.3) and (4.4). Electromagnetic potentials and field strengths (A.1) coincide with the terms in (3.5) and (3.6) containing usual Bessel functions.

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