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ON ANALYTICAL SOLUTIONS
TO THE PROBLEM OF THE COULOMB AND
CONFINING POTENTIALS

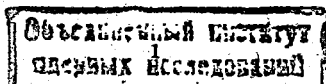
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1 Introduction

The combination of the Coulomb potential with different type of confining potentials is a typical problem in quantum mechanics for many applications. The two-dimensional electron gas with the Coulomb interaction confined in a parabolic potential is a model of a quantum dot (see for example [1, 2] and references therein), i.e. an artificial atom the properties of which can be controlled by a man (for review on quantum dots see [3, 4]). The Coulomb and the Yukawa potentials, called the exponentially screened Coulomb potential (ESCP), describe the effective two-particle interaction for charged particles in an ionized gas or in a metal (see [5] and references therein). The model of quark confinement of charmed quarks is successfully simulated by a simple superposition of the linear and Coulomb term [6].

Whereas many approximate analytical and numerical methods have been developed for the last years for the analysis of such a superposition of potentials [7], we would like to suggest a method which is called the oscillator representation method (ORM) [8]: The ORM, based on the ideas and methods of quantum field theory, has been proposed to calculate the binding energy of different systems with fairly arbitrary potentials described by the Schrödinger equation (SE). A spectrum of the one-dimensional power-law potential



or of the quartic oscillator, which are test benches for approximate methods (see for example [9, 10]), can be determined analytically in the ORM with accuracy very closed to the best, available nowadays, computational methods (see for the comparison [8]). One of our goals is to present the mathematical formalism of the ORM in detail making it available for further applications, in particular, for calculations of ground and excited states of the systems with the Coulomb interaction.

In situation, when different terms of the effective potential of a system can compete, and there is no basis for the perturbative approach to be applied, a solution of the SE becomes a nontrivial problem. However, the lowest levels of such a system, especially ones closest to the ground state, can be described with a good approximation by the oscillator states defined in the expanded hyperspace which includes the original one. For the first time, Fock demonstrated this result for a nonrelativistic hydrogen atom by making a stereographic projection of the momentum space onto a four-dimensional unit sphere [11]. For an arbitrary smooth potential, admitting the existence of system bound states, its eigenfunctions have exponential asymptotic behavior at large distances [12, 13]. Therefore, the variables in the original SE must be changed so that the modified equation should have solutions with the oscillator behavior at large distances. Schrödinger pointed out at the existence of such a transformation which turned the three-dimensional Coulomb system into the oscillator one in the four-dimensional space [14]. The explicit form of this transformation has been found in [15] and used to solve the classical Kepler problem. Since this transformation is not a canonical one, after the transformation we have a new system with another set of quantum numbers and wave functions which contains, however, a subset of the original wave functions. The transformation of variables, leading to the Gaussian asymptotic behavior in the expanded space, is one of the basic elements of the ORM.

As the next step, it is necessary to represent the canonical variables (coordinate and momentum) of the Hamiltonian through the creation and annihilation operators a^+ and a . The pure oscillator part with some, yet unknown, frequency ω is extracted from the Hamiltonian, i.e. $H \rightarrow H_0 + H_I = \omega a^+ a + \text{higher order terms}$. The remaining part, i.e. the interaction Hamiltonian H_I , is represented in terms of normal products over a^+ and a . In addition, it is required that the interaction Hamiltonian does not contain terms

quadratic in the canonical variables. This condition is equivalent to the equation

$$\frac{d\epsilon_0}{d\omega} = 0 \quad (1)$$

which determines ω , the oscillator frequency, in the ORM and is called the *oscillator representation condition* (ORC) [8]. Similar ideas are used in the Hartee-Fock-Bogoliubov theory to describe different correlations between nucleons moving in an average nuclear potential (see for review [16]), in order to include particle correlations into a new quasi-particle vacuum. In fact, the ORC coincides with the variational equation arising after averaging of the total Hamiltonian over the Gaussian probe functions in the expanded configuration space. In addition, the dimension of the hyperspace is the variational parameter in the ORM. Using the ORM, we will calculate the energy spectrum and wave functions for the Coulomb and power-law potentials and for the exponentially screened Coulomb potential.

The paper is organized as follows: In Sect. 2, basic formulae of the ORM for spherical potentials are presented. In Sect. 3 the method is applied to calculate the energy spectrum for the Coulomb and power-law potentials, for the Coulomb and the Yukawa potentials. In Sect. 4 we discuss the main results. In the appendices, useful relations used in the ORM are presented.

2 ORM in the Space R^D

Let us consider the Schrödinger equation in 3-dimensions

$$\int d^3\mathbf{r} \Psi(\mathbf{r}) \left[-\frac{1}{2} \Delta + V(r) - E \right] \Psi(\mathbf{r}) = 0 \quad (2)$$

where $V(r)$ is a potential of the Coulomb and/or Yukawa type, decreasing at large distances ($V(r) \rightarrow 0$ as $r \rightarrow \infty$). If $\Psi(\mathbf{r}) = \psi_{nl}(r) Y_{nl}(\theta, \phi)$, this equation for the wave function of the l -th orbital excitation is [12, 13]

$$\int_0^\infty dr (r \psi_{nl}(r)) \left[-\frac{1}{2} \left(\frac{d}{dr} \right)^2 + \frac{l(l+1)}{2r^2} + V(r) - E \right] (r \psi_{nl}(r)) = 0 \quad (3)$$

where the function $\psi_{nl}(r)$ depends on a radial variable only. General discussion on the transition to arbitrary dimensions in the radial SE is given in [17]. In this case, calculation of the wave function $\psi_{0l}(r)$ would be equivalent to that of the ground state wave function

of the modified Hamiltonian in another dimension. The radial excitation wave functions $\psi_{nl}(r) = |n\rangle$ ($n = n_r + l + 1$) are equivalent to the highest oscillator states.

At large distances, for the Coulomb and Yukawa type potentials the asymptotic behaviour of the wave function $\psi(r)$ as $r \rightarrow \infty$ is well known, i.e. $\psi(r) \sim \exp(-a(r))$. The new variable $r = r(q)$ should be introduced so that $a(r(q)) \sim q^2$ as $q \rightarrow \infty$ and $\psi(r(q)) \sim \exp(-q^2)$. Using the substitutions

$$r = q^{2\rho} \quad \text{and} \quad r\psi_{nl}(r) = q^\alpha \phi(q), \quad (4)$$

after simple calculations we obtain

$$\int_0^\infty dq q^{D-1} \phi(q) \left[-\frac{1}{2} \left(\frac{d}{dq} \right)^2 + \frac{D-1}{q} \frac{d}{dq} + W_I(q^2; E) \right] \phi(q) = 0, \quad (5)$$

where

$$W_I(q^2; E) = -\frac{K(l, \rho, D)}{2q^2} + 4\rho^2(q^2)^{(2\rho-1)}(V(q^2) - E), \quad (6)$$

$$D = 2a - 2\rho + 2 \quad (7)$$

$$K(l, \rho, D) = \frac{1}{4}((D-2)^2 - 4\rho^2(2l+1)^2) \quad (8)$$

We can identify Eq.(5) with the SE in the expanded space R^D for the wave function $\phi(q)$ depending on the radial variable q only

$$\int d^D \mathbf{q} \phi(\mathbf{q}) \left[-\frac{1}{2} \Delta_D + W_I(q^2; E) - \varepsilon(E) \right] \phi(\mathbf{q}) = 0, \quad (9)$$

where the function

$$\varepsilon(E) = \varepsilon(l, \rho, D; E) \quad (10)$$

should be considered as an eigenvalue of the SE in D -dimensions

$$\left[-\frac{1}{2} \Delta_D + W_I(q^2; E) \right] \phi(q) = \varepsilon(E) \phi(q) \quad (11)$$

We note that the potential $W_I(q^2; E)$ contains the attractive term which can compensate the repulsive part of the $V(q^2)$.

The energy spectrum E_{nl} of the original system is contained in the radial excitation spectrum $\varepsilon^{[nr]}$ of the Hamiltonian of Eq.(11)

$$H(E) \phi^{[nr]}(q) = \varepsilon^{[nr]}(E) \phi^{[nr]}(q), \quad (n_r = 0, 1, 2, \dots) \quad (12)$$

and it is determined by the equation

$$\varepsilon^{[nr]}(E) = \varepsilon^{[nr]}(l, \rho, D; E) = 0. \quad (13)$$

The additional parameters ρ and D can be found, for example, by the minimization of the energy in the zeroth approximation

$$\varepsilon_0(E) = \min_{\{\rho, D\}} \varepsilon(l, \rho, D; E).$$

On the other hand, the parameter ρ is connected with the condition of the Gaussian asymptotic behavior for the wave function (see the discussion above). The parameter D is connected with the behaviour of the wave function at short distances. If the potential $V(r)$ has no a repulsive character as $r \rightarrow 0$ in the ORM we choose $K(l, \rho, D) = 0$ and

$$D = 2 + 2\rho(2l+1). \quad (14)$$

In the opposite case, the parameter D is chosen to suppress a repulsion produced by the potential $V(r)$ at $r \rightarrow 0$. In fact, the parameter D is used to improve the zeroth approximation at the minimization of the ground state energy.

One can see that the radial quantum number n_r does not enter into Eq.(11) in an explicit form. The orbital quantum number l is absorbed by the parameter D . From the point of view of the expanded space R^D the functions

$$\phi_n(q) = q^{2\rho-n} \psi_{nl}(q^{2\rho}) \quad \text{or} \quad \psi_{nl}(r) = r^{\frac{D-2\rho-2}{4\rho}} \phi_n(r^{1/(2\rho)})$$

for any n and for fixed l are eigenfunctions of the basic series of radial excitations in the space R^D with radial quantum number n and zeroth orbital momentum. Consequently, the solution of the SE in 3 dimensions for l orbital excitation is equivalent to the solution of the SE in the space R^D for states with the zeroth angular momentum. As a result, the original SE is represented in the form of Eq.(11) in which the wave function of the ground state $\phi(q)$ has

- a Gaussian asymptotic behaviour at large distances $\phi(q) \sim \exp(-q^2)$,
- a maximum at the point $q = 0$.

In addition, the integration over the expanded space R^D and the Laplacian are defined by:

$$i) \int d^D q = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dq \cdot q^{D-1}$$

$$ii) \Delta_q = \left(\frac{d^2}{dq^2} + \frac{D-1}{q} \cdot \frac{d}{dq} \right).$$

The next step is to introduce the oscillator representation which can be done in the following way. The Hamiltonian H can be rewritten in the form

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \left(W(q) - \frac{1}{2}\omega^2 q^2 \right), \quad (15)$$

where ω is an unknown oscillator frequency. Let us substitute the creation and annihilation operators (Appendix A) into Eq.(15) and go to the normal product of the operators a_j^\dagger and a_j . Hereafter we use the notation $d \equiv D$. One can obtain

$$\frac{1}{2}(p^2 + \omega^2 q^2) = \omega \sum_j a_j^\dagger a_j + \frac{d}{2}\omega = \omega(a^\dagger a) + \frac{d}{2}\omega, \quad (16)$$

$$W(q) - \frac{\omega^2}{2} q^2 = \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega}\right) : e^{i(kq)} : - \frac{\omega^2}{2} (: q^2 : + \frac{d}{2\omega}),$$

where $: * :$ is the symbol of the normal ordering and $(qk) = \sum_j k_j q_j$,

$$\bar{W}_d(k^2) = \int (d\rho)^d W(\rho) e^{i(k\rho)}. \quad (17)$$

We require that the interaction part of the Hamiltonian should not contain the term with $: q^2 :$ because this term is postulated to be included into the oscillator part completely. This requirement gives the equation for the frequency ω :

$$\omega^2 - \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega}\right) \frac{k^2}{d} = 0. \quad (18)$$

Using these equations, we can rewrite the Hamiltonian, Eq.(15), in the form:

$$H = H_0 + H_I + \epsilon_0, \quad (19)$$

$$H_0 = \omega(a^\dagger a), \quad (20)$$

$$H_I = \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega}\right) : e^{i(kq)} - 1 + \frac{k^2 q^2}{2d} : \\ = \int \left(\frac{d\rho}{\sqrt{\pi}} \right)^d e^{-\rho^2} W\left(\frac{\rho}{\sqrt{\omega}}\right) : \exp(-q^2 + 2(\rho q)) - 1 + q^2 \left(1 - \frac{2\rho^2}{d}\right) :. \quad (21)$$

$$\epsilon_0 = \frac{d\omega}{4} + \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega}\right). \quad (22)$$

Details of calculations of different integrals appearing in the above expressions can be found in Appendices. In the ORM the solution of the SE Eq.(11) has the form

$$\phi_n(q) = \exp\left(-\frac{\omega}{2} q^2\right) \sum_m c_{nm} P_m^{(D)}(q^2 \omega),$$

where $\{P_m^{(D)}(t)\}$ is the class of orthogonal polynomials which are orthogonal within the interval $0 < t < \infty$ with the weight function

$$\rho_D(t) = t^{\frac{D}{2}-1} \exp(-t),$$

i.e.,

$$\int_0^\infty dt t^{\frac{D}{2}-1} e^{-t} P_n^{(D)}(t) P_m^{(D)}(t) = \delta_{nm}.$$

These orthogonal polynomials can be constructed by using the formalism of creation and annihilation operators a_j and a_j^\dagger in the space R^D (see Appendices A, B). Using the definition for the ground wave function in the R^D space

$$|0\rangle = \prod_{j=1}^D \frac{\omega^{1/4}}{\pi^{1/4}} e^{-\frac{\omega}{2} q_j^2} = \left(\frac{\omega}{\pi}\right)^{D/4} e^{-\frac{\omega}{2} q^2} \quad (23)$$

we can write for radial excitations $n = 1, 2, \dots$

$$\Phi_n \sim (a^\dagger a^\dagger)^n |0\rangle \sim P_n^{(D)}(\omega q^2) e^{-\frac{\omega}{2} q^2} \\ \sim P_n^{(D)}(\omega r^{1/\rho}) \exp\left(-\frac{\omega}{2} r^{1/\rho}\right), \quad (24)$$

where $P_n^{(D)}(t)$ is a polynomial of the n th order. The parameter D in this representation can be considered to be any positive number. These polynomials satisfy the orthogonal condition

$$(\Phi_n, \Phi_m) \sim \langle 0 | (aa)^n (a^\dagger a^\dagger)^m | 0 \rangle \\ \sim \int_0^\infty dq q^{D-1} \exp(-q^2) P_n^{(D)}(q^2) P_m^{(D)}(q^2) \\ \sim \int_0^\infty dt t^{D/2-1} \exp(-t) P_n^{(D)}(t) P_m^{(D)}(t) \sim \delta_{nm}.$$

Thus, the ORC, Eq.(1), is written as

$$\frac{\partial}{\partial \omega} \varepsilon_0(E; \omega, d) = 0. \quad (25)$$

This equation determines the parameter $\omega = \omega(E, d)$ as a function of the energy E , d and other parameters defining the potential $V(r)$ in Eq.(3).

According to Eq.(13), the ground state energy E of the original problem in the N -th perturbation order of the ORM is defined by Eq.(25) and by the equation

$$\varepsilon_{(N)}(E, d) = \varepsilon_0(E, d) + \varepsilon_2(E, d) + \dots + \varepsilon_N(E, d) = 0. \quad (26)$$

Eq.(26) determines the energy $E_{(N)}(d)$ in the N -th perturbation order as a function of d or ρ and other parameters defining the potential. The function $\varepsilon(E; \omega, d)$ in Eq.(19) depends on two parameters d and ω . Let us denote the parameter d and other additional parameters by $\{\alpha_j\}$. The ground state energy in the expanded space up to second order is

$$\varepsilon_{(2)}(E; \omega, \alpha_j) = \varepsilon_0(E; \omega, \alpha_j) + \varepsilon_2(E; \omega, \alpha_j).$$

According to Eqs.(22), (17) and (6), the function $\varepsilon_0(E; \omega, \alpha_j)$ can be written in the form

$$\varepsilon_0(E; \omega, \alpha_j) = A(\omega, \alpha_j) - E \cdot B(\omega, \alpha_j), \quad (27)$$

where $A(\omega, \alpha_j)$ and $B(\omega, \alpha_j)$ are known functions.

In the zeroth order Eqs.(26) and (27) lead to equations

$$\frac{\partial}{\partial \alpha_k} E_0(\omega, \alpha_j) = \frac{\partial}{\partial \alpha_k} \left(\frac{A(\omega, \alpha_j)}{B(\omega, \alpha_j)} \right) = 0 \quad \text{for all } k, \quad (28)$$

which determine the parameters $\{\alpha_j\}$ as functions of E_0

$$\alpha_j = \alpha_j(E). \quad (29)$$

As a result, in the zeroth approximation we have

$$E_0 = \min_{\{\omega, \alpha_j\}} \frac{A(\omega, \alpha_j)}{B(\omega, \alpha_j)} = \frac{A(\omega_0, \alpha_j^0)}{B(\omega_0, \alpha_j^0)}, \quad (30)$$

where the parameters ω_0 and α_j^0 determine the minimum.

In the second approximation we solve the following equation

$$\varepsilon_{(2)}(E; \omega, \alpha_j) = \varepsilon_0(E; \omega, \alpha_j) + \varepsilon_2(E; \omega, \alpha_j) = 0. \quad (31)$$

We assume that the second correction is small one and the energy is $E_{(2)} = E_0 + E_2$. Consequently, we have

$$\varepsilon_0(E_{(2)}; \omega_0, \alpha_j^0) = A(\omega_0, \alpha_j^0) - E_{(2)} B(\omega_0, \alpha_j^0), \quad (32)$$

where $\omega_0 = \omega(E_0)$. Therefore, the second correction is

$$E_2 = \frac{\varepsilon_2(E_0; \omega_0, \alpha_j^0)}{B(\omega_0, \alpha_j^0)} + O(E_2^2).$$

Finally, we have

$$\begin{aligned} E_{(2)} &= E_0 + E_2 = \\ &= \min_{\{\omega, \alpha_j\}} \frac{A(\alpha_j)}{B(\alpha_j)} + \frac{\varepsilon_2(E_0; \omega_0, \alpha_j(E_0))}{B(\omega_0, \alpha_j(E_0))} + E_0 O\left(\left|\frac{E_2}{E_0}\right|^2\right) \\ &= \frac{A(\omega_0, \alpha_j^0) + \varepsilon_2(E_0; \omega_0, \alpha_j^0)}{B(\omega_0, \alpha_j^0)} + E_0 O\left(\left|\frac{E_2}{E_0}\right|^2\right). \end{aligned} \quad (33)$$

In our calculations we use essentially these equations. The accuracy of the oscillator representation can be evaluated as

$$\delta \sim \left| \frac{\varepsilon_2}{E_0} \right|.$$

For the radial excitations $|n_r\rangle$ ($n_r = 1, 2, \dots$) the matrix element

$$\langle n_r | H_I | n_r \rangle = A^{[n_r]}(\omega, \alpha_j) - E B^{[n_r]}(\omega, \alpha_j) \neq 0.$$

The energy $\varepsilon^{[n_r]}$ in the lowest approximation looks like

$$\begin{aligned} \varepsilon_1^{[n_r]}(E) &= \langle n_r | H | n_r \rangle = \varepsilon_0(E) + 2n_r \omega + \langle n_r | H_I | n_r \rangle \\ &= A_1^{[n_r]}(\omega, \alpha_j) - E B_1^{[n_r]}(\omega, \alpha_j), \end{aligned} \quad (34)$$

where

$$\begin{aligned} A_1^{[n_r]}(\omega, \alpha_j) &= A(\omega, \alpha_j) + 2n_r \omega + A^{[n_r]}(\omega, \alpha_j), \\ B_1^{[n_r]}(\omega, \alpha_j) &= B(\omega, \alpha_j) + B^{[n_r]}(\omega, \alpha_j). \end{aligned}$$

Two equations

$$\begin{aligned} \frac{\partial}{\partial \omega} A(\omega, \alpha_j) - E \frac{\partial}{\partial \omega} B(\omega, \alpha_j) &= 0, \\ A_1^{[n_r]}(\omega, \alpha_j) - E B_1^{[n_r]}(\omega, \alpha_j) &= 0, \end{aligned} \quad (35)$$

determine the functions $\omega(\alpha_j)$ and $E(\alpha_j)$. The energy of the n_r -th excited state in the first approximation of the oscillator representation is determined as

$$E^{[n_r]} = \min_{\{\alpha_j\}} \frac{A_1^{[n_r]}(\omega(\alpha_j), \alpha_j)}{B_1^{[n_r]}(\omega(\alpha_j), \alpha_j)}$$

Further steps should be done according to the rules formulated above.

3 The Coulomb type potentials

3.1 The Coulomb and power-law confining potentials

In the general case, the potential under consideration can be written as follows

$$V(r) = -\frac{1}{r} + g \cdot r^\nu. \quad (36)$$

At $\nu = 1$ this potential corresponds to the well-known "Cornell" potential [6], used to study the energy spectrum of a many-quark system. In addition, at $\nu = 1$ and 2 the potential can be used to study the Stark problem and the Zeeman effect in a spherical system, respectively. Due to various applications of this kind of potential, there are many attempts to solve the eigenvalue problem with a good quantitative accuracy (see, for example [7, 18, 19] and references therein). In fact, the main approach is based on ideas of the perturbation theory. As a result, the problem of summing the perturbation series is not solved completely.

We apply the ORM to calculate the energy spectrum for an arbitrary value of the parameter ν . Using the transformation Eq.(4), after some simple calculations the SE with the potential Eq.(36) can be written in the following form

$$\left[-\frac{1}{2} \left(\frac{\partial^2}{\partial q^2} + \frac{d-1}{q} \frac{\partial}{\partial q} \right) - 4\rho^2 E q^{2(2\rho-1)} - 4\rho^2 q^{2(\rho-1)} \right. \quad (37)$$

$$\left. + 4\rho^2 g q^{2(\nu\rho+2\rho-1)} \right] \Phi(q) = \varepsilon(E) \Phi(q).$$

$$d = 2 + 2\rho + 4\rho l \quad (38)$$

According to the ORM, the Hamiltonian has the form of Eq.(19) where

$$H_0 = \omega(a^\dagger a), \quad (39)$$

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x}\omega(q\eta)} :$$

$$\left[-\frac{4\rho^2 E}{\omega^{2\rho-1} \Gamma(1-2\rho)} x^{-2\rho} - \frac{4\rho^2}{\omega^{\rho-1} \Gamma(1-\rho)} x^{-\rho} + \frac{4g\rho^2}{\omega^{\nu\rho+2\rho-1} \Gamma(1-2\rho-\nu\rho)} x^{-2\rho-\nu\rho} \right]$$

and the function ε_0 is

$$\varepsilon_0(E) = \min_{\{\omega, \rho\}} \varepsilon_0(E; \omega, \rho),$$

Here

$$\varepsilon_0(E; \omega, \rho) = \frac{d\omega}{4} - \frac{4E\rho^2 \Gamma(d/2 + 2\rho - 1)}{\omega^{2\rho-1} \Gamma(d/2)} \quad (40)$$

$$- \frac{4\rho^2 \Gamma(d/2 + \rho - 1)}{\omega^{\rho-1} \Gamma(d/2)} + \frac{4g\rho^2 \Gamma(d/2 + 2\rho + \nu\rho - 1)}{\omega^{2\rho+\nu\rho-1} \Gamma(d/2)}$$

The energy of the system and the new oscillator frequency are determined from Eq.(13) and from the ORC, Eq.(25), respectively. After some simplifications, we obtain the following equation from the system of two equations:

$$E = \min_{[l]} \left\{ \frac{Z^2 \Gamma(d/2 + 1)}{8\rho^2 \Gamma(d/2 + 2\rho - 1)} - \frac{Z\Gamma(d/2 + \rho - 1)}{\Gamma(d/2 + 2\rho - 1)} \right. \quad (41)$$

$$\left. + \frac{g}{Z^\nu} \frac{\Gamma(d/2 + 2\rho + \nu\rho - 1)}{\Gamma(d/2 + 2\rho - 1)} \right\},$$

The parameter Z is determined from Eq.(25)

$$Z^{2+\nu} - Z^{1+\nu} 4\rho^2 \frac{\Gamma(d/2 + \rho - 1)}{\Gamma(d/2 + 2\rho - 1)} \quad (42)$$

$$- g\nu 4\rho^2 \frac{\Gamma(d/2 + 2\rho + \nu\rho - 1)}{\Gamma(d/2 + 2\rho - 1)} = 0$$

and the oscillator frequency is

$$\omega = Z^{1/\rho}.$$

In the strong coupling limit, i.e. as $g \rightarrow \infty$, we obtain from Eq.(41) the following result

$$E = g^{\frac{2}{2+\nu}} \cdot C, \quad (43)$$

where

$$C = \min_{[l]} \left\{ \left(\frac{1}{2} + \frac{1}{\nu} \right) \right. \quad (44)$$

$$\left. \frac{\Gamma(d/2 + 1)}{4\rho^2 \Gamma(d/2 + 2\rho - 1)} \left[4\nu\rho^2 \frac{\Gamma(d/2 + 2\rho + \nu\rho - 1)}{\Gamma(d/2 + 1)} \right]^{\frac{2}{2+\nu}} \right\}.$$

In Table 1 we present, as an illustration, the results at the zeroth approximation of the ORM for $\nu = 1$.

3.2 The exponentially screened Coulomb potential

The ESCP describes various physical phenomena. It was used to evaluate the cohesive energy of alkali metals [20], to calculate of energy bands of sodium by empirical pseudopotential method [21] and to describe of the exciton and donor spectra in ionic crystals and polar semiconductors [22, 23].

The Schrödinger equation for the ESCP has the following form:

$$\left[\frac{1}{2\mu} p^2 - \frac{A}{r} + B \cdot \frac{e^{-cr}}{r} \right] \Psi(r) = E \Psi(r), \quad (45)$$

where μ is relative mass, A , B and c are some constants. We use the following units: the length unit is $\hbar^2/(A\mu)$ and the energy unit is $\mu A^2/(2\hbar^2)$. The SE in the dimensionless units can be written

$$\left[-\frac{1}{2} \Delta - \frac{1}{r} + \frac{B}{2} \frac{e^{-cr}}{r} - \frac{E}{2} \right] \Psi(r) = 0 \quad (46)$$

Using the transformation Eq.(4), we obtain the modified SE

$$\begin{aligned} & \left[-\frac{1}{2} \left(\frac{\partial^2}{\partial q^2} + \frac{d-1}{q} \frac{\partial}{\partial q} \right) - 2\rho^2 E q^{2(2\rho-1)} - 4\rho^2 q^{2(\rho-1)} \right. \\ & \left. + 2B\rho^2 q^{2(2\rho-1)} \exp\{-cq^{2\rho}\} \right] \Phi(q) = \varepsilon(E) \Phi(q); \\ & \varepsilon(E) = 0, \quad d = 2 + 2\rho + 4\rho\ell. \end{aligned} \quad (47)$$

Applying the ORM to the Hamiltonian Eq.(47), we obtain

$$H = H_0 + H_I + \varepsilon_0,$$

where

$$\begin{aligned} H_0 &= \omega (a^\dagger a), \\ H_I &= \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x}\omega(\eta)} : \\ & \left[\frac{-2\rho^2 E}{\omega^{2\rho-1}} \frac{x^{-2\rho}}{\Gamma(1-2\rho)} - \frac{4\rho^2}{\omega^{\rho-1}} \frac{x^{-\rho}}{\Gamma(1-\rho)} \right] \\ & + 2B\rho^2 \sum_{k=0}^\infty \frac{(-c)^k}{k! \omega^{k\rho+1}} \int_0^\infty dx \frac{x^{-\rho(1+k)}}{\Gamma(1-\rho(1+k))} \\ & \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x}\omega(\eta)} : \end{aligned} \quad (48)$$

$$\begin{aligned} \varepsilon_0(E) &= \frac{d\omega}{4} - \frac{2E\rho^2}{\omega^{2\rho-1}} \frac{\Gamma(d/2 + 2\rho - 1)}{\Gamma(d/2)} - \frac{4\rho^2}{\omega^{\rho-1}} \frac{\Gamma(d/2 + \rho - 1)}{\Gamma(d/2)} \\ & + \frac{2B\rho^2}{\Gamma(d/2)} \int_0^\infty du u^{\frac{d}{2}-1} e^{-u} \left(\frac{u}{\omega} \right)^{\rho-1} \exp\left\{-c \left(\frac{u}{\omega} \right)^\rho\right\}. \end{aligned} \quad (49)$$

The energy E and the new oscillator frequency of the system are determined from Eq.(13) and Eq.(25), respectively. After some algebraic calculations, the solution of the system of these two equations has the form

$$\begin{aligned} E &= \min_{|\rho|} \left\{ \frac{Z^2}{4\rho^2} \frac{\Gamma(2 + \rho + 2\rho\ell)}{\Gamma(3\rho + 2\rho\ell)} - 2Z \frac{\Gamma(2\rho + 2\rho\ell)}{\Gamma(3\rho + 2\rho\ell)} \right. \\ & \left. + \frac{ZB}{\Gamma(3\rho + 2\rho\ell)} \int_0^\infty du u^{2\rho+2\rho\ell-1} \exp\left\{-u - \frac{c}{Z} u^\rho\right\} \right\}, \end{aligned} \quad (50)$$

Here the parameter Z is determined from Eq.(25)

$$\begin{aligned} & Z^2 - 4Z\rho^2 \frac{\Gamma(2\rho + 2\rho\ell)}{\Gamma(2 + \rho + 2\rho\ell)} \\ & + \frac{2B\rho^2}{\Gamma(2 + \rho + 2\rho\ell)} \int_0^\infty du u^{2\rho+2\rho\ell-1} (Z + cu^\rho) \exp\left\{-u - \frac{c}{Z} u^\rho\right\} = 0 \end{aligned} \quad (51)$$

and the oscillator frequency is

$$\omega = Z^{1/\rho}.$$

In Tables 2 and 3 we present the result of calculations of the ESCP energy for different sets of parameters. The comparison with the computational results [5] is very good and it gives a strong support to the ideas of the ORM.

4 Summary

We presented the main ideas and the mathematical formalism of the ORM which is very efficient in getting the analytical solution to bound states of a stationary quantum-mechanical problem.

Using the ORM, the discrete eigenvalues for the superposition of the Coulomb and power-law potentials have been calculated. The results are obtained for arbitrary values of the strength and the degree of the power-law potential. It is a well-known fact that many methods fail to solve the problem of transition from a weak to a strong coupling limit. The advantage of the ORM is in the consistent treatment of the problem, while different approaches run into a problem of summing of the perturbation series. In addition, the

ground state and the orbital and radial excitations are treated in a similar way within the ORM. We note that our results are in the remarkable agreement with the ones, obtained for example in [24] at different limits. The ORM has been applied successfully to calculate the energy spectrum of the Coulomb and the Yukawa potentials for different sets of their strengths. The results show good agreement with the quantitative solutions of the problem [5]. In fact, in both cases our solutions have been obtained avoiding the question of the Dyson phenomenon [25] which causes the main difficulties in the perturbation approach. We hope that the results obtained for the considered problems can be useful for an analysis of different questions arising, in particular, in atomic physics.

In conclusion, the ORM allows in many cases to obtain analytical results, which are available only numerically or partially analytically within the perturbation approach or the WKB approximation. It is clear, that any progress in the development of analytical methods can provide a deeper insight into solution and understanding of the quantum mechanical problem, making further contribution to quantum mechanics.

A Calculation of some products of the annihilation and creation operators

The canonical variables q_j and p_j are coordinates and momentum, respectively, in the d -space R^d and are expressed through a_j^\dagger and a_j

$$q_j = \frac{1}{\sqrt{2\omega}} (a_j + a_j^\dagger), \quad p_j = \sqrt{\frac{\omega}{2}} \cdot \frac{a_j - a_j^\dagger}{i}. \quad (\text{A.1})$$

The operators a_j^\dagger and a_j fulfill the standard commutation relation

$$[a_i, a_j^\dagger] = \delta_{i,j}, \quad j = 1, 2, \dots, d. \quad (\text{A.2})$$

Hereafter $d \equiv D$. From (A.1) and (A.2) we can obtain for q^2, p^2

$$q^2 = \frac{d}{2\omega} + :q^2: , \quad p^2 = \frac{d\omega}{2} + :p^2: , \quad (\text{A.3})$$

where the symbol $: * :$ means the normal ordering and ω is the oscillator frequency.

For a_j^\dagger and a_j we have

$$e^{i\vec{k}\vec{a}} e^{i\vec{p}\vec{a}^\dagger} = e^{i\vec{p}\vec{a}^\dagger} e^{i\vec{k}\vec{a}} \cdot e^{-(k\cdot p)}, \quad (\text{A.4})$$

where \vec{k} and \vec{p} are vectors in the d -space.

Let consider the expression

$$Y_j(\vec{k}) = e^{i\vec{k}\vec{a}} a_j^\dagger e^{-i\vec{k}\vec{a}}. \quad (\text{A.5})$$

At $\vec{k} = 0$ from Eq.(A.5) we have

$$Y_j(0) = a_j^\dagger. \quad (\text{A.6})$$

Taking into account (A.2), we obtain from (A.5)

$$\frac{dY_j(\vec{k})}{dk_i} = i\delta_{ji}. \quad (\text{A.7})$$

Integrating over k_i and taking into account (A.6), we have

$$Y_j(\vec{k}) = e^{i\vec{k}\vec{a}} a_j^\dagger e^{-i\vec{k}\vec{a}} = a_j^\dagger + ik_j. \quad (\text{A.8})$$

Similar expressions can be obtained for

$$\begin{aligned} e^{-i\vec{p}\vec{a}^\dagger} a_j e^{i\vec{p}\vec{a}^\dagger} &= a_j + ip_j, \\ e^{\alpha a^\dagger \vec{a}} a_j e^{-\alpha a^\dagger \vec{a}} &= a_j e^{-\alpha} \\ e^{\alpha a^\dagger \vec{a}} a_j^\dagger e^{-\alpha a^\dagger \vec{a}} &= a_j^\dagger e^{\alpha}. \end{aligned} \quad (\text{A.9})$$

According to the ORM, we transform the Hamiltonian rewriting all variables into the d -space in a normal order. As a result, the potential can be expressed as a polynomial of different powers of q . Some typical cases of potentials are the following:

a) a power-law potential

$$\begin{aligned} q^{2n} &= (-1)^n \frac{d^n}{dx^n} \cdot e^{-xq^2} \Big|_{x=0} \\ &= (-1)^n \frac{d^n}{dx^n} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x/\omega)} : e^{-2i\sqrt{x}\eta(q\eta)} : \Big|_{x=0} \\ &= \frac{1}{\omega^n} \cdot \frac{\Gamma(\frac{d}{2} + n)}{\Gamma(\frac{d}{2})} + :q^2: \frac{n}{\omega^{n-1}} \cdot \frac{\Gamma(\frac{d}{2} + n)}{\Gamma(\frac{d}{2} + 1)} \\ &\quad + \frac{(-1)^n}{\omega^n} \frac{d^n}{dx^n} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x}\omega(q\eta)} : \Big|_{x=0}, \end{aligned} \quad (\text{A.10})$$

where $n = 1, 2, \dots$ are integer and positive numbers.

b) an inverse power-law potential

$$q^{2r} = \int_0^\infty \frac{dx}{\Gamma(-r)} x^{-1-r} \cdot e^{-xq^2} \quad (\text{A.11})$$

$$\begin{aligned}
&= \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x/\omega)} : e^{-2i\sqrt{x}\omega(q\eta)} : \\
&= \frac{1}{\omega^\tau} \cdot \frac{\Gamma(\frac{d}{2} + \tau)}{\Gamma(\frac{d}{2})} + : q^2 : \frac{\tau}{\omega^{\tau-1}} \cdot \frac{\Gamma(\frac{d}{2} + \tau)}{\Gamma(\frac{d}{2} + 1)} \\
&+ \frac{1}{\omega^\tau} \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e^{-2i\sqrt{x}\omega(q\eta)} :
\end{aligned}$$

Here $\tau \neq n$; we use the notation $e_x^\tau = e^x - 1 - x - x^2/2$.

c) In the general case we use the Fourier transform

$$\begin{aligned}
W(q^2) &= \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) e^{ikq} \quad (A.12) \\
&= \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(ik \frac{a+a^+}{\sqrt{2\omega}} \right) \\
&= \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega} \right) \exp\left(ik \frac{a^+}{\sqrt{2\omega}} \right) \exp\left(ik \frac{a}{\sqrt{2\omega}} \right) \\
&= \int \left(\frac{dk}{2\pi} \right)^d \bar{W}_d(k^2) \exp\left(-\frac{k^2}{4\omega} \right) : \exp(ikq) :,
\end{aligned}$$

where $(kq) = \sum k_j q_j$ and

$$\bar{W}_d(k^2) = \int (d\rho)^d W(\rho^2) e^{i(k\rho)}.$$

The normal order form of an arbitrary potential can be defined by Eqs.(A.10)-(A.12).

In particular, we have

for $n = 1, 2, 3$

$$\begin{aligned}
q^2 &= \frac{d}{2\omega} + : q^2 : , \quad (A.13) \\
q^4 &= \frac{d(d+2)}{4\omega^2} + \frac{d+2}{\omega} : q^2 : + q^4 : , \\
q^6 &= \frac{d(d+2)(d+4)}{8\omega^3} + \frac{3(d+2)(d+4)}{4\omega^2} : q^2 : + \frac{3(d+4)}{2\omega} : q^4 : + : q^6 : .
\end{aligned}$$

For a calculation of the energy and different matrix elements we need to know the result of action of a combination $(a^+ a^+)^n$ or $(aa)^n$. The following representation is useful

$$\begin{aligned}
(a^+ a^+)^n &= (-1)^n \frac{d^n}{d\beta^n} \cdot e^{-\beta(a^+ a^+)} \Big|_{\beta=0} \quad (A.14) \\
&= (-1)^n \frac{d^n}{d\beta^n} \cdot \int \left(\frac{d\xi}{\sqrt{\pi}} \right)^d e^{-\xi^2 - 2i\sqrt{\beta}(a^+ \xi)} \Big|_{\beta=0}
\end{aligned}$$

According to Eqs.(A.10)-(A.12), the interaction Hamiltonian contains the term proportional to $\exp_2\{-i(\vec{k}\vec{a})\}$. Let us introduce the following operators:

$$e_2^{-i(\vec{k}\vec{a})} = P_\nu : e^{-i\nu(\vec{k}\vec{a})} : \quad (A.15)$$

where P_ν is the operator defined according to the following rules:

$$P_\nu \cdot Const = 0 \quad (A.16)$$

$$P_\nu \cdot \nu^n = 0, \quad \text{nat } n \leq 2$$

$$P_\nu \cdot \nu^n = 1, \quad \text{nat } n > 2$$

B Normalization

In the ORM, the wave function is defined

$$|n_r\rangle = C_{n_r} (a_j^+ a_j^+)^{n_r} |0\rangle, \quad j = 1, \dots, d, \quad (B.1)$$

where C_{n_r} is the normalization constant. Hereafter we use the notation $n \equiv n_r$. The constant can be determined from the condition

$$1 = \langle n|n\rangle = C_n^2 \langle 0| (a_i a_i)^n (a_j^+ a_j^+)^n |0\rangle. \quad (B.2)$$

Using the definition (B.1), the representation (A.10), and the normalization condition $\langle 0|0\rangle = 1$ from (B.2) we obtain

$$\begin{aligned}
1 &= C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi}{\sqrt{\pi}} \right)^d e^{-\eta^2 - \xi^2} \\
&\times \langle 0| e^{-2i\sqrt{\alpha}(a\xi)} \cdot e^{-2i\sqrt{\beta}(a^+\eta)} |0\rangle \Big|_{\alpha, \beta=0} \\
&= C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi}{\sqrt{\pi}} \right)^d e^{-\eta^2 - \xi^2 - 4\sqrt{\alpha\beta}(\xi\eta)} \Big|_{\alpha, \beta=0} \\
&= C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \cdot \frac{1}{(1 - 4\alpha\beta)^{d/2}} \Big|_{\alpha, \beta=0}.
\end{aligned} \quad (B.3)$$

Finally, for C_n we can write

$$C_n = \left(\frac{\Gamma(d/2)}{4^n n! \Gamma(d/2 + n)} \right)^{1/2} \quad (B.4)$$

Let consider the result of the action of some operators on the wave function (B.1). By definition ($\langle 0|a^+ = 0, a|0\rangle = 0$, where $\langle 0|0\rangle = 1$), it follows that

$$\begin{aligned} e^{-B(a^+a)}|n\rangle &= C_n e^{-B(a^+a)} (a^+ a^+)^n |0\rangle & (B.5) \\ &= C_n (-1)^n \frac{\partial^n}{\partial \beta^n} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2} e^{-B(a^+a)} e^{-2i\sqrt{\beta}(a^+\eta)} |0\rangle \Big|_{\beta=0} \\ &= C_n (-1)^n \frac{\partial^n}{\partial \beta^n} \cdot e^{-\beta e^{-2B}} |0\rangle \Big|_{\beta=0} = e^{-2nB} |n\rangle \end{aligned}$$

C Calculation of corrections to the energy spectrum

In the ORM the Hamiltonian has the form

$$H = H_0 + H_I + \varepsilon_0 \quad (C.1)$$

The contribution of the interaction Hamiltonian H_I is considered in the perturbation approach. The energy and wave function are defined

$$\begin{aligned} E_n &= E_0^{(0)} + E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots & (C.2) \\ \Psi_n &= \Psi_n^{(0)} + \Psi_n^{(1)} + \Psi_n^{(2)} + \dots \end{aligned}$$

where $E_0^{(0)} = \varepsilon_0$ is the ground state energy. Within the perturbation approach the SE has the form

$$H_0 \Psi_n^{(k)} + H_I \Psi_n^{(k-1)} = \sum_{k_1+k_2=k} E_n^{(k_1)} \Psi_n^{(k_2)}, \quad (C.3)$$

where n is a radial quantum number and k is an order of the perturbation approach. In the zeroth approximation, we have from Eq.(C.3)

$$H_0 \Psi_n^{(0)} = E_n^{(0)} \Psi_n^{(0)}. \quad (C.4)$$

According to the ORM, $H_0 = \omega(a^+a)$ and $\Psi_n^{(0)} = |n\rangle$. From Eq.(C.4) it follows that

$$\begin{aligned} E_n^{(0)} &= (\Psi_n^{(0)} H_0 \Psi_n^{(0)}) = \langle n | \omega(a^+a) | n \rangle, & (C.5) \\ H_0 &= \omega(a^+a), \quad \Psi_n^{(0)} = |n\rangle. \end{aligned}$$

For $k=1$ we obtain from Eq.(C.3)

$$H_0 \Psi_n^{(1)} + H_I \Psi_n^{(0)} = E_n^{(0)} \Psi_n^{(1)} + E_n^{(1)} \Psi_n^{(0)}. \quad (C.6)$$

Taking into account Eq.(C.5), we obtain from Eq.(C.6)

$$\begin{aligned} E_n^{(1)} &= (\Psi_n^{(0)} H_I \Psi_n^{(0)}), & (C.7) \\ \Psi_n^{(1)} &= -\frac{1}{H_0 - 2n\omega} [H_I - E_n^{(1)}] \Psi_n^{(0)}. \end{aligned}$$

Following the same procedure, from Eq.(C.3) we obtain

$$E_n^{(2)} = -(\Psi_n^{(0)} [H_I - E_n^{(1)}] \frac{1}{H_0 - 2n\omega} [H_I - E_n^{(1)}] \Psi_n^{(0)}). \quad (C.8)$$

• Calculation of $E_n^{(0)}$.

Taking into account Eq.(B.1) and Eq.(A.14), we obtain from Eq.(C.5)

$$\begin{aligned} E_n^{(0)} &= \omega C_n^2 \langle 0 | (a_i a_i)^n (a_i^+ a_i^+)^n | 0 \rangle = \omega C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \\ &\int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi}{\sqrt{\pi}} \right)^d e^{-\eta^2 - \xi^2} \langle 0 | e^{-2i\sqrt{\alpha}(a\xi)} (a_i^+ a_i)^n e^{-2i\sqrt{\beta}(a^+\eta)} | 0 \rangle \Big|_{\alpha, \beta=0} \end{aligned} \quad (C.9)$$

Using Eqs.(A.4), (A.8) and (A.9), after some manipulations we have

$$\begin{aligned} E_n^{(0)} &= \omega C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \cdot \frac{\partial}{\partial \tau} \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi}{\sqrt{\pi}} \right)^d e^{-\eta^2 - \xi^2 - 4\tau \sqrt{\alpha\beta}(\xi\eta)} \Big|_{\alpha, \beta=0; \tau=1} \\ &= \omega C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \cdot \frac{\partial}{\partial \tau} \frac{1}{(1 - 4\tau^2 \alpha\beta)^{d/2}} \Big|_{\alpha, \beta=0; \tau=1} \end{aligned} \quad (C.10)$$

Finally, we obtain from Eq.(C.10)

$$E_n^{(0)} = \langle n | H_0 | n \rangle = 2n\omega. \quad (C.11)$$

In order to demonstrate the principles of a calculation of the first-order, let us consider the interaction Hamiltonian in the form

$$H_I = g \frac{(-1)^\tau}{\omega^\tau} \frac{\partial^\tau}{\partial x^\tau} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} : e_2^{-2i\sqrt{x\omega}(q\eta)} : \Big|_{x=0}, \quad (C.12)$$

where g is the interaction constant and τ is some constant.

• Calculation of $E_n^{(1)}$.

Taking into account Eq.(C.12), we obtain from (C.7)

$$E_n^{(1)} = g \frac{(-1)^\tau}{\omega^\tau} \frac{\partial^\tau}{\partial x^\tau} \cdot \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} \langle n | : e_2^{-2i\sqrt{x\omega}(q\eta)} : | n \rangle \Big|_{x=0} \quad (C.13)$$

For calculations of the first order contributions, we very often treat the following term

$$T_n(x) = \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} |n\rangle : e^{-2i\sqrt{x}\omega(q\eta)} : |n\rangle. \quad (C.14)$$

Let us consider in more detail the procedure of calculating this term. Taking into account Eqs.(A.15), (B.1), (A.14), we obtain for Eq.(C.14)

$$\begin{aligned} T_n(x) &= P_\nu \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x)} |n\rangle : e^{-2i\nu\sqrt{x}\omega(q\eta)} : |n\rangle \quad (C.15) \\ &= P_\nu C_n^2 \frac{\partial^{2n}}{\partial \alpha^n \partial \beta^n} \int \left(\frac{d\eta}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi_1}{\sqrt{\pi}} \right)^d \int \left(\frac{d\xi_2}{\sqrt{\pi}} \right)^d e^{-\eta^2(1+x) - \xi_1^2 + \xi_2^2} \\ &\quad \langle 0 | e^{-2i\sqrt{\alpha}(\alpha\xi_1)} e^{-i\nu\sqrt{2x}(\alpha^+\eta)} e^{-i\nu\sqrt{2x}(\alpha\eta)} e^{-2i\sqrt{\beta}(\alpha^+\xi_2)} | 0 \rangle \Big|_{\alpha, \beta=0} \end{aligned}$$

Finally, we have

$$T_n(x) = \sum_{k=2}^{2n} \sum_{s=0}^n (-1)^k \frac{x^k}{(1+x)^{k+d/2}} \frac{\Gamma(1+n)}{\Gamma(n+d/2)} \frac{2^{2s-k}}{\Gamma(n-s+1)} \frac{\Gamma(k+n-s+d/2)}{\Gamma^2(k-s+1)\Gamma(2s-k+1)} \quad (C.16)$$

According to Eq.(C.13), the first order correction to the energy of a state is

$$E_n^{(1)} = g \frac{(-1)^r}{\omega^r} \frac{\partial^r}{\partial x^r} T_n(x) \Big|_{x=0} \quad (C.17)$$

Also, in different cases it is useful to know the result of calculations of the following quantity

$$Q_n(\tau) = \int_0^\infty \frac{dx}{\Gamma(-\tau)} x^{-1-\tau} \cdot T_n(x). \quad (C.18)$$

Taking into account Eq.(C.16), after integration over x and some manipulation we obtain for Eq.(C.18)

$$\begin{aligned} Q_n(\tau) &= \frac{\Gamma(d/2 - \tau)}{\Gamma(\tau)} \frac{\Gamma(1+n)}{\Gamma(n+d/2)} \sum_{k=2}^{2n} \frac{\Gamma(\tau+k)}{\Gamma(k+d/2)} \quad (C.19) \\ &\quad \sum_{s=0}^n (-1)^k \frac{2^{2s-k}}{\Gamma(n-s+1)} \frac{\Gamma(k+n-s+d/2)}{\Gamma^2(k-s+1)\Gamma(2s-k+1)}. \end{aligned}$$

	[26]	[6]	[24]	ORM
$g = 0.976562$	$\ell = 0$ 0.556764	0.556767	0.557	0.5580
	$\ell = 1$			1.9375
$g = 4.0$	$\ell = 0$ 2.79573	2.79575	2.796	2.7972
	$\ell = 1$			5.65135
$g = 62.5$	$\ell = 0$ 24.79	24.856	24.86	24.8569
	$\ell = 1$			39.4096
$g = 100$	$\ell = 0$ 34.75		34.9	34.9049
	$\ell = 1$			54.4364
$g = 500$	$\ell = 0$ 107.0	108.366	108.4	108.3854
	$\ell = 1$			162.898
$g = 1000$	$\ell = 0$ 174.8			174.8698
	$\ell = 1$			260.295
$g \rightarrow \infty$	$\ell = 0$ $C = 1.764$	1.85576	1.856	1.8559
	$\ell = 1$			0.668
	$\ell = 3$			0.009

Table 1. The energy of the ground state and orbital excitations

c	$\ell = 0$		$\ell = 1$		$\ell = 2$	
	ρ	$-E$	ρ	$-E$	ρ	$-E$
0.001	1.0	0.251 (0.251)	1.0	0.06350 (0.06350)	1.0	0.02877 (0.02877)
0.005	1.0	0.25497 (0.25496)	1.0	0.06740 (0.06738)	1.0	0.032525 (0.032520)
0.01	1.0	0.25985 (0.25895)	1.0	0.07204 (0.07202)	0.99	0.036813 (0.036810)
0.05	1.0	0.29650 (0.29650)	0.99	0.10219 (0.10242)	0.91	0.060521 (0.060520)
0.1	0.99	0.33694 (0.33694)	0.91	0.12931 (0.12931)	0.88	0.077046 (0.077050)
0.2	0.97	0.404244 (0.40424)	0.88	0.164292 (0.16429)	0.88	0.093731 (0.093730)
0.5	0.91	0.54246 (0.54243)	0.87	0.211998 (0.21203)	0.93	0.107695 (0.10770)
1.0	0.89	0.67482 (0.67482)	0.89	0.236543 (0.23665)	.98	0.110671 (0.11067)
2.0	0.89	0.80554 (0.80566)	0.97	0.24709 (0.24711)	1.00	0.111084 (0.11108)
10.0	0.90	0.97223 (0.97424)	1.00	0.249996 (0.24998)	1.00	0.111111 (0.11111)

Table 2. Energy of ESCP for $B = 1$ as a function of the parameter c . Result of calculations [5] are presented in brackets.

c	$\ell = 0$		$\ell = 1$		$\ell = 2$	
	ρ	$-E$	ρ	$-E$	ρ	$-E$
0.001	1.0	0.00168 (0.00178)	1.0	0.00160 (0.00168)	1.0	0.001538 (0.00160)
0.005	0.96	0.00807 (0.00814)	0.87	0.00735 (0.00739)	0.88	0.006746 (0.00678)
0.01	0.88	0.015264 (0.01540)	0.72	0.013575 (0.01360)	0.755	0.012125 (0.01215)
0.05	0.69	0.063234 (0.06325)	0.655	0.050177 (0.05019)	0.695	0.040366 (0.04039)
0.1	0.63	0.11136 (0.11130)	0.67	0.081862 (0.08188)	0.71	0.060999 (0.06100)
0.2	0.64	0.1885 (0.18847)	0.705	0.12482 (0.12486)	0.77	0.083568 (0.08360)
0.5	0.69	0.34885 (0.34891)	0.755	0.18899 (0.18915)	0.88	0.104978 (0.10502)
1.0	0.72	0.51243 (0.51246)	0.86	0.226511 (0.22668)	0.965	0.110258 (0.11027)
2.0	0.7863	0.6877 (0.68856)	0.94	0.244485 (0.24458)	0.995	0.111057 (0.11106)
10.0	0.93	0.95111 (0.95197)	1.00	0.24999 (0.24997)	1.00	0.11111 (0.11111)

Table 3. Similar to Table.2 for $B = 2$.

c	$\ell = 0$		$\ell = 1$		$\ell = 2$	
	ρ	$-E$	ρ	$-E$	ρ	$-E$
0.005	0.815	0.0033 (0.00359)	0.495	0.0034 (0.00355)	0.455	0.00345 (0.00351)
0.01	0.50	0.0062 (0.00700)	0.465	0.0065 (0.00694)	0.655	0.0066 (0.00681)
0.05	0.50	0.0311 (0.03235)	0.465	0.0305 (0.03090)	0.53	0.02835 (0.02840)
0.1	0.51	0.0605 (0.06088)	0.415	0.0555 (0.05570)	0.535	0.04780 (0.04782)
0.2	0.52	0.110 (0.11160)	0.48	0.0943 (0.09451)	0.625	0.07261 (0.07260)
0.5	0.52	0.233 (0.23359)	0.64	0.16370 (0.16380)	0.81	0.10098 (0.10098)
1.0	0.53	0.3790 (0.37984)	0.77	0.21265 (0.21279)	0.94	0.10951 (0.10954)
2.0	0.65	0.56403 (0.56407)	0.90	0.2402 (0.24029)	0.995	0.11104 (0.11101)
10.0	0.94	0.91317 (0.91540)	1.00	0.250 (0.24993)	0.667	0.11111 (0.11111)

Table 4. Similar to Table.2 for $B = 4$.

Eqs.(C.16) and (C.18) allow one to calculate any potential matrix elements. In particular, using Eq.(C.16), we have very simple expressions for the power-law potential

$$\begin{aligned} \langle n_r | : q^2 : | n_r \rangle &= \frac{2n_r}{\omega} & (C.20) \\ \langle n_r | : q^4 : | n_r \rangle &= \frac{n_r}{\omega^2} [d + 6n_r - 4], \\ \langle n_r | : q^6 : | n_r \rangle &= \frac{2n_r(n_r - 1)}{\omega^3} [3d + 10n_r - 8]. \end{aligned}$$

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