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EXACTLY SOLVABLE MODELS FOR INVESTIGATION OF DYNAMICAL QUANTUM SYSTEMS

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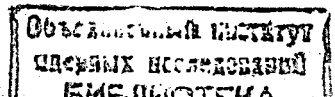
1 Introduction

The method of adiabatic representation is very useful for investigation of many real quantum systems with some degrees of freedom the separation of which is not valid, so far as it allows one to take into account the mutual influence of slowly changing external and rapidly changing internal fields. The direct scattering problem treated in the adiabatic approach has had a rich history that dates back to the first studies of Born and Oppenheimer [1] and Born and Fock [2]. Then it has been intensely studied in many works by Landau [3] and Zener [4, 5], Hill and Wheeler [6], Demkov [7] (for references, see [8] – [12]. However, the inverse scattering problem in the adiabatic representation was proposed relatively recently [13, 14]. One of the main advantages of the inverse scattering method is the possibility of constructing a wide class of Hamiltonians for which analytic solutions can be found in both linear and nonlinear wave scattering theory [15, 16, 17]. In its turn, the inverse problem in the adiabatic representation extends the possibilities of the inverse scattering method in the investigation of many real quantum systems with complicated dynamics [20, 21, 14].

The procedure of the adiabatic representation is a variant of dimensional reduction of space, because it leads to two the effective scattering problems in the spaces of a lower dimension, than the original one. The method of analytic modelling in this approach is based on the consistent formulation in analytic form of both mutually connected problems: the parametric problem and the multichannel problem for the system of equations with a covariant derivative. The main peculiarity of this approach is that the potential and the basis wave functions have to be found from the scattering data parametrically depending on the spatial variables. Generalization of the Bargmann technique to the parametric family of inverse problems is based on the choice of Jost functions that have to be rational, as usual, but parametrically depending on the adiabatic variables through the dependence of spectral characteristics on them. In the consistent statement this dependence is determined by solving the inverse problem for the system of equations.

When the functional dependence of scattering data on the external adiabatic variable is given in advance, one initially recovers the potential by solving the parametric inverse problem and determines the basis functions. The matrix elements of the induced vector and scalar potentials can be constructed and studied in terms of first obtained exact solutions of the parametric problem. The full solution of the original problem is then obtained by solving the system of multichannel equations with respect to the expansion coefficients. In the general case, the solution of the system of equations has not to be an exact analytic one and demands numerical calculation.

By specifying the functional dependence of spectral characteristics on time through the dependence on it of the external dynamical variable $x(t)$ we elaborate algebraic procedure for constructing time-dependent multidimensional potentials and corresponding solutions of the parametric problem in a closed analytical form. Here we have considered a set of certain examples of exactly solvable models within the parametric problem on the entire line and on the half line. In particular, for a given functional dependence of spectral characteristics on the external coordinate variable, we reconstruct transparent symmetric and nonsymmetric potentials in another spatial coordinate with the pertinent solutions of the parametric task. In terms of the analytic basis functions, the matrix elements A_{nm} of exchange interactions are calculated within a straightforward procedure at different moments of time. The approach suggested permits investigation of adiabatically driven



quantum systems with prescribed dependence on parametric adiabatic variables, as well.

2 The Adiabatic Approach and Inverse Problem

Consider the system evolving according to the Schrödinger equation

$$i\hbar\frac{d|\Psi(t)>}{dt} = H(x(t))|\Psi(t)> \tag{1}$$

where the Hamiltonian $H(x(t))$ is given in the form

$$H(x(t)) = -\partial^2/\partial x^2 + h(x(t)).$$

If $\phi_n(x(t);y)$ are solutions to the equation

$$h(x(t))|\phi_n(x(t);y)>=\mathcal{E}_n(x(t))|\phi_n(x(t);y)>,\quad h(x(t)) = -\partial^2/\partial y^2 + V(x(t),y) \tag{2}$$

and form a complete orthonormal set $|\phi_n(x;y) >< \phi_m(x;y')| = \delta(y-y')$, $<\phi_n(x)\mid\phi_m(x)>=\delta_{nm}\ \forall x$, then Ψ can be expressed by the expansion

$$|\Psi(x,y)>=|n><n|\Psi>=\sum_n\int\phi_n(x;y)F_n(x) \tag{3}$$

over eigenstates of the self-adjoint parametric Hamiltonian (2). Upon substituting the expansion (3) into the initial Schrödinger equation (1) and using relations of orthonormalization, we arrive multichannel system of gauge equations

$$[-(\nabla_x + A(x))^2 + V(x)]F(x) = i\hbar\partial/\partial t F(x)\ . \tag{4}$$

Here $A(x)$ and $V(x)$ is the effective vector and scalar potentials, respectively, matrix elements of which are induced by the basis functions $\phi(x;y)$ of the parametric problem (2)

$$A_{nm}(x) =< \phi_n(x;y)|\nabla_x|\phi_m(x;y)>, \tag{5}$$

$$\begin{aligned} V_{nm}(x) &= <\phi_n(x;y)|h(x)|\phi_m(x;y)> - i\hbar <\phi_n(x;y)|\dot\phi_m(x;y)> \\ &\equiv \mathcal{E}_n(x)\delta_{nm} - i\hbar B_{nm}(x(t)). \end{aligned} \tag{6}$$

In the case of a slow and smooth evolution in time of the collective coordinates $x(t)$, the second derivative is neglected in (1), $H(x(t)) \to h(x(t))$, and the solution of equation (1) is sought in the form of the expansion (see, for instance, [22])

$$|\Psi(t,x(t),\mathrm{y})>=\sum_n c_n(x(t))exp\Big(-\frac{i}{\hbar}\int_0^t\mathcal{E}_n(x(t'))dt'\Big)|\phi_n(x(t);y)>. \tag{7}$$

With account of (2) the system of equations for $c_n(t)$ can be written in the form

$$\dot c_n(t) = \sum_m B_{nm}(x(t))exp[-\frac{i}{\hbar}\int_0^t(\mathcal{E}_n(t') - \mathcal{E}_m(t'))dt']c_m(t). \tag{8}$$

The matrix elements of exchange interaction

$$B_{nm}(x(t)) =< n|\dot m >= A_{nm}(x(t))\cdot\dot x(t)$$

are generated by basis functions $|n>$ of the "instantaneous" Hamiltonian (2).

Thus, the initial problem is reduced to the consistent solution of two problems (2) and (4) or (2) and (8). Here, we assume that $H(x(t))$ is real, limited and continuous in t. Because for each t the eigenfunctions are real valued and orthonormal

$$<\phi_n(x)\mid\phi_m(x)>=\delta_{nm}\quad\forall x$$

then the nonadiabatic couplings $A_{nm} = -A_{mn}$ in (5) are real and antisymmetric in n and m. The matrix elements (5) of the induced connection A can be computed in terms of the analytic eigenfunctions of equations (2) for a given functional dependence of scattering data $\{\mathcal{E}_n(x),\gamma_n^2(x),\mathcal{S}(x,k)\}$ on the slow coordinate variables $x(t)$ $(\mathcal{E}_n(x(t)) = -\kappa_n^2(x(t)))$. After that the transition amplitudes $c(t)$ can be determined from (8). In the consistent adiabatic approach, the parametric dependence of spectral characteristics on "slow" variables should be determined by solving the inverse problem for a system of gauge equations (4).

In accordance with the general definition of the inverse problem [18, 19], the parametric inverse problem [20] consists of the reconstruction of the potentials and corresponding solutions from known spectral data $\{\rho(x,k),N^2(x),\mathcal{E}(x)\}$ or the scattering data $\{S(x,k),\gamma_n^2(x),\mathcal{E}_n(x)\}$ parametrically depending on the coordinate variables x. This dependence reflects the peculiarity of the nonstandard parametric inverse problem. Specifying this dependence and employing the algebraic methods of inverse scattering problem, we present a wide class of Hamiltonians for which one can construct exactly solvable models and, consequently, derive solutions in a closed analytic form. These Hamiltonians with generalized Bargmann potentials [20] are defined by the rational Jost functions

$$f(x;k) = \prod\frac{k-i\alpha(x)}{k+i\beta(x)} \tag{9}$$

parametrically depending on the "slow" dynamical variables x through the dependence of spectral parameters on them. This situation is to a certain extent analogous to the theory of nonlinear evolution equations. The parametric Jost function (9) has N curves $k = -i\beta_j(x)$ of simple poles and N curves of simple zeros on the $k = i\alpha_j(x)$ defined as functions of the parametric variable x. In $\alpha(x)$ there are not only zeros on the imaginary semiaxis corresponding to bound states $\Re\ \kappa_j(x) = 0$, $\Im\ \kappa_j(x) > 0$ for each values of x but also zeros in the lower half-plane with $\Im\ \nu_j(x) < 0$ (the number of simple pole curves of $\beta_j(x)$ equals the total number of κ_j and ν_j). In this case the scattering matrix and spectral function assume the form

$$S(x;k) = \prod\frac{(k+i\alpha(x))(k+i\beta(x))}{(k-i\beta(x))(k-i\alpha(x))},\ \rho(x;k) = \prod\frac{(k-i\beta(x))(k+i\beta(x))}{(k+i\alpha(x))(k-i\alpha(x))}. \tag{10}$$

For such $S(x;k)$ and $\rho(x;k)$ the kernels of integral equations of the parametric inverse problem can be represented as sums of terms with a factorized dependence on the fast

variable y: $Q(x;y,y') = \sum_i^N B_i(x;y)B_i(x;y')$. When the kernel Q is inserted into the base parametric equation of the inverse problem

$$K(x;y,y') + Q(x;y,y') + \int_{y(0)}^{\infty(y)} K(x;y,y'')Q(x;y'',y')dy' = 0, \tag{11}$$

it is evident that the kernel of the generalized shift $K(x;y,y')$ also becomes degenerate: $K(x;y,y') = \sum_i^N K_i(x;y)B_i(x;y')$. As a consequence, the system of integral equations of the inverse problem is reduced to the system of algebraic equations. Then, the spherically nonsymmetric potential and solutions corresponding to it can be expressed in a closed analytic form in terms of the known solutions and spectral characteristics by using the generalized equations of the parametric inverse problem

$$V(x;y) = \overset{\circ}{V}(x,y) \mp 2\frac{d}{dy}K(x;y,y), \tag{12}$$

$$\phi(x;k,y) = \overset{\circ}{\phi}(k,y) + \int_{y(0)}^{\infty(y)} K(x;y,y')\,\overset{\circ}{\phi}(k,y')dy'. \tag{13}$$

Integration limits in (11), (13) and signs in (12) depend on the particular approach to the inverse problem. Limits from y to ∞ (from y to a) and minus sign correspond to the Marchenko approach. Limits $[0,y]$ and plus sign represent the Gelfand–Levitan approach.

2.1 Exactly solvable models in the Marchenko approach on the semi-axis

For the parametric inverse problem, radial or on a semi-axis, when $\overset{\circ}{V}(y) = 0$ and $\overset{\circ}{S}(k) = 1$ the kernel of the basic integral equation (11) in the Marchenko approach

$$\begin{aligned} Q(x;(y+y')) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}[1 - \mathcal{S}(x;k)]\exp[ik(y+y')]\,dk \\ &+ \sum_n^N M_n^2(x)\exp[-\kappa_n(x)(y+y')] \end{aligned} \tag{14}$$

with the scattering matrix (10) is rewritten as follows

$$Q(x;(y+y')) = -i\sum_n^N Res\ \mathcal{S}(k = ib_n(x))\exp[-b_n(x)(y+y')] \tag{15}$$

$$+\sum_n^N\{-iRes\ \mathcal{S}(k = i\kappa_n(x))\exp[-\kappa_n(x)(y+y')] + M_n^2(x)\exp[-\kappa_n(x)(y+y')]\}\ .$$

Following the procedure of constructing phase-equivalent potentials suggested in [17] for the one-dimensional problem and in [20] for the parametric problem, one can cancel out

the second summation in the right-hand side of (15) if the normalization functions $M_n^2(x)$ are chosen to be equal to $i\ Res\ \mathcal{S}(k)$ at $k = i\kappa_n(x)$

$$\overset{\circ}{M}{}_n^2(x) = i\ Res\ \mathcal{S}(k)|_{k=i\kappa_n(x)} \tag{16}$$

$$= -\frac{2\kappa_n(x)(\kappa_n(x)+b_n(x))}{(\kappa_n(x)-b_n(x))}\prod_{n'\neq n}^N \frac{(\kappa_n(x)+b_{n'}(x))(\kappa_n(x)+\kappa_{n'}(x))}{(\kappa_n(x)-b_{n'}(x))(\kappa_n(x)-\kappa_{n'}(x))}.$$

As a result, we obtain the simpler expression for the kernel $Q = \overset{\circ}{Q}$

$$\begin{aligned} \overset{\circ}{Q}(x;(y+y')) &= -i\sum_n^N Res\ \mathcal{S}(k)|_{k=ib_n(x)}\exp[-b_n(x)(y+y')] \\ &\equiv \sum_n^N A_n(x)\exp[-b_n(x)(y+y')], \end{aligned} \tag{17}$$

where

$$A_n(x) = \frac{2b_n(x)(b_n(x)+\kappa_n(x))}{(b_n(x)-\kappa_n(x))}\prod_{n'\neq n}^N \frac{(b_n(x)+\kappa_{n'}(x))(b_n(x)+b_{n'}(x))}{(b_n(x)-b_{n'}(x))(b_n(x)-\kappa_{n'}(x))}.$$

Inserting the kernel $\overset{\circ}{Q}(x;y,y')$ (17) into the parametric Marchenko equation (11), we obtain

$$\overset{\circ}{K}(x;y,y') + \sum_n^N A_n(x)\Big\{e^{-b_n(x)y} + \int_y^\infty \overset{\circ}{K}(x;y,y'')e^{-b_n(x)y''}dy''\Big\}e^{-b_n(x)y'} = 0,$$

where the expression in braces is the Jost solution $\overset{\circ}{f}_+(ib_n(x),y)$ for the sought potential $\overset{\circ}{V}(x;y)$. So that $\overset{\circ}{K}(x;y,y')$ has a form similar to that of $\overset{\circ}{Q}(x;y,y')$ from (17) with a separabilized dependence on y and y' and with a parametric dependence on x

$$\overset{\circ}{K}(x;y,y') = -\sum_n^N A_n(x)\,\overset{\circ}{f}_+(k = ib_n(x),y)\exp[-b_n(x)y']. \tag{18}$$

Substituting this kernel of generalized shift from the free wave to the Jost solution into the triangular integral equation

$$\overset{\circ}{f}(x;k,y) = \exp(iky) + \int_y^\infty \overset{\circ}{K}(x;y,y')\exp(iky')dy' \tag{19}$$

at $k = ib_n(x)$ we get a set of equations for $\overset{\circ}{f}(ib_n(x),y)$:

$$\overset{\circ}{f}(ib_n(x),y) = \sum_{n'}^N P_{nn'}^{-1}(x;y)\exp[-(b_{n'}(x)y]$$

where $P_{nn'}(x;y)$ is defined as follows

$$P_{nn'}(x;y) = \delta_{nn'} + A_n(x)\frac{\exp[-(b_n(x)+b_{n'}(x))y]}{b_n(x)+b_{n'}(x)}.$$

Then by substituting (18) into the parametric equations of the inverse problem (11) - (13) we obtain

$$\overset{\circ}{V}(x;y) = -2\frac{d^2}{dy^2}\ln\det||P(x;y)||; \tag{20}$$

$$\overset{\circ}{f}_{\pm}(x;k,y) = \exp(\pm iky) + \sum_{nn'}^{N} A_n(x)P^{-1}_{nn'}(x;y)\frac{\exp[-(b_n(x)+b_{n'}(x)\mp ik)y]}{(b_n(x)\mp ik)}. \tag{21}$$

The corresponding algebraic formulae for the one–dimensional Bargmann potentials and their solutions ([17]) can be directly obtained if we put $\kappa_n(x) \equiv \kappa_n$ and $b_n(x) \equiv b_n$.

Let us now present a case with two potential curves in the problem on the semi-axis. The Jost function (9) can be written in the form

$$f(x;k) = \frac{(k-i\kappa_1(x))(k-i\kappa_2(x))}{(k+i\beta_1(x))(k+i\beta_2(x))}. \tag{22}$$

We take the normalizations of bound state wave functions in the form (16)

$$M_1^2(x) = -\frac{2\kappa_1(x)(\kappa_1(x)+b_1(x))}{(\kappa_1(x)-b_1(x))}\frac{(\kappa_1(x)+b_2(x))(\kappa_1(x)+\kappa_2(x))}{(\kappa_1(x)-b_2(x))(\kappa_1(x)-\kappa_2(x))};$$

$$M_2^2(x) = -\frac{2\kappa_2(x)(\kappa_2(x)+b_2(x))}{(\kappa_2(x)-b_2(x))}\frac{(\kappa_2(x)+b_1(x))(\kappa_2(x)+\kappa_1(x))}{(\kappa_2(x)-b_1(x))(\kappa_2(x)-\kappa_1(x))}.$$

Thereby the potential $V(x,y)$ is determined only by the spectral data $\kappa_i(x)$ and $\beta_i(x)$, $i=1,2$ and corresponds to one of the phase–equivalent potentials. From normalizations $M_n^2(x)$ being positively definite the conditions $\beta_2(x) \geq \kappa_2(x)$ and $\beta_1(x) \geq \kappa_1(x)$ follow. The spectral data have been chosen in the following way

$$\kappa_1(x) = \frac{\sqrt{2}\kappa_1}{\cosh(\kappa_1 x - 4.8)}, \qquad \beta_1(x) = \frac{\sqrt{2}\beta_1}{\cosh(\beta_1 x - 5)} + 0.3,$$

$$\kappa_2(x) = \frac{\sqrt{2}\kappa_2}{\cosh(\kappa_2 x - 5)} + 0.4, \qquad \beta_2(x) = \frac{\sqrt{2}\beta_2}{\cosh(\beta_2 x - 5)} + 0.5,$$

$$\kappa_1 = 0.6,\ \beta_1 = \kappa_2 = \beta_2 = 0.7.$$

From the relations (20), (21) we obtain the two–dimensional potential $V(x,y)$ (Fig.1c) and the corresponding normalized wave functions $\psi_{1,2}(x,y)$ (Fig.1d,e) of the self-energy curves $\mathcal{E}_{1,2}(x)$. The behavior of the nonadiabatic connection (5), accomplished by the matrix element of the effective vector potential $A_{12}(x)$ is shown in Fig.1b under the potential curves Fig.1a.

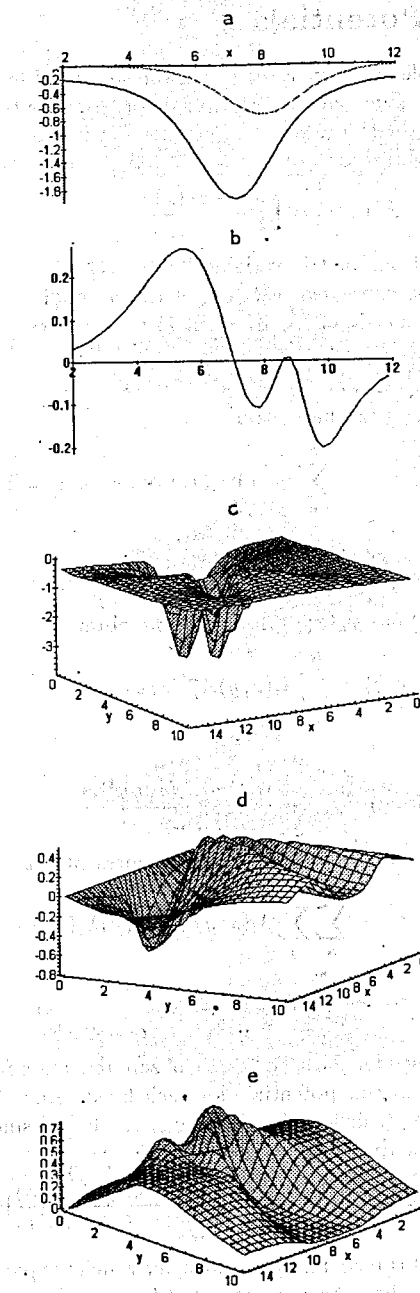


Fig.1. (**c**) Potential $V(x,y)$ with $\mathcal{S}(x;k)$ with 4 pole curves in the upper k half–plane, (**d,e**) the normalized eigenfunctions $\psi_{1,2}(x;y)$ at $k=-i\kappa_{1,2}(x)$. (**a**) The energy–level curves $\mathcal{E}_{1,2}(x) = -\kappa^2_{1,2}(x)$ and (**b**) matrix element $A_{12}(x)$ are shown in the upper figures.

3 Transparent Potentials

Let us consider quite a simple example of the use of the suggested technique. Reflectionless (transparent) potentials along the fast variable describe the one-dimensional inverse problem along the whole axis with the zero-th reflection coefficient, $\mathcal{S}^{ref} = 0$. The transmission coefficient $\mathcal{S}^{tr}$ with the absolute value equal to unity is a rational function

$$\mathcal{S}^{tr}(x;k) = \prod \frac{k + i\kappa(x)}{k - i\kappa(x)} \tag{23}$$

depending on the dynamical parametric variable "$x = x(t)$" The relations for the potentials and solutions can be expressed in terms of the normalized eigenfunctions and represented in the most symmetric and convenient form. Following [15], introduce the function

$$\lambda_n(x;y) = \gamma_n(x)\exp(-\kappa_n(x)y).$$

Then the formula for $K(x;y,y')$ can be written as

$$\begin{aligned} K(x;y,y') &= -\sum_n^N \gamma_n(x)\psi_n(x;y)\exp(-\kappa_n(x)y') \\ &= -\sum_n^N \psi_n(x;y)\lambda_n(x;y'). \end{aligned} \tag{24}$$

For the normalized eigenfunctions $\psi_n(x;y)$ from (13), we obtain

$$\psi_n(x;y) = \sum_j^N \lambda_j(x;y)A^{-1}_{jn}(x;y) \tag{25}$$

with the matrix $A_{jn}(x;y)$ given by

$$A_{jn}(x;y) = \delta_{jn} + \frac{\lambda_j(x;y)\lambda_n(x;y)}{\kappa_n(x) + \kappa_j(x)}. \tag{26}$$

Finally, the kernel $K(x;y,y')$ and the potential can be represented as

$$\begin{aligned} K(x;y,y') &= -\sum_n^N \sum_j^N \lambda_j(x;y)A^{-1}_{jn}(x;y)\lambda_n(x;y'), \\ V(x;y) &= -4\sum_n^m \kappa_n(x)\psi_n^2(x;y). \end{aligned} \tag{27}$$

Note, these relations are obtained for the specific case of zero reflection function $\mathcal{S}^r(x;k) = 0 \ \forall x$. Note, symmetric transparent potentials for each fixed value of x and appropriate wave functions are completely defined by the energy levels [23] since the normalized functions can be determined by the energy levels

$$\gamma_n^2(x) = iRes\mathcal{S}^{tr}(k)_{/k=i\kappa_n(x)} = 2\kappa_n(x)\prod_{m\neq n}\left|\frac{\kappa_m(x) + \kappa_n(x)}{\kappa_m(x) - \kappa_n(x)}\right|. \tag{28}$$

The set of time-independent transparent potentials, for which the parametric Schrödinger equations have exact solutions, have been constructed by us in [23], [24]. Here we give some examples of the reconstruction of time-dependent potentials that are symmetric and transparent in y (Figs.2) and nonsymmetric transparent potentials (Figs.5).

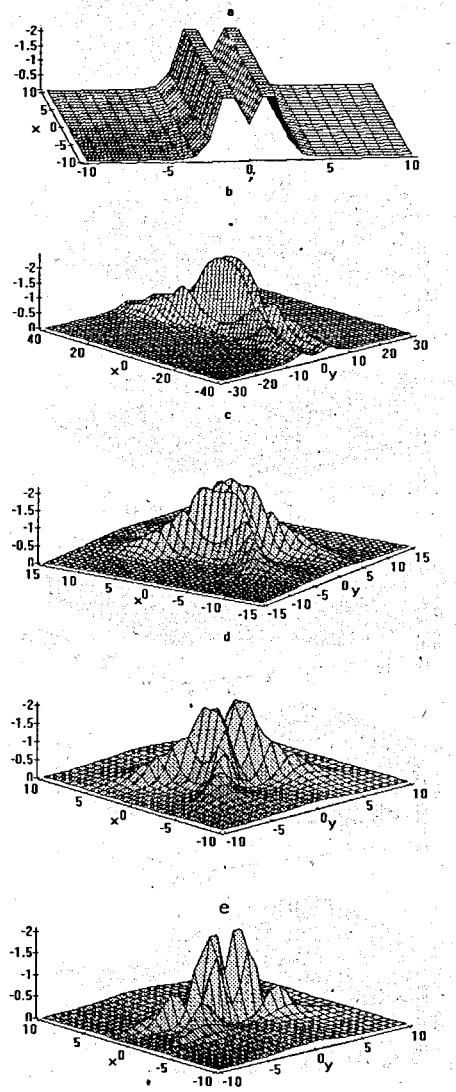


Fig.2. Potentials $V(x(t),y)$, transparent and symmetrical in y, calculated for different values of ωt: (**a**) $\omega t = 0$, (**b**) $\omega t = \pi/6$, (**c**) $\omega t = \pi/3$, (**d**) $\omega t = \pi/2$, (**e**) $\omega t = \pi$. To achieve a clearer presentation, the potential is inverted.

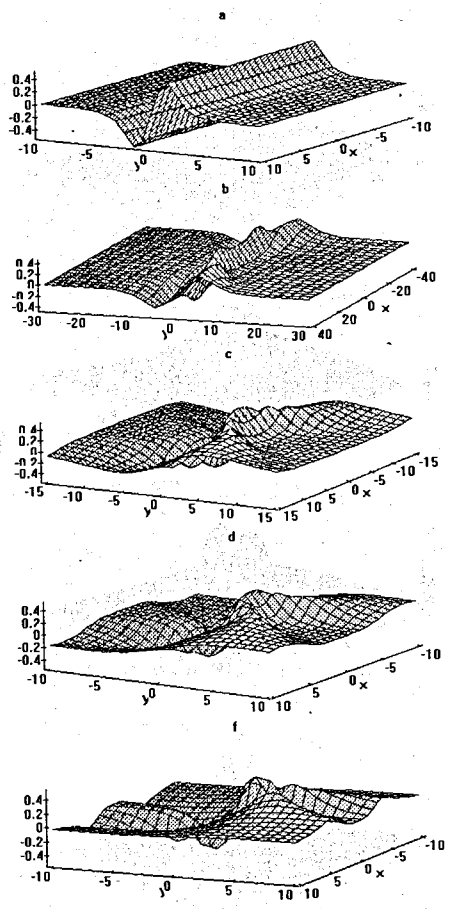


Figure 3: Eigenfunctions $\phi_1(x(t), y)$ at (a) $\omega t = 0$, (b) $\omega t = \pi/6$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

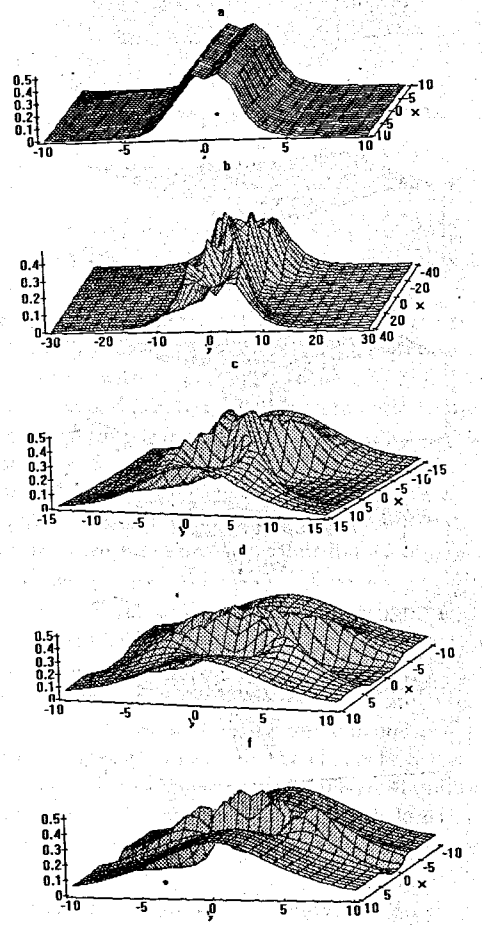


Figure 4: Eigenfunctions $\phi_2(x(t), y)$ at (a) $\omega t = 0$, (b) $\omega t = \pi/6$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

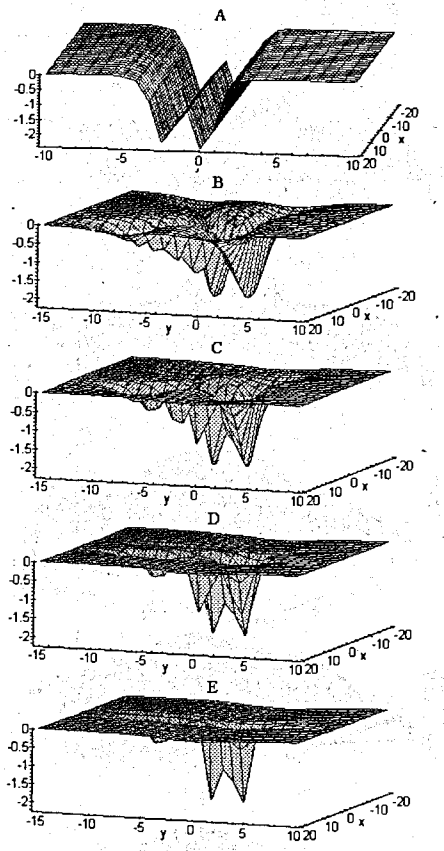


Fig.5. Potentials $V(x(t), y)$ calculated for different values of ωt: (a) $\omega t = 0$, (b) $\omega t = \pi/4$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

3.1 Exactly solvable models with time-dependent symmetric potentials

Consider a simple example of a two-dimensional exactly solvable model for a two-level system with a periodical dependence of the dynamical variable $x(t)$ on time. We define two terms in the following way

$$\mathcal{E}_1 = -1/ch^2(x/3), \quad \mathcal{E}_2 = -(1/ch(x/2) + 0.25)^2$$

with

$$x(t) = \overset{\circ}{x}\,(1 - a\ cos(\omega t)),$$

where $\overset{\circ}{x}$ corresponds to the time-independent case. We reconstruct symmetric transparent potentials and appropriate basis wave functions that are determined by the energy levels due to choosing the normalization functions $\gamma_n^2(x(t))$ in the form (28). The dynamical behavior of the potentials and the pertinent eigenfunctions is presented for ωt equal to $0, \pi/6, \pi/3, \pi/2, \pi$ in Figs.(2a,b,c,d,e) – (4a,b,c,d,e), respectively, and $a = 1$. Since their behavior is mirror-symmetric with respect to the line $\omega t \in (0, \pi)$, it is not shown. It is easy to see that the potential and functions change from very simple one-dimensional ones for $\omega t = 0$, Figs.2a – 4a, to quite complicate two-dimensional potentials and functions for all other values of $\omega t \neq 0$. In accordance with our choice of the potential curves, when $t = 0$, $x(0) = 0$ and bound state energies $\mathcal{E}_1 = -1$ and $\mathcal{E}_2 = -1.5625$ define the one-dimensional potential. The one-dimensional transparent potential with two bound states and corresponding wave functions are immediately obtained from more general relations [20] of the parametric task. When $\omega t \neq 0$, we have two-dimensional potentials and functions. At $\omega t = \pi/2$ and $\omega t = 3\pi/2$, $x(t)$ coincides with $\overset{\circ}{x}$ and we have the two-dimensional time-independent case, while at $\omega t = 2\pi$, the system comes back into the one-dimensional position with the initial states $|\phi_n(x(t); y) > \equiv |\phi_n(y) >$. Note, the eigenfunctions $\psi_2(x; y)$ are symmetric (Fig.3) and $\psi_1(x; y)$ are antisymmetric (Fig.4) in y at each fixed values of $x(t)$ as it is required for the problem on the entire axis $-\infty < y < \infty$ with a potential $V(x; y)$ symmetric in y. Matrix elements of exchange interaction A_{12} induced by basis functions of the parametric instantaneous Hamiltonian are equal to zero in this case. The explanation is very simple, one of the basis functions is strong symmetric in y and another is strong antisymmetric in "y" (Figs.3–4) and integrals

$$\int_{-\infty}^{\infty} \phi_1(x; y)\partial_x\phi_2(x; y)dy = -\int_{-\infty}^{\infty} \phi_2(x; y)\partial_x\phi_1(x; y)dy = 0.$$

The systems of equations (4) and (8) decompose on two systems uncoupled with each other. The transition amplitudes between even and odd states equal zero because they are defined by the zero matrix elements A_{nm}. As a consequence we received an important result: the choice of normalized functions influences on the behavior of dynamical quantum systems. Obviously, another parametric dependence of the spectral characteristics can be chosen, and other initial spectral data corresponding not only to the one– but also to two– or three-dimensional dependence on the extra coordinate variables can occur. In the next section we shall consider the influence of normalized functions on the behavior of potentials, basis wave functions and matrix elements $A(x)$ of an exchange interaction.

3.2 Nonsymmetric transparent potentials

The choice of normalizing functions $\gamma_n^2(x)$ of energy-level curves, that do not obey the condition (28), leads to loss of symmetry in "y" of potentials and to another behavior of parametric basis functions. As a result, the coupling between neighboring states (and also between even and odd states) takes place and the transitions takes place, too.

Let us define two potential curves as in the previous case: $\mathcal{E}_1 = -1/ch^2(x/3)$, $\mathcal{E}_2 = -(1/ch(x/2) + 0.25)^2$ with $x(t) = \overset{\circ}{x}(1 - a\ cos(\omega t))$, but normalized functions as $\gamma_n^2(x) = 2\kappa_n(x)$. The behavior of the potentials $V(x(t), y)$ and the pertinent normalized eigenfunctions $\phi_n(x(t), y) = \gamma_n(x(t))f(i\kappa_n(x(t), y))$, $n = 1, 2$ is presented for ωt equal to $0, \pi/4, \pi/3, \pi/2, \pi$ in Figs.(5a,b,c,d,e) – (7a,b,c,d,e), respectively, $a = 1$. It is easy to see that the potential and functions loose their symmetry in y as it takes place in choosing normalized functions in the form (28).

The behavior of the matrix elements of the nonadiabatic coupling $A_{12}(x(t))$ and $B_{12}(x(t))$ calculated at different instants by using (5) are pictured in Figs.8b ÷ 8e. Naturally, when $\omega t = 0$, A_{ij} and B_{ij} are absent, since the dependence on x vanishes. For $\omega t \neq 0$ all functions $A_{12}(x) = -A_{12}(-x)$ are antisymmetric and all $B_{12}(x) = B_{12}(-x)$ are symmetric with respect to the origin of the coordinates. The behavior of A is defined by our choice of spectral data and of the dependence of x on time. It is evident, $B(x)$ are symmetric functions since they are obtained as the product of both antisymmetric functions: $A(x)$ and $\dot{x}(t)$. Remember that the matrix elements of $A_{12}(x(t)) = -A_{21}(x(t))$ and $B_{12}(x(t)) = -B_{21}(x(t))$ are antisymmtric in the index of the state (here 1 and 2). The tendency of changing A as a function of ωt is the following: the amplitude of changing of A is the larger, the smaller the ωt. When ωt is small, the second pick to the right of the origin is comparable with the first pick. With increasing ωt, the second pick is decreasing. From our point of view this behavior of the exchange interaction A can be explained by the mutual influence of eigenstates. At small values of ωt the potential curves are close to each other on a larger interval than at large values of ωt. It is interesting to note, at $\omega t = \pi$ the matrix elements $B_{12}(x) = 0$ in spite of the adiabatic coupling $A_{12}(x) \neq 0$.

The full solution of the original problem (1) is then obtained by solving the systems of multichannel equations (4) or (8) with respect to the expansion coefficients. In general case the solutions of the systems of equations have not to be exact analytic solutions.

In such a way one may also investigate properties adiabatically driven quantum systems, Hamiltonians for which are slowly varying functions of time. In the case quantum systems are described by the equations (1) and (8). According to equation (8) transitions from an initial state $\Psi(t^o) = \phi(x(t^o))$ to a final state $\phi(x(t))$ are determined by the matrix elements of the evolution operator $\mathcal{U}(t, t^o)$ which satisfies the integral equation

$$\mathcal{U}_{nm}(t, t^o) = \delta_{nm} + \sum_{m'} \int_{t^o}^{t} B_{nm'}(x(s)) exp\Big[-\frac{i}{\hbar}\int_{t^o}^{s}(\mathcal{E}_n(t') - \mathcal{E}_{m'}(t'))dt'\Big]\mathcal{U}_{m'm}(s, t^0)ds. \quad (29)$$

One can solve this equation by iteration

$$\mathcal{U}_{nm}(t, t^o) = \delta_{nm} + \int_{t^o}^{t} B_{nm}(x(s)) exp\Big[-\frac{i}{\hbar}\int_{t^o}^{s}(\mathcal{E}_n(t') - \mathcal{E}_m(t'))dt'\Big]ds + \dots . \quad (30)$$

The transition amplitudes in this case are defined by the matrix elements of an exchange interaction $B_{nm}(x)$ that can be calculated in terms of analytic basis functions (8) and (5).

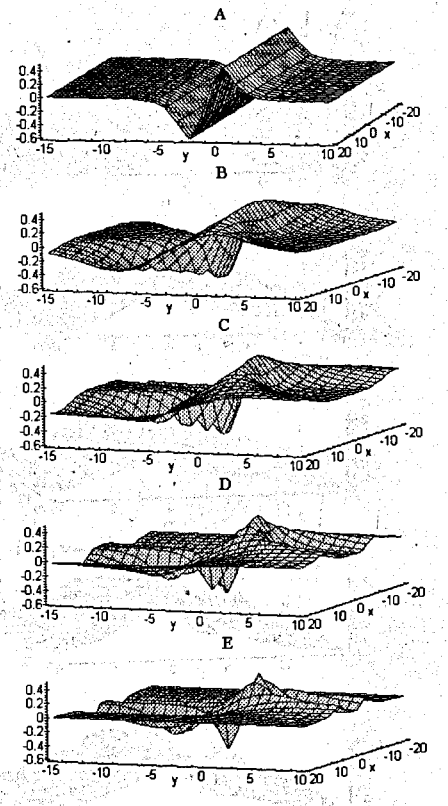


Figure 6: Eigenfunctions $\phi_1(x(t), y)$ at (a) $\omega t = 0$, (b) $\omega t = \pi/4$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

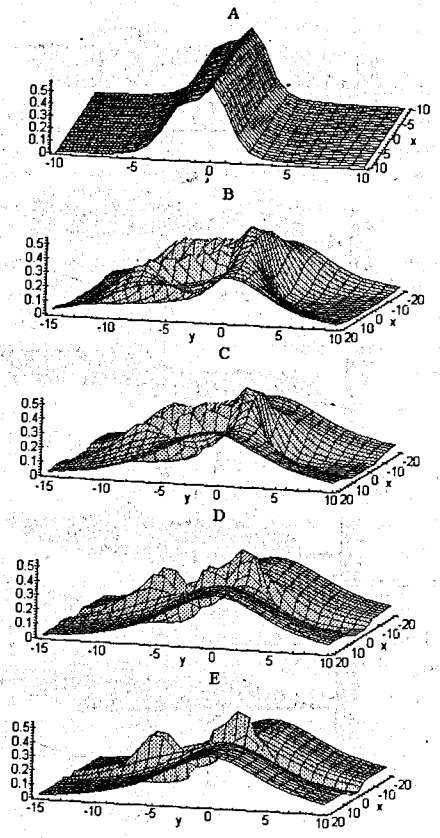


Figure 7: Eigenfunctions $\phi_2(x(t), y)$ at (a) $\omega t = 0$, (b) $\omega t = \pi/4$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

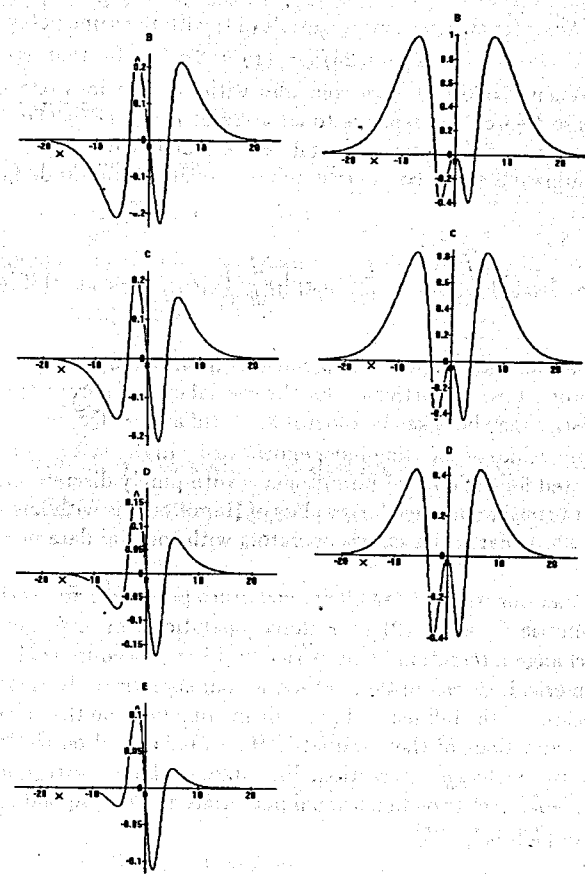


Fig.8 Matrix elements of exchange interections $A_{12}(x)$ (the left column) and $B_{12}(x)$ (the right column) calculated for different values of ωt: (b) $\omega t = \pi/4$, (c) $\omega t = \pi/3$, (d) $\omega t = \pi/2$, (e) $\omega t = \pi$.

Consider the concrete example of the smooth behavior of $x(t)$ studied in [10]

$$x(t) = \overset{\circ}{x}\,(1 + a\ exp(-1/t)), \tag{31}$$

but spectral data we take as for transparent potentials (23) with the same potential curves $\mathcal{E}_1 = -1/ch^2(x/3)$, $\mathcal{E}_2 = -(1/ch(x/2) + 0.25)^2$, $\gamma_n^2(x) = 2\kappa_n(x)$. So that $x(t=0) = \overset{\circ}{x}$, $x(\infty) = (1+a)\ \overset{\circ}{x}$. When $t = 0$ this case coincides with the previous one at $\omega t = \pi$ (Figs. 5d – 7d), the case $t \to \infty$ corresponds to $\omega t = 2\pi$ at $a = 1$ (Figs. 5e – 7e). It is evident from the comparison of the figures 8d and 8e the matrix element $A_{12}(t)$ changes very smoothly in the interval $0 \le t < \infty$. Therefore the transition amplitude takes a very simple form

$$\mathcal{U}_{nm}(\infty, 0) = \delta_{nm} + A_{nm}(\overset{\circ}{x}) \int_0^\infty \partial_s (1 + e^{-1/s}) exp\Big[-\frac{i}{\hbar}\int_0^s (\mathcal{E}_n(t') - \mathcal{E}_m(t'))dt'\Big] ds. \tag{32}$$

It is quite evident, we can choose another dependence on time, and other initial spectral data with prescribed properties. In particular, for the special case of parametric variation, the spectral characteristics may be taken in a factorized form as in [10] $\mathcal{E}_n(x) = x\widetilde{\mathcal{E}}_n$ where $x = x(t)$ and $\widetilde{\mathcal{E}}_n$ are eigenvalues of the time-independent task. In the last paper, transition amplitudes were evaluated for the class of Hamiltonians with purely discrete spectra. Our approach permits one to consider a much larger class of Hamiltonians with known discrete and continuous parts, whose particular case is operators with spectral data in a factorized form.

We take attention that our method permits to construct potentials and corresponding solutions of the parametric equation (2) in a closed analytic form and computing the matrix elements of exchange interaction by numerically. First procedure is algebraic one, but second step is numerical, therefore the method is semianalytical. Nevertheless, this approach allows to investigate the influence of normalizing functions on the behavior of the potentials, on the basis functions of the parametric Hamiltonian and on the behavior of the matrix elements of the exchange interaction. Examples of the reconstruction of time-dependent and time-independent two–dimensional potentials and corresponding solutions have been considered in [23], [24], [25].

4 Conclusions

In conclusion, we would like to note that this approach allows one to construct a very wide class of two-dimensional exactly solvable models with time-dependent and time-independent potentials. Some concrete examples of two-dimensional potentials with corresponding analytic solutions of the nonstandard parametric problem are considered on the entire axis and on the semi-axis. We have presented an examples of the two–dimensional time-dependent transparent symmetric and nonsymmetric potentials with their basis functions for two-level systems. We applied these exact wave functions for calculating the matrix elements of nonadiabatic couplings determining the exchange interaction.

We find out the exchange interaction between the neighboring states is equal to zero for transparent symmetric potentials with respect to fast coordinate variables at each fixed value of adiabatic variable. In fact, we can trace the behavior of the matrix elements of

exchange interaction (5) at any moment of time and recommend our approach for the investigation of the Landau–Zener transitions and level crossing problems.

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