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SPIN PHENOMENA IN THE CURVED SPACE-TIME WITH THE METRIC IN THE KERR–SCHIELD FORM. – DEPOLARIZATION OF THE MASSIVE NEUTRINO IN THE GRAVITATIONAL FIELD

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The study of spin phenomena in the curved space-time is one of the most important problems of the modern astrophysics. The study of the massive neutrino spin behavior in the gravitational fields with various geometries presents particular interest. That is why we shall base the following statement of the theory of spin phenomena in the curved space-time on the neutrino example.

The Dirac equation in the curved space-time has the following form [1]:

$$\left[i(e^{\mu}_{(a)}\gamma^{a}\frac{\partial}{\partial x^{\mu}} - \frac{1}{4}\gamma_{abc}\gamma^{c}\gamma^{a}\gamma^{b}) - m\right]\Psi = 0, \tag{1}$$

where $\gamma_{abc} = e^{\mu}_{(c)}e_{(a)\nu;\mu}e^{\nu}_{(b)}$, $e^{\mu}_{(a)}$ are tetradic vectors ; $\nu, \mu = 0, 1, 2, 3$ are space indices; a, b, c = 0, 1, 2, 3 are tetradic indices, γ^{a} are the Dirac constant matrices; we chosen them according to Ref.[2]. The metric tensor $g^{\mu\nu}$ is connected with tetradic vectors $e^{\mu}_{(a)}$ in the standard way [3] :

$$g^{\mu\nu} = e^{\mu}_{(a)}e^{\nu}_{(b)}\eta^{ab}, \tag{2}$$

η^{ab} is Minkowski metric tensor, $\gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a} = 2\,\eta^{ab}$.

Kerr metric [4] presents the most interesting example of studying the solution of equation (1) . Kerr was looking for a special type solution of Einstein equations , which would be suitable for describing algebraically special manifold. This form of metrics is currently referred to as Kerr-Schield form [5] :

$$g^{\mu\nu} = \eta^{\mu\nu} + \xi^{\mu}\xi^{\nu}, \tag{3}$$

where ξ^{μ} is an isotropic vector in relation to the Minkowski metric $\eta^{\mu\nu}$. Kerr metric belongs to this class (3). It is in the Kerr-Schield form that the structure of Kerr space-time is most obviously revealed. As it will be shown in this work , spin phenomena in the curved space-time are most simply described by the Dirac equation (1) for the class of metrics presented in the Kerr-Schield form (3).

Let's present the product of the Dirac matrices in the following form :

$$\gamma^{a}\gamma^{b}\cdots\gamma^{c} = \beta^{n}T_{l}^{ab\cdots c}\sigma^{l}, \tag{4}$$

where σ^{l} are 4-dimensional Pauli matrices.Then, (1) can be presented in the form of Schrödinger equation [6]:

$$i\frac{\partial\Psi}{\partial t} = \hat{\mathbf{H}}\Psi, \tag{5}$$

where

$$\hat{\mathbf{H}} = \frac{1}{g^{00}}[ie^0_{(a)}e^k_{(b)}T^{ab}_l\sigma^l\frac{\partial}{\partial x^k} - \frac{i}{4}\gamma_{abc}e^0_{(d)}T^{abcd}_l\sigma^l - me^0_{(a)}T^a_l\beta\sigma^l]. \tag{5a}$$

For the metric (3), tetradic vectors $e^\mu_{(a)}$ are easily expressed through isotropic vectors ξ^μ:

$$e^\mu_{(a)} = \delta^\mu_a + \tfrac{1}{2}\xi^\mu\xi_a,$$

$$\xi_a = \eta_{ab}\xi^b; \qquad \xi^0 = \xi, \quad \xi^\mu = (\xi, \boldsymbol{\xi}). \tag{6}$$

We shall look for the solution of equation (5) in the form of a wave package localized in a spatial area with a diameter l. We shall designate the length on which ξ is essentially changed as L_g. Obviously $L_g \sim \frac{1}{\max|\nabla\xi|}$. Practically L_g is the order of the sizes R of the gravitational field source. For black holes L_g is the order of the gravitational radius R_g of the collapsing astrophysical object. While $R_g$stars with the mass of the Sun order is 1.5 km, so $\gamma_{abc} \sim L_g$: and these members in (5a) may be neglected with the accuracy of $(\frac{L_p}{L_g})$-th order. In this case, (5a) allows an essential simplification:

$$\hat{\mathbf{H}} = -i(1+\xi^2)^{-1}[-(1+\frac{1}{2}\xi^2)\gamma^5(\boldsymbol{\sigma}\nabla) + (\xi + \frac{1}{2}\gamma^5(\boldsymbol{\sigma}\boldsymbol{\xi}))(\boldsymbol{\xi}\nabla) + \frac{i}{2}\xi(\boldsymbol{\sigma}\,[\boldsymbol{\xi}\times\nabla]) +$$

$$+im\beta(1+\frac{1}{2}\xi^2 + \frac{1}{2}\gamma^5\xi(\boldsymbol{\sigma}\boldsymbol{\xi}))]. \tag{7}$$

We shall present Ψ in the form of combination of conditions with left and right helicities:

$$\Psi = \Psi_L + \Psi_R, \quad \Psi_L = \frac{1}{2}(1+\gamma^5)\Psi, \quad \Psi_R = \frac{1}{2}(1-\gamma^5)\Psi. \tag{8}$$

Then

$$i\frac{\partial\Psi_{L,R}}{\partial t} = -i(1+\xi^2)^{-1}\{[\mp(1+\frac{1}{2}\xi^2)(\boldsymbol{\sigma}\nabla) + (\xi \pm \frac{1}{2}(\boldsymbol{\sigma}\boldsymbol{\xi})(\boldsymbol{\xi}\nabla) +$$

$$+\frac{i}{2}\xi(\boldsymbol{\sigma}\,[\boldsymbol{\xi}\times\nabla])]\Psi_{L,R} + +im\beta(1+\frac{1}{2}\xi^2 \mp \frac{1}{2}\xi(\boldsymbol{\sigma}\boldsymbol{\xi}))\Psi_{R,L}\}. \tag{9}$$

The wave function Ψ will be normalized according to the following condition:

$$\int \overline{\Psi} e^0_{(a)}\gamma^a\Psi\sqrt{-g}d^3x = 1. \tag{10}$$

As follows from (10), the possibility of neutrino being in the conditions of right and left helicities equals

$$W_R = \int \overline{\Psi}_R e^0_{(a)}\gamma^a\Psi_R\sqrt{-g}d^3x, \quad W_L = \int \overline{\Psi}_L e^0_{(a)}\gamma^a\Psi_L\sqrt{-g}d^3x, \tag{11}$$

$$W_L + W_R = 1.$$

We shall search the solution of (9) in the form of a wave package,with a width $L_p \gg L_g$:

$$\Psi_{L,R} = \frac{1}{\pi^{\frac{1}{4}}d^{\frac{3}{2}}}\int d^3q \exp(-\frac{(\mathbf{q}-\mathbf{p}_0)^2}{2d^2} + i\int_0^t E(\mathbf{q},\tau)d\tau - i(\mathbf{q}\mathbf{r})\ )\ u(\mathbf{q},t)_{L,R},$$

$$u_L = \begin{pmatrix} F \\ -F \end{pmatrix}, \quad u_R = \begin{pmatrix} H \\ H \end{pmatrix}. \tag{12}$$

Substituting (12) in (9), we shall have

$$\frac{d}{dt}F(H) = -i(1+\xi^2)^{-1}\{(E(1+\xi^2) - \xi(\boldsymbol{\xi}\mathbf{q}) \pm (\boldsymbol{\sigma}\mathbf{Q}_{\mp}))F(H) +$$

$$+m(1+\frac{1}{2}\xi^2 \mp \frac{1}{2}\xi(\boldsymbol{\sigma}\boldsymbol{\xi}))H(F)\},$$

$$\mathbf{Q}_{\mp} = \mathbf{q} - \frac{1}{2}[\boldsymbol{\xi}\,[\boldsymbol{\xi}\times\mathbf{q}]] \mp \frac{i}{2}\xi\,[\boldsymbol{\xi}\times\mathbf{q}]. \tag{13}$$

In (13) $\boldsymbol{\xi}$ is taken in the maximum of wave package amplitude (12). Here we have an accuracy of order $\left(\frac{L_p}{L_g}\right)$. The maximum of the wave package amplitude (12) is defined by the following condition:

$$\mathbf{r} = \int_0^t (\nabla_{\mathbf{q}}E(\mathbf{q},\tau))d\tau + \mathbf{r}_0 \tag{14}$$

which corresponds to the movement of the package's gravity center along the classical trajectory. The system of differential equations (13) describes the behavior of massive neutrino spin in the curved space-time with the Kerr-Schield metric. Contrary to the classical theory of spin in gravitational field, the quantum system of equations (13) obviously includes the mass of the particle m. Besides, in (13) there are members which are stipulated by non-hermiticity of the Hamiltonian $\hat{\mathbf{H}}$ for particles with spin $\frac{1}{2}$ in the curved space-time. This circumstance defines the character of neutrino spin behavior in the gravitational field.

Let's present (13) in a more suitable form :

$$\frac{d}{dt}F = -i\hat{\mathbf{H}}_1^{(+)}F - i\hat{\mathbf{H}}_2H + i\hat{\mathbf{V}}_cH, \quad \frac{d}{dt}H = -i\hat{\mathbf{H}}_1^{(-)}H - i\hat{\mathbf{H}}_2F - i\hat{\mathbf{V}}_cF, \tag{15}$$

where

$$\hat{\mathbf{H}}_1^{(+-)} = (1+\xi^2)^{-1}(E(1+\xi^2) - \xi(\boldsymbol{\xi}\mathbf{q})_{-}^{+}(\boldsymbol{\sigma}\mathbf{Q}_{+})),$$

$$\hat{\mathbf{H}}_2 = \frac{m(1+\frac{1}{2}\xi^2)}{1+\xi^2}; \qquad \hat{\mathbf{V}}_c = \frac{m\xi}{2}(\boldsymbol{\sigma}\boldsymbol{\xi}).$$

The member $\hat{\mathbf{V}}_c$ in (15) leads to the change of spin conditions, i.e. $\frac{d}{dt}F \neq 0$ and $\frac{d}{dt}H \neq 0$. This is why we solve (15) for a stationary case $\hat{\mathbf{V}}_c = 0$. Then, we have

$$\hat{\mathbf{H}}_1^{(+)}F_0 + \hat{\mathbf{H}}_2H_0 = 0, \qquad \hat{\mathbf{H}}_1^{(-)}H_0 + \hat{\mathbf{H}}_2F_0 = 0, \tag{16}$$

where F_0 and H_0 is solution of a linear algebraic equation system (16). From the conditions of compatibility of the system equations (16) it follows that

$$\begin{vmatrix} \hat{\mathbf{H}}_1^{(+)}, \hat{\mathbf{H}}_2 \\ \hat{\mathbf{H}}_2, \quad \hat{\mathbf{H}}_1^{(-)} \end{vmatrix} = 0. \tag{17}$$

The value of $E(\mathbf{q},\boldsymbol{\xi})$ can be found from (17) . Due to the presence of the member $\frac{i}{2}\xi\,[\boldsymbol{\xi}\times\mathbf{q}]$ in $\hat{\mathbf{H}}_1^{(+)}$ and $\hat{\mathbf{H}}_1^{(-)}$ the values $E(\mathbf{q},\boldsymbol{\xi})$ are complex and $ImE(\mathbf{q},\boldsymbol{\xi}) \sim m\xi^2\sin\Theta$ (Θ is the angle between the vectors $\mathbf{q}$ and $\boldsymbol{\xi}$). In (14) we need to substitute only $ReE(\mathbf{q},\boldsymbol{\xi})$. We shall not need the explicit form of F_0 and H_0. These solutions are normalized according to the conditions

$$F_0^{+}(1+\frac{1}{2}\xi(\boldsymbol{\sigma}\boldsymbol{\xi}))F_0 = 1, \quad H_0^{+}(1-\frac{1}{2}\xi(\boldsymbol{\sigma}\boldsymbol{\xi}))H_0 = 1. \tag{18}$$

We divide F_0 and H_0 into two linearly independent members:

$$F_0 = f_{\,0} - \frac{m(1+\frac{1}{2}\xi^2)}{\hat{\mathbf{H}}_1^{(+)}}h_0, \qquad H_0 = h_0 - \frac{m(1+\frac{1}{2}\xi^2)}{\hat{\mathbf{H}}_1^{(-)}}f_{\,0}. \tag{19}$$

Then the solution of the initial system (15) is to be searched in the form

$$F = C_L(t)f_{\,0} - C_R(t)\frac{m(1+\frac{1}{2}\xi^2)}{\hat{\mathbf{H}}_1^{(+)}}h_0,$$

$$H = C_R(t)h_0 - C_L(t)\frac{m(1+\frac{1}{2}\xi^2)}{\hat{\mathbf{H}}_1^{(-)}}f_{\,0}. \tag{20}$$

Substituting (20) in(15) and neglecting the members of the $(\frac{m}{E})$ order, we obtain a system of differential equations for C_R and C_L:

$$\frac{d}{dt}C_L(t) = \frac{i}{2}m\xi(f_{\,0}^{\,+}(\boldsymbol{\sigma}\boldsymbol{\xi})h_0)C_R(t),$$

$$\frac{d}{dt}C_R(t) = -\frac{i}{2}m\xi(h_0^{+}(\boldsymbol{\sigma}\boldsymbol{\xi})f_0)C_L(t). \tag{21}$$

We indicate $\frac{m}{2}\xi(f_0^{+}(\boldsymbol{\sigma}\boldsymbol{\xi})h_0) = \omega$. The case of $C_R(0) = 0$ presents particular interest . In this case, we have the following equation system :

$$\frac{d}{dt}C_L = i\omega C_R, \qquad \frac{d}{dt}C_R = -i\omega^{+}C_L, \qquad C_R(0) = 0. \tag{22}$$

On integrating (22),we get

$$C_L = Ach\int_0^t |\omega|d\tau, \qquad C_R = -iA\frac{\omega^{+}}{\omega}sh\int_0^t |\omega|d\tau. \tag{23}$$

This kind of $C_L$and C_R behavior is radically different from the case of electromagnetic field influence over the spin conditions of a neutral particle, with a spin of $\frac{1}{2}$ and anomalous magnetic moment. There, we deal with spin precession into electromagnetic field which, unlikely (23), is described in trigonometric functions.

For defining $W_L$and W_R we need to find $\exp(-2Im\int_0^t E(\mathbf{q},\boldsymbol{\xi})d\tau)$ in the explicit form, which is presented together with $C_L$and C_R. The equation (17) gives several values of $ImE(\mathbf{q},\boldsymbol{\xi})$. The concrete values $ImE(\mathbf{q},\boldsymbol{\xi})$ are estimated by the functions $C_L(t)$ and $C_R(t)$ which determine the particle's spin conditions . That is why we express $E(\mathbf{q},\boldsymbol{\xi})$ by F and H, using the system (16):

$$ImE = \frac{\xi}{2(1+\xi^2)}\frac{H^+(\boldsymbol{\sigma}\,[\boldsymbol{\xi}\times\mathbf{q}])H + F^+(\boldsymbol{\sigma}\,[\boldsymbol{\xi}\times\mathbf{q}])F}{(F^+F)+(H^+H)}. \tag{24}$$

In general case,the calculations in (24) are complicated and bulky. However, for $\xi^2 \ll 1$ and $\frac{m}{E} \ll 1$ calculations (24) are considerably simplified:

$$ImE(\mathbf{q},\boldsymbol{\xi}) = \frac{1}{2}\frac{d}{dt}\ln(C_L^+C_L + C_R^+C_R), \qquad |\omega| = \frac{1}{2}m\xi^2\sin\Theta. \tag{25}$$

As a result, for a weak gravitational field $\xi^2 \ll 1$ with the help of (11),(23) and (25) we find the dependence W_L and W_R on time in Kerr-Schield metric:

$$W_L = [1 + th^2(\int_0^t |\omega|d\tau)]^{-1}, \qquad W_R = [1 + cth^2(\int_0^t |\omega|d\tau)]^{-1}. \tag{26}$$

From (26) an effect of massive neutrino depolarization in gravitational field follows. The gravitational field of a rotating gravitating body can be described by the following Kerr axial-symmetrical stationary metric:

$$\xi = \sqrt{\frac{R_g\rho^3}{\rho^4 + (\mathbf{ra})^2}}, \qquad \boldsymbol{\xi} = \xi(\rho^2+a^2)^{-1}(\rho\mathbf{r} + [\mathbf{a}\times\mathbf{r}] + \rho^{-1}(\mathbf{ar})\mathbf{a}),$$

$$\rho^2 = \frac{1}{2}(r^2-a^2)[1 + (1+4(\mathbf{ar})^2(r^2-a^2)^{-2})^{\frac{1}{2}}], \tag{27}$$

$a = \frac{J}{M}$, J is gravitating body's impulse moment, M is its mass. For the Sun $a_\odot = 0,28km$, $R_{g\odot} = 2,96km$. Under $r \gg a$ (27) is simplified:

$$\xi = \sqrt{\frac{R_g}{r}}, \qquad \boldsymbol{\xi} = \frac{\xi}{r}(\mathbf{r} + \frac{[\mathbf{ar}]}{r}). \tag{28}$$

Assume that a particle is moving along the radius $\mathbf{r}$. Then $|\omega| = \frac{1}{2}mR_g \cdot ar^{-2}\sin\alpha$ (α is the angle between the vectors $\mathbf{r}$ and $\mathbf{a}$). In this case, the effect of changing massive neutrino helicity can be explained by the fact that the gravitating body has its impulse, or rotation,moment. Since (28) describes the external gravitational field, we shall calculate it with $t = 0$ $r = R$ (R is the gravitating body's radius). Then:

$$\int_0^t |\omega|d\tau = \int_R^r |\omega|r^{-1}dr. \tag{29}$$

However, in (29) $\frac{dr}{dt}$ can be equaled to 1 with big precision. Having integrated in(29) and calculated, we have the evaluation by $r \gg R$:

$$\int_0^\infty |\omega|d\tau = \Omega = \frac{1}{2}mR_gaR^{-1}\sin\alpha. \tag{30}$$

The exact formula for Ω will evidently differ from (30) with a factor of the 1-st order. It will also consider the particular geometry in the gravitational field of the rotating gravitating body . Let's assume that $\Omega = \ln 10$. Then, $W_L = 0,51$; $W_R = 0,49$, and $m = m_0 = \frac{1}{2}\ln 10(\frac{R}{aR_g})\sin\alpha$. Hence, if $m \geq m_0$ the flows of right and left neutrino emitted by a star will be practically equal. W_L and W_R change most considerably, if $0,1m_0 \leq m \leq 0,7m_0$. Here $0,92 \geq W_L \geq 0,55$, and $0,45 \geq W_R \geq 0,08$.

For a solar neutrino, $m_0 = 3.14 \cdot 10^{-4}$ eV at $\alpha = \frac{\pi}{2}$. If the neutrino is emitted by a neutron star, having the same moment and mass as those of the Sun and a radius of $R = 10km$, then, if $\alpha = \frac{\pi}{2}$, m_0 will be much greater than $m_0 = 3.14 \cdot 10^{-8}$ eV. The deficit left solar neutrinos on the Earth D_{ν_L} evidently amounts to W_R.The effect described in this work allows establishing considerable similarity between the theoretical and observed solar neutrino flows, if $m_\nu \geq 10^{-4}$ eV. The effect of the massive neutrino depolarizations will also considerably influence the rotating neutron stars cooling, especially at the moment of their formation when supernova stars explode.

To gain a better understanding of the nature of the massive neutrino depolarization in the gravitational field, we shall express m_0 in the explicit form, preserving the Planck constant h and the velocity of light c:

$$m_0 = (\frac{h}{4\pi ac}) \ln 10(\frac{R}{R_g}) = (\frac{hM}{4\pi J}) \ln 10(\frac{R}{R_g}) = \frac{5\pi \ln 10}{(\frac{v_e}{c})} \frac{M_p^2}{M}, \quad (31)$$

where $M_p = \sqrt{\frac{hc}{16\pi^2 G}}$ is Planck mass, $v_e$is the revolving star equatorial velocity. From (31) it obviously follows that the massive neutrino depolarization in gravitational field is of quantum nature.

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