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LOW-ENERGY EXPANSIONS FOR THE ONE-DIMENSIONAL SHRÖDINGER SCATTERING PROBLEM

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1. Introduction

In the present work, we continue our previous study [1] of the one-dimensional Schrödinger scattering problem for the superposition of the repulsive Coulomb potential and any central potential vanishing more rapidly than the centrifugal one. We construct now the low-energy asymptotics for all the functions characterizing the scattering for the above potential superposition.

The knowledge of the low-energy dependence of the functions characterizing the collision of quantum mechanical objects allows one to solve a series of important practical and theoretical problems. For example, an experimental problem of extrapolating the characteristics to the low-energy region which is inaccessible for direct experimental study, and a theoretical problem of choosing the form and parameters of the interaction to describe the considered collision in the low-energy limit.

For these reasons, a construction of the low-energy approximations and a correct definition of the coefficients for these approximations are the important problems of scattering theory. The scattering length is one of the coefficients.

First, the concept of the scattering length has been introduced [2] for the scattering by the central potential $V^s(r)$ decreasing at large distances r as (or more rapidly),

$$V^s(r) \sim V^Y(r) \quad \Big(\text{or} \quad V^s(r) = o\big(V^Y(r)\big) \Big), \quad r \to \infty , \tag{1}$$

than the Yukawa potential

$$V^Y(r) \equiv V_0 \exp(-\mu r)/r , \quad \mu > 0 , \quad V_0 = \text{ const} . \tag{2}$$

Usually [3, 4] the potentials of that kind are called the short-range potentials, whereas the potentials vanishing as an inverse power of the distance,

$$V^l(r) \sim V_0^d\, r^{-d} , \quad V_0^d = \text{const} , \quad d = 1, 2, 3, \ldots , \quad r \to \infty , \tag{3}$$

are called the long-range potentials.

For the phase-shift $\delta_\ell^s(k)$ generated by the short-range potential V^s in the state $|k, \ell\rangle$ with the scattering momentum k and angular momentum ℓ, the limit

$$a_\ell^s \equiv - \lim_{k \to 0} \tan \delta_\ell^s(k)/k^{2\ell+1} \tag{4}$$

is finite and the corresponding effective-range function

$$K_\ell^s(E) \equiv k^{2\ell+1} \text{ cotan } \delta_\ell^s(k) \tag{5}$$

is the entire function of k^2. The coefficients a_ℓ^s, $r_{0\ell}^s$ and P_ℓ^s in the low-energy ($E = k^2 \to 0$) expansion [2] of this function

$$K_\ell^s(E) \sim -1/a_\ell^s + k^2\, r_{0\ell}^s/2 - k^4\, r_{0\ell}^s\, P_\ell^s\,, \quad k \to 0\,, \tag{6}$$

are called the scattering length, effective range and shape-parameter, respectively, for the scattering in the state $|k,\ell\rangle$. Owing to (4) and (5), the phase-shift $\delta_\ell^s(k)$ vanishes as the power function $-a_\ell^s\, k^{2\ell+1}$ of k as $k \to 0$, and the scattering length has a meaning of the coefficient defining a slope of this function of the variable $-k^{2\ell+1}$.

In the low-energy limit, the total scattering phase $\delta_\ell^{cs}(k)$ generated by the superposition $V^{cs} \equiv V^c + V^s$, where $V^c = 1/rR$ is the repulsive Coulomb potential and $R > 0$ is the Bohr radius [5], rises to infinite as the pure Coulomb phase-shift $\delta_\ell^c(k)$:

$$\delta_\ell^{cs}(k) \sim \delta_\ell^c(k) \sim \eta(\ln\eta - 1)\,, \quad \eta \equiv 1/2kR\,, \quad k \to 0\,.$$

Therefore, the analog of the limit (4), i.e.,

$$a_\ell^{cs} \equiv -\lim_{k\to 0} \tan\delta_\ell^{cs}(k)/k^{2\ell+1} = -\lim_{k\to 0}\left(2Rk^{2\ell+2}\right)^{-1},$$

is equal to $-\infty$. This is the first reason why, for the superposition V^{cs}, the concepts of the scattering length and effective-range function should be redefined. The second reason is the following. The phase-shift $\delta_\ell^{cs}(k)$ is the result of the joint influence of V^c and V^s on the scattering. To extract the contribution of the short-range potential, one has obviously to subtract from the total phase-shift $\delta_\ell^{cs}(k)$ the pure Coulomb one $\delta_\ell^c(k)$ and then to redefine the effective-range function in the form of the entire function of k^2. This method was used by Breit, Condon and Present [6] to define of scattering length in the proton-proton scattering for the superposition V^{cs}. These authors introduced the concept of the Coulomb-nuclear phase-shift

$$\delta_\ell^{c,s}(k) = \delta_\ell^{cs}(k) - \delta_\ell^c(k) \tag{7}$$

and proved that the relevant Coulomb-nuclear effective-range function

$$K_\ell^{c,s}(E) \equiv \left(k^\ell C_\ell(\eta)\right)^2\left(k\cot\delta_\ell^{c,s}(k) + h^c(\eta)\right) \tag{8}$$

should contain the analytically known Coulomb factors [7]

$$C_\ell(\eta) \equiv 2^\ell \exp(-\pi\eta/2)\,|\Gamma(\ell+1+i\eta)|/\Gamma(2\ell+2)\,, \tag{9}$$

$$h^c(\eta) \equiv h(\eta)/RC_0^2(\eta)\,, \quad h(\eta) \equiv \mathrm{Re}\,\Psi(i\eta) - \ln\eta \tag{10}$$

providing the desired asymptotics

$$K_\ell^{c,s}(E) \sim -1/a_\ell^{c,s} + k^2\, r_{0\ell}^{c,s}/2 - k^4\, r_{0\ell}^{c,s}\, P_\ell^{c,s}\,, \quad k \to 0\,. \tag{11}$$

As far as both the expansions (6) and (11) have an analogous functional dependence on k^2, the coefficients

$$a_\ell^{c,s}(k) \equiv -\lim_{k\to 0}\tan\delta_\ell^{c,s}(k)/\left[k(k^\ell C_\ell(\eta))^2\right], \tag{12}$$

$r_{0\ell}^{c,s}$ and $P_\ell^{c,s}$ are usually called [3, 8] the Coulomb-nuclear scattering length, effective range and shape-parameter, respectively.

The problem of defining the scattering length for the superposition $V^{cls} \equiv V^c + V^l + V^s$ of the Coulomb and long- and short-range potentials occurred to be very complicated. The total scattering phase-shift $\delta_l^{cls}(k)$ generated by V^{cls} can be expanded in two different ways:

$$\delta_\ell^{cls}(k) = \delta_\ell^c(k) + \delta_\ell^{c,ls}(k)\,; \tag{13}$$

$$\delta_\ell^{cls}(k) = \delta_\ell^{cl}(k) + \delta_\ell^{cl,s}(k)\,, \quad \delta_\ell^{cl}(k) = \delta_\ell^c(k) + \delta_\ell^{c,l}(k)\,. \tag{14}$$

It should be emphasized that both these decompositions are mathematically equivalent. However, from the physical point of view, there is a nonequivalence of the following conceptual meaning. The phase-shift $\delta_\ell^{c,ls}(k)$ characterizes the joint influence of the two interactions V^l and V^s on the scattering in the Coulomb field V^c, whereas the phase-shift $\delta_\ell^{cl,s}(k)$ is the contribution generated only the short-range potential V^s acting in the long-range field $V^{cl} \equiv V^c + V^l$. Therefore, to extract the information on the structure of V^s from the measured scattering phase $\delta_\ell^{cls}(k)$, one needs a reliable method of separating of the phase-shift $\delta_\ell^{cl,s}(k)$ and constructing its low-energy asymptotics. These problems are very general and important. In fact, these problems arise in studying a role of a given short-range potential responsible for nuclear forces in molecular, atomic and nuclear collisions running at extremely low energies and, vice versa, in extracting the information on the nuclear forces from the measured cross-sections in the corresponding collisions.

The low-energy scattering of a charged projectile (atom, ion, molecule or nucleus) by a target with an extended charge or magnetic distributions can be described using the effective two-body approximation. In this approximation, the total interaction is represented as the relevant superposition V^{cls} of the pure Coulomb interaction V^c, long-range electromagnetic correction V^l to this interaction, and short-range potential V^s responsible for the pure nuclear interaction.

The electromagnetic corrections are due to the long-range potentials having the asymptotics (3) with $d = 2, 3, 4 \ldots$. For example, the magnetic-moment correction [9]

$$V^m(r) = -\Big(3(\vec{\mu}_1 \cdot \vec{r})\,(\vec{\mu}_2 \cdot \vec{r})\, r^{-2} - (\vec{\mu}_1 \cdot \vec{\mu}_2) \Big)\, r^{-3}, \tag{15}$$

describes the interaction of the magnetic moments $\vec{\mu}_1$ and $\vec{\mu}_2$ of two nucleons and has the inverse-cubic r-dependence; the polarization correction V^p describes the interaction of a charged projectile with the electric moment of the deuteron (considered as a point-like particle) and has the asymptotics [10]

$$V^p(r) \sim -\alpha_e/2Rr^4, \quad r \gg R, \tag{16}$$

where α_e stands for the deuteron electric polarizability.

A detail theoretical investigation of the scattering by the superposition V^{cls} for a class of the long-range potentials with asymptotics (3) was made by Berger and Spruch [11] and by Berger, Snodgrass, and Spruch [12]. As was proved by Berger and Spruch, in the case $V^c > 0$ and $d \geq 3$, the phase-shifts $\delta_\ell^{c,ls}(k)$ and $\delta_\ell^{c,l}(k)$ of (13) and (14) vanish as $k \to 0$ more slowly than the Coulomb-nuclear phase-shift $\delta_\ell^{c,s}(k)$ of (7),

$$\delta_\ell^{c,ls}(k) \sim \delta_\ell^{c,l}(k) \sim (-V_0^d/2R^{1-d})\, k^{2d-3}\, B(d-1, 1/2), \quad k \to 0, \tag{17}$$

where B is the beta-function [7], and, therefore, the direct analog

$$a_\ell^{c,ls} \equiv -\lim_{k\to 0} \tan \delta_\ell^{c,ls}/k^{2\ell+1} C_\ell^2(\eta) = -\lim_{k\to 0} k^{2(d-\ell-1)} \exp(\pi/kR) \tag{18}$$

of the Coulomb-nuclear scattering length (12) is infinite and a physically meaningless value. Thus, the problem of defining the scattering length for V^{cls} has arisen. To resolve this problem, Berger, Snodgrass, and Spruch used decomposition (14), introduced the appropriate analog

$$K_\ell^{cl,s}(E) \equiv \big(k^\ell C_\ell^{cl}(\eta) \big)^2 \big(k \cot \delta_\ell^{cl,s}(k) + h^{cl}(\eta) \big) \tag{19}$$

of the Coulomb-nuclear effective-range function (8), proved the asymptotics

$$K_\ell^{cl,s}(E) \sim -1/a_\ell^{cl,s} + k^2\, r_{0\ell}^{cl,s}/2 - k^4\, r_{0\ell}^{cl,s}\, P_\ell^{cl,s}, \quad k \to 0, \tag{20}$$

and defined the relevant modified scattering length

$$a_\ell^{cl,s} \equiv -\lim_{k\to 0} \tan \delta_\ell^{cl,s}(k) / \Big[k (k^\ell\, C_\ell^{cl}(\eta))^2 \Big], \tag{21}$$

effective range $r_{0\ell}^{cl,s}$, and shape-parameter $P_\ell^{cl,s}$ as the coefficients in (20). The modified scattering parameters thus defined characterize the influence of V^s alone on the scattering for V^{cls}.

Though the above problem was resolved from the conceptual point of view, the Berger and Spruch and the Berger, Snodgrass, and Berger formulae for the coefficients in expansion (20) are too complicated for practical calculations. Moreover, in their method, the factors $C_\ell^{cl}(\eta)$ and $h_\ell^{cl}(\eta)$ are expressed in terms of the defined integrals containing the regular and irregular scattering wavefunctions for V^{cl}. These integral representations are also very complicated to be used for both the analytical and numerical studies of $C_\ell^{cl}(\eta)$ and $h_\ell^{cl}(\eta)$ as $k \to 0$.

The consideration of the scattering length for V^{cls} was renewed by Kvitsinsky and Merkuriev [13] in 1984. As they noted in [14], due to the proton-deuteron (pd) polarization interaction (16) the dublet and triplet pd-scattering lengths defined by (18) have no physical meaning. After this, the polarization effects in low-energy nuclear collisions where analyzed in an number of papers (see [15] and [16] and references therein). The most complete low-energy scattering theory for the Coulomb plus the long-range polarization potentials was constructed by Bencze et al. [17]. As a power method they used the variable phase approach [18, 19]. For the S-wave scattering ($\ell = 0$) by V^{cls} with $V^l = V^p$, Benze et al. gave a mathematically rigorous proof that, when at least the modified scattering length is considered, it is possible to replace $C_\ell^{cl}(\eta)$ by $C_\ell(\eta)$ in (21) and then use the resulting relation

$$\tilde{a}_\ell^{cl,s} \equiv -\lim_{k\to 0} \tan \delta_\ell^{cl,s}(k) / \Big[k (k^\ell\, C_\ell(\eta))^2 \Big] \tag{22}$$

as a physically correct definition for the modified scattering length.

However, the Benze et al. theory is incomplete, because the main theoretical questions were unsettled. These questions are: what is the structure of the factors C_ℓ^{cl} and h^{cl} in (19) and how should one calculate the coefficients $r_{0\ell}^{cl,s}$ and $P_\ell^{cl,s}$ in (20). There are still no reliable and simple methods for calculating these values.

The main goal of this work is to present such a method in the Coulomb repulsion ($V^c > 0$) case. Following our previous work [1], we use the system of units $\hbar = 2\mu = 1$. We also introduce the dimensionless variable $x \equiv r/R$ and parameter $q \equiv kR$ and present the studied Schrödinger scattering problem as differential equations

$$\big(\partial_x^2 - \ell(\ell+1)x^{-2} - V^c(x) - V(x) + q^2 \big)\, u_\ell^\pm(x,q) = 0, \quad x \in \mathcal{R}^+, \tag{23}$$

for the sought regular (u_ℓ^+) and irregular (u_ℓ^-) wavefunctions obeying the corresponding boundary conditions:

$$u_\ell^\pm(x,q) = O(x^{\pm(\ell+1/2)+1/2}), \quad x \to 0, \qquad (24)$$

$$u_\ell^\pm(x,q) \to \sin(\rho - \eta \ln 2\rho - (2\ell+1\mp 1)\pi/4 + \delta_\ell^c(q) + \delta_\ell(q)), \, x \to \infty, \qquad (25)$$

Here, $\mathcal{R}^+$ denotes the positive half-axis, $\rho = kr = qx$, $\eta = 1/2q$, and $\delta_\ell(q)$ stands for the phase-shift generated by the potential V in the repulsive Coulomb field $V^c = 1/x$. We assume that V is an arbitrary central potential so that

$$I_\ell(b,x) \equiv \left(2\pi/(\ell+1)\right)^{1/2} \int_b^x t|V(t)|dt < \infty, \quad 0 \le b \le x \le \infty. \qquad (26)$$

This relation is sufficiently general [4] and valid for a wide class of the short-range potentials obeying (1), long-range ones having asymptotics (3) with $d > 2$ and, certainly, for the sum V^{ls} of the above potentials. When it is necessary, we will specify the potential V as V^s, V^l or V^{ls}.

As far as we be dealing with the specifics low-energy ($q \to 0$) expansions for the auxiliary (Coulomb and amplitude) functions and the wavefunctions, it is necessary to make some things clear. All the expansions will asymptotic infinite series of the form

$$S(x,q) = N(q) \sum_{n=0}^{\infty} q^{2n} S_n(x), \quad q \to 0, \quad x \in \mathcal{R}^+, \qquad (27)$$

where $N(q)$ denotes the normalization factor and, what is very important, the argument x is separated from the vanishing parameter q. We expand each studied series S of (27) in two parts: the finite sum $S^{(M)}$ and residual term ${}^{(M)}S$:

$$S = S^{(M)} +^{(M)} S, \quad S^{(M)}(x,q) \equiv N(q) \sum_{n=0}^{M} q^{2n} S_n(x). \qquad (28)$$

We would like to point out that such a reliable and simple (from the computational point of view) method for constructing expansions (27) for the two-body scattering wavefunctions is unknown in the present scattering theory [4]. We would like also to stress that the low-energy representations for the three-body wavefunctions and their phase-shifts are not constructed [20] yet. To derive these representations, one need to know the low-energy expansions for the two-body subsystem wavefunctions, because these functions are contained in the physical boundary conditions for the Faddeev components of the studied three-body wavefunction.

For these reasons, the analysis of the ordinary two-body scattering problem (23)-(26) with $q \to 0$ is also important. Therefore, in Sect.2, we will pay a special attention to the **complete** study of the structure of both the solutions $u_\ell^\pm$ of this problem. We hope that this analysis will contributes to present scattering theory and also will be useful to treat analytically the three-body problem at comparatively small total energy. One of the interesting problems of such a kind is a study of the role of magnetic-moment interaction (15) in the neutron-deuteron (nd) and proton-deuteron (pd) scatterings. The matter is that this interaction generates the neutron analog [21] of the Ramsauer effect in the triplet nn-scattering [22]. Therefore, it seems to be quite reasonable to expect that some new peculiarities caused by the long-range part (15) of the total nucleon-nucleon interaction will be found in the slow nd- and pd-collisions.

However, before discussing the asymptotical method for the three-body problem, we have to finish our investigation of the ordinary two-body problem (23)-(26). In Sect.3, we will exemplify how the expansions for the auxiliary and wavefunctions $u_\ell^\pm$ can be used in solving some problems in two-body low-energy scattering theory. As the examples, we will give a new proof for expansions (11) and (20) and propose simple relations for calculating their coefficients.

2. Low-energy expansions for auxiliary and wavefunctions

The linear form (in fact it is equivalent with the varying constant coefficients [23, 24]) of the variable phase approach [18] is a powerful method for studying the various solutions of the Schrödinger equation. In [25] Calogero formulated the linear form in terms of the amplitude functions for calculating the regular scattering wavefunction and its phase-shift. Recently, this form was completed in a simple way for constructing the irregular scattering wavefunction [1], in studying of the artificial and physical resonances [26] and was combined with the complex coordinate rotation method [27] for calculating the Jost function [28].

We extend now the linear form of the variable phase approach for constructing all low-energy expansions for problem (23)-(26). First, we will remind how this problem is reformulated using this form and recall some basic formulae and facts proved in [1].

2.1. Reformulation of the initial problem

The sought wavefunctions $u_\ell^\pm$ are represented in terms of the amplitude ($c_\ell^\pm$ and $s_\ell^\pm$) and Coulomb (F_ℓ and G_ℓ) functions as follows:

$$u_\ell^+(x,q) = N_\ell^+(q)\, U_\ell^+(x,q),$$

$$u_\ell^-(x,q) = \tilde{u}_\ell^-(x,q) + \alpha_\ell(q)\, u_\ell^+(x,q)\,, \quad \tilde{u}_\ell^-(x,q) = N_\ell^-(q)\, U_\ell^-(x,q)\,. \tag{29}$$

$$U_\ell^\pm(x,q) \equiv c_\ell^\pm(x,q)\, F_\ell(\rho,\eta) + s_\ell^\pm(x,q)\, G_\ell(\rho,\eta)\,. \tag{30}$$

The cotangent of the phase-shift $\delta_\ell(q)$ and the normalization factors $N_\ell^\pm(q)$ and $\alpha_\ell(q)$ providing the asymptotics (25) are defined as the limits

$$A(q) \equiv \lim_{x\to\infty} A(x,q) \tag{31}$$

of the relevant functions $A(x,q)$ given by

$$\text{cotan}\ \delta_\ell(x,q) \equiv c_\ell^+(x,q)/s_\ell^+(x,q)\,, \tag{32}$$

$$N_\ell^\pm(x,q) \equiv \left(\left(c_\ell^+(x,q)\right)^2 + \left(s_\ell^+(x,q)\right)^2\right)^{\mp 1/2}, \tag{33}$$

$$\alpha_\ell(x,q) \equiv -c_\ell^+(x,q)\, c_\ell^-(x,q) - s_\ell^+(x,q)\, s_\ell^-(x,q)\,. \tag{34}$$

By definition, the amplitude functions obey the Lagrange identity [24]

$$F_\ell(\rho,\eta)\, \partial_x c_\ell^\pm(x,q) + G_\ell(\rho,\eta)\, \partial_x s_\ell^\pm(x,q) \equiv 0$$

and satisfy the two sets of the ordinary first-order differential equations (the first one for c_ℓ^+, s_ℓ^+ and the second one for c_ℓ^-, s_ℓ^-):

$$\partial_x \left\{ \begin{array}{c} c_\ell^\pm(x,q) \\ s_\ell^\pm(x,q) \end{array} \right\} = q^{-1}\, V(x)\, U_\ell^\pm(x,q) \left\{ \begin{array}{c} +G_\ell(\rho,\eta) \\ -F_\ell(\rho,\eta) \end{array} \right\} \tag{35}$$

with the corresponding asymptotical ($x \to 0$) boundary conditions

$$\left\{ \begin{array}{c} c_\ell^+(x,q) \\ s_\ell^+(x,q) \end{array} \right\} \sim \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} + q^{-1} \int_0^x V(t)\, F_\ell(\rho,\eta) \left\{ \begin{array}{c} +G_\ell(\rho,\eta) \\ -F_\ell(\rho,\eta) \end{array} \right\} dt\,, \tag{36}$$

$$c_\ell^-(x,q) \sim c_\ell^-(x_0,q) + q^{-1} \int_{x_0}^x V(t)\, G_\ell^2(\rho,\eta)\, dt\,,$$

$$s_\ell^-(x,q) \sim 1 - q^{-1} \int_0^x V(t)\, F_\ell(\rho,\eta)\, G_\ell(\rho,\eta)\, dt \tag{37}$$

that ensure the asymptotics (24) of functions (29). In (36) and (37), $\rho = tq$ and if $V(x) G_\ell^2(\rho,\eta) \in \mathcal{L}^1_{[0,b]}$, then $x_0 = 0$ and $c_l^-(0,q) = 0$, otherwise x_0 is an arbitrary, however fixed parameter so that $x_0 > x$ and $x_0 q \ll 1$. In this special case, the value $c_\ell^-(x_0,q)$ is defined from the Wronskian relation

$$c_\ell^+(x,q)\, s_\ell^-(x,q) - c_\ell^-(x,q)\, s_\ell^+(x,q) \equiv 1 \tag{38}$$

for problem (35)-(37) as follows. The values $c_\ell^+(x_0,q)$ and $s_\ell^\pm(x_0,q)$ found using (36) and (37) are substituted into (38). Then the obtained equation is resolved in respect to $c_\ell^-(x_0,q)$.

So, for constructing the solutions $u_\ell^\pm$ of the initial problem (23)-(26) by formulae (29) and (30), one should solve problems (35)-(37) and then find the limit (31) for each function (32)-(34). Equations (35) are the linear first order differential equations. Such equations are comparably simple [23] for the analysis using various asymptotical methods [29, 30], e.g. using the method of separating of an argument from a parameter. To apply this method to the total problem (35)-(37), we should firstly represent the Coulomb functions as series of (27). Therefore, we will rewrite the two known formulae for these functions in the form that we need.

2.2. The Coulomb functions

In nonrelativistic quantum mechanics [5], the Coulomb functions $F_\ell(\rho,\eta)$ and $G_\ell(\rho,\eta)$ with real $\eta > 0$ and integer $\ell = 0, 1, \ldots$ are well-known as the exact regular and irregular solutions of the Schrödinger scattering problem (23)-(25) with $V \equiv 0$ and the repulsive Coulomb potential $V^c = 2\eta/\rho$.

As was shown by Lambert [31], the function G_ℓ is the sum containing the entire function Θ_ℓ of the squared scattering momentum (k^2) and the function F_ℓ. We rewrite this sum (Eq.(3.25) of Ref. [31]) as

$$G_\ell(\rho,\eta) = \tilde{G}_\ell(\rho,\eta) + h^c(q)\, F_\ell(\rho,\eta)\,, \quad \tilde{G}_\ell(\rho,\eta) \equiv \Theta_\ell(x,q)/C_\ell^2(q)\,, \tag{39}$$

$$C_\ell(q) \equiv q^\ell C_\ell(\eta)\,, \quad h^c(q) \equiv h(\eta)/q C_0(q)\,, \tag{40}$$

where $C_\ell(\eta)$ and $h(\eta)$ are given by (9).

The known Bessel-Clifford expansions (see the formulae 14.4.1-14.4.4 of [7]) for F_ℓ and G_ℓ are the infinite series in the polynomials $b_n(\eta)$ of k^2 and the modified Bessel functions $I_n(z)$ and $K_n(z)$ of the variable $z \equiv 2x^{1/2}$. Combining the terms with the same k-dependence in these series, we obtain the series in the form of (27):

$$F_\ell(\rho,\eta) = q C_\ell(q) \sum_{n=0}^{\infty} q^{2n} f_{\ell n}(x)\,, \quad \tilde{G}_\ell(\rho,\eta) = C_\ell^{-1}(q) \sum_{n=0}^{\infty} q^{2n} g_{\ell n}(x)\,, \tag{41}$$

$$\left\{ \begin{array}{c} 2 f_{\ell n}(x) \\ (2l+1)\, g_{\ell n}(x) \end{array} \right\} \equiv 2^{-2n} \sum_{m=2n}^{3n} a_{nm}\, z^{m+1} \left\{ \begin{array}{c} I_{2\ell+m+1}(z) \\ (-1)^{-m} K_{2\ell+m+1}(z) \end{array} \right\}, \tag{42}$$

where the coefficients a_{nm} are energy-independent and obey the recursive chains ($m = 2n, \ldots, 3n$ for each $n = 1, 2, \ldots$) of equations

$$2m\, a_{nm} + 2(2\ell + m)\, a_{n-1,m-2} + a_{n-1,m-3} = 0\,, \tag{43}$$

in which $a_{00} \equiv 1$ and $a_{nm} \equiv 0$, if $n > 0$ and $m < 2n$ or $m > 3n$.

The finite sums (28) $S^{(M)} = F_\ell^{(M)}$, $\tilde{G}_\ell^{(M)}$ of the series $S = F_\ell$, $\tilde{G}_\ell$ of (41) can be easily constructed via (41)-(43). As is known in WKB-approach [29], the estimates [7]

$$^{(M)}F_\ell(\rho, \eta) = O(\, q^{2M+3}\, C_\ell(q)\,)\,, \quad ^{(M)}\tilde{G}_\ell(\rho, \eta) = O(\, q^{2M+2}/C_\ell(q)\,) \tag{44}$$

of the residual terms of these decompositions are uniform in ρ, if

$$\rho \ll \rho_\ell^c \equiv \eta(1 + (1 + \ell(\ell+1)/\eta)^{1/2})\,, \tag{45}$$

where ρ_ℓ^c stands for the Coulomb turning point.

In the above case of $V \equiv 0$, the solutions $u_\ell^+ \equiv F_\ell$ and $u_\ell^- \equiv G_\ell$ of problem (23)-(25) are related by (39) and the functions F_ℓ and $\tilde{u}_\ell \equiv \tilde{G}_\ell$ are series (41) in even powers of the parameter q and functions (42) of the variable x. Do the solutions $u_\ell^\pm$ of problem (23)-(26) have the analogous properties in the case $V \neq 0$? How can one generalize the Lambert and Bessel-Clifford representations in this case? In further, we will answer both these questions for the potential V obeying (26).

2.3. Amplitude functions

According to (39)-(41), the Coulomb functions contain the factors $h(q)$ and $C_\ell(q)$ with nonanalytical q-dependence ($\ln(\eta)$ and $\exp(-\pi/2\eta)$). Evidently, the objects of that nature should generate the nonanalytical terms or factors in the sought low-energy expansions for the solutions of problem (35)-(37). Hence, to construct these expansions, we first have to resolve an essential problem: How to separate explicitly all the nonanalytical terms and factors? For this purpose, we use the special ansatz

$$c_\ell^\pm(x, q) = \tilde{c}_\ell^\pm(x, q) - h^c(q)\, s_\ell^\pm(x, q) \tag{46}$$

for $c_\ell^\pm$ and look for $\tilde{c}_\ell^\pm$ and $s_\ell^\pm$ as the series

$$\left\{ \begin{array}{c} \tilde{c}_\ell^\pm(x,q) \\ s_\ell^\pm(x,q) \end{array} \right\} = \left\{ \begin{array}{c} (qC_\ell^2(q))^{(-1\pm1)/2} \\ (qC_\ell^2(q))^{(+1\pm1)/2} \end{array} \right\} \sum_{n=0}^{\infty} q^{2n} \left\{ \begin{array}{c} c_{\ell n}^\pm(x) \\ s_{\ell n}^\pm(x) \end{array} \right\}. \tag{47}$$

Now, we substitute F and G in the form of (39) and (41) and $c_\ell^\pm$ and $s_\ell^\pm$ as (46) and (47) into (30) and (35)-(37). Then, separating q and x in the obtained relations, we derive the representations

$$\begin{aligned} U_\ell^\pm(x, q) &= q^{(1\pm1)/2}\, C_\ell^{\pm1}(q) \sum_{n=0}^{\infty} q^{2n}\, U_{\ell n}^\pm(x)\,, \\ U_{\ell n}^\pm(x) &\equiv \sum_{m'+m=n} \left(\, c_{\ell m'}^\pm(x)\, f_{\ell m}(x) + s_{\ell m'}^\pm(x)\, g_{\ell m}(x)\,\right) \end{aligned} \tag{48}$$

and obtain the infinite ($n = 0, 1, \ldots$) set of the equations

$$\partial_x \left\{ \begin{array}{c} c_{\ell n}^\pm(x) \\ s_{\ell n}^\pm(x) \end{array} \right\} = V(x) \sum_{m'+m=n} U_{\ell m'}^\pm(x) \left\{ \begin{array}{c} +g_{\ell m}(x) \\ -f_{\ell m}(x) \end{array} \right\} \tag{49}$$

for the new sought functions $c_{\ell n}^\pm$ and $s_{\ell n}^\pm$ and the explicit formulae

$$\left\{ \begin{array}{c} c_{\ell n}^+(x) \\ s_{\ell n}^+(x) \end{array} \right\} \sim \left\{ \begin{array}{c} \hat{\delta}_{n0} \\ 0 \end{array} \right\} + \sum_{m'+m=n} \int_0^x V(t)\, f_{\ell m'}(t) \left\{ \begin{array}{c} +g_{\ell m}(t) \\ -f_{\ell m}(t) \end{array} \right\} dt\,. \tag{50}$$

$$\begin{aligned} c_{\ell n}^-(x) &\sim c_{\ell n}^-(x_0) + \sum_{m'+m=n} \int_{x_0}^x V(t)\, g_{\ell m'}(t)\, g_{\ell m}(t)\, dt\,. \\ s_{\ell n}^-(x) &\sim \hat{\delta}_{n0} - \sum_{m'+m=n} \int_0^x V(t)\, f_{\ell m'}(t)\, g_{\ell m}(t)\, dt \end{aligned} \tag{51}$$

describing the asymptotics of these functions as $x \to 0$.

In (50) and (51), $\hat{\delta}_{nm}$ is the Kronecker symbol, $x_0 = 0$ and $c_{\ell n}^-(0) = 0$ if $x^{-2\ell}V(x) \in \mathcal{L}^1_{[0,b]}$, otherwise x_0 is an arbitrary, however, fixed parameter so that $x < x_0 \ll 1$.

Employing the Peano theorem [23], one can show that problem (49)-(51) for $c_{\ell n}^+$, $s_{\ell n}^+$ (or for $c_{\ell n}^-$, $s_{\ell n}^-$, in the case of $c_{\ell n}^-(0) = 0$) has the unique solution belonging to the $C^0_{[0,\infty)}$-class of functions. In the case of $|c_{\ell n}^-(0)| = \infty$, an appropriate and unique set of the functions $c_{\ell n}^-$ and $s_{\ell n}^-$ is constructed by using the special recipe [32], which is based on the shift of boundary conditions. In our case, this recipe is realized as follows. We find the values $c_{\ell n}^+(x_0)$, $s_{\ell n}^+(x_0)$, and $s_{\ell n}^-(x_0)$ by solving (49), (50) and integrating (51). Then, we substitute these values in the relation

$$\sum_{m'+m=n} \left(\, c_{\ell m'}^+(x)\, s_{\ell m}^-(x) - c_{\ell m'}^-(x)\, s_{\ell m}^+(x)\,\right) \equiv \hat{\delta}_{n0}\,.$$

which we derived by inserting (47) into (38) and subsequent separating q. As a result, we obtain the equation defining $c_{\ell n}^-(x_0)$. Thus, we find $c_{\ell n}^-(x_0)$ and $s_{\ell n}^-(x_0)$. For $x \leq x_0$, we represent the functions $c^-_{\ell n}$ and $s^-_{\ell n}$ as (51). For $x \geq x_0$, we define them as the solution of Eqs. (49) with the boundary conditions at $x = x_0 > 0$.

Let us discuss the structure of the total problem (49)-(51). Each particular ($n = 0, 1, \ldots$) problem of (49)-(51) for $c^+_{\ell n}$, $s^+_{\ell n}$ (or for $c^-_{\ell n}$, $s^-_{\ell n}$) with the fixed n contains neither q nor $c^\pm_{\ell m}$, $s^\pm_{\ell m}$ at $m > n$. Moreover, Eqs. (49) for $c^\pm_{\ell n}$, $s^\pm_{\ell n}$ and $c^\pm_{\ell m}$, $s^\pm_{\ell m}$ at $n \neq m$ differ only by the inhomogeneous terms. Due to this structure of the total problem (49)-(51), some analytical properties of $c^\pm_{\ell n}$ and $s^\pm_{\ell n}$ (for example, their asymptotics as $x \to \infty$) could be established by induction and, moreover, the calculation of $c^\pm_{\ell n}$ and $s^\pm_{\ell n}$ at $n = 0, 1, \ldots, M < \infty$ is reduced to the subsequent ($n = 0, 1, \ldots, M$) integration of the coupled pairs of Eqs. (49) written in the increasing order of the index n and completed with the corresponding boundary conditions (50) or (51). Hence, both the analytical and numerical investigations of the finite sums of series (47) can be easily performed. However, mathematically, these series are only the formal asymptotic expansions [29] with the form of (27). Hence, to complete the study, we have to find the sufficient conditions for the approximations $S \approx S^{(M)}$ and estimate the residual terms ${}^{(M)}S$.

For these purposes, we decompose the series $S = \tilde{c}^\pm_\ell$, $s^\pm_\ell$ of (47) as (28) and reduce problem (35)-(37) for c^+_ℓ, s^+_ℓ (or for c^-_ℓ, s^-_ℓ) to the two mutually coupled problems: the problem (49)-(51) for $c^+_{\ell n}$, $s^+_{\ell n}$ (or for $c^-_{\ell n}$, $s^-_{\ell n}$) at $n \leq M$ and the problem for ${}^{(M)}c^+_\ell$, ${}^{(M)}s^+_\ell$ (or for ${}^{(M)}c^-_\ell$, ${}^{(M)}s^-_\ell$). The latter, being written in a more compact way, has a form of the two sets of equations

$$\partial_x \left\{ \begin{array}{c} {}^{(M)}c^\pm_\ell \\ {}^{(M)}s^\pm_\ell \end{array} \right\} = q^{-1}\, V\, U^\pm_\ell \left\{ \begin{array}{c} +G_\ell \\ -F_\ell \end{array} \right\} - \partial_x \left\{ \begin{array}{c} c^{\pm(M)}_\ell \\ s^{\pm(M)}_\ell \end{array} \right\}, \qquad (52)$$

with the corresponding boundary conditions

$${}^{(M)}S = S - S^{(M)}, \quad S \equiv c^\pm_\ell\,, s^\pm_\ell\,, \quad x \to 0\,, \qquad (53)$$

that can be easily disentangled using (36), (37), (50) and (51).

Now, we reformulate problems (52), (53) under conditions (45) as follows. In (52), we represent S-series (27) for F_ℓ, $\tilde{G}_\ell$, $c^\pm_\ell$, and $s^\pm_\ell$ as the relevant decomposition (28). Then, we replace $F^{(M)}_\ell$, $\tilde{G}^{(M)}_\ell$ and $c^{\pm(M)}_\ell$, $s^{\pm(M)}_\ell$ by the sums of the first $(M+1)$ terms of the corresponding series (41) and (47). To estimate the residual terms ${}^{(M)}F_\ell$ and ${}^{(M)}\tilde{G}_\ell$, we employ (44) and take into consideration the inequalities $|c^\pm_{\ell n}|$, $|s^\pm_{\ell n}| < \infty$ valid, when

$0 < x < \infty$ and $n \leq M < \infty$. After the estimation we use the substitution

$$\left\{ \begin{array}{c} {}^{(M)}c^\pm_\ell(x,q) \\ {}^{(M)}s^\pm_\ell(x,q) \end{array} \right\} = q^{2M+2} \left\{ \begin{array}{c} (qC^2_\ell(q))^{(-1\pm1)/2}\, y^\pm_1(x,q) \\ (qC^2_\ell(q))^{(+1\pm1)/2}\, y^\pm_2(x,q) \end{array} \right\} \qquad (54)$$

and, in the obtained equations for the functions $y^\pm_1$ and $y^\pm_2$, we omit all the residual term of the order of q^2 as $q \to 0$. Thus, we reduce problems (52), (53) to the asymptotical ($q \to 0$, $xq \ll \rho^c_\ell$) equations

$$\partial_x \left\{ \begin{array}{c} y^\pm_1(x,q) \\ y^\pm_2(x,q) \end{array} \right\} \approx V(x)\,(\, f_{\ell 0}(x)\, y^\pm_1(x) + g_{\ell 0}(x)\, y^\pm_2(x)\,) \left\{ \begin{array}{c} +g_{\ell 0}(x) \\ -f_{\ell 0}(x) \end{array} \right\}$$
$$+ \; V(x) \left\{ \begin{array}{c} z^\pm_1(x) \\ z^\pm_2(x) \end{array} \right\} \qquad (55)$$

with the q-independent functions

$$\left\{ \begin{array}{c} z^\pm_1(x) \\ z^\pm_2(x) \end{array} \right\} \equiv (\, f_{\ell,M+1}(x)\, c^\pm_{\ell 0}(x) + g_{\ell,M+1}(x)\, s^\pm_{\ell 0}(x)\,) \left\{ \begin{array}{c} +g_{\ell 0}(x) \\ -f_{\ell 0}(x) \end{array} \right\}$$
$$+ \sum_{m'+m=M+1} U^\pm_{\ell m'}(x) \left\{ \begin{array}{c} +g_{\ell m}(x) \\ -f_{\ell m}(x) \end{array} \right\} \qquad (56)$$

and explicit asymptotical ($q, x \to 0$) boundary conditions

$$\left\{ \begin{array}{c} y^+_1(x,q) \\ y^+_2(x,q) \end{array} \right\} \sim \sum_{m'+m=M+1} \int_{x_0}^{x} V(t)\, f_{\ell m'}(t) \left\{ \begin{array}{c} +g_{\ell m}(t) \\ -f_{\ell m}(t) \end{array} \right\} dt\,, \qquad (57)$$

$$y^-_1(x,q) \sim y(x_0,q) + \sum_{m'+m=M+1} \int_{x_0}^{x} V(t)\, g_{\ell m'}(t)\, g_{\ell m}(t)\, dt\,,$$
$$y^-_2(x,q) \sim - \sum_{m'+m=M+1} \int_{0}^{x} V(t)\, f_{\ell m'}(t)\, f_{\ell m}(t)\, dt\,. \qquad (58)$$

In (55) and (56)-(58), $x_0 = 0$ and $y^-_1(0,q) = 0$, if $x^{-2\ell}V(x) \in \mathcal{L}^1_{[0,b]}$, in the other case, x_0 is an arbitrary however, fixed parameter so that $x < x_0 \ll 1$.

Following the known asymptotical method [30], we write Eqs. (55) in an integral form and iterate the obtained integral equations. As V obeys (26) and $f_{\ell n}$, $g_{\ell n}$ of (42) and $z^\pm_i$ of (56) are finite, when $0 < x < \infty$, the iterations of these integral equations converge uniformly on the finite interval $0 < x \leq \rho^c_\ell$. Therefore, all the functions $y^\pm_i$ are limited on this interval and do not identically vanish as $q \to 0$. Hence, if $q \to 0$ and $qx \ll \rho_\ell^c$, then the residual terms (54) can be estimated as

$$\left\{ \begin{array}{l} {}^{(M)}c_\ell^{\pm}(x,q) \\ {}^{(M)}s_\ell^{\pm}(x,q) \end{array} \right\} = O\Big(q^{2M+2} \left\{ \begin{array}{l} (qC_\ell^2(q))^{(-1\pm 1)/2} \\ (qC_\ell^2(q))^{(+1\pm 1)/2} \end{array} \right\}\Big). \tag{59}$$

2.4. Wavefunctions

Substituting $U_\ell^{\pm}$ of (48) into (29), we obtain the desirable formal asymptotic expansions for the wavefunctions:

$$u_\ell^+(x,q) = qC_\ell(q)\, N_\ell^+(q) \sum_{n=0}^{\infty} q^{2n}\, U_{\ell n}^+(x)\,, \tag{60}$$

$$\tilde{u}_\ell^-(x,q) = \big(C_\ell(q)\, N_\ell^+(q)\big)^{-1} \sum_{n=0}^{\infty} q^{2n}\, U_{\ell n}^-(x)\,, \tag{61}$$

$$U_{\ell n}^{\pm}(x) \equiv \sum_{m'+m=n} \Big(c_{\ell m'}^{\pm}(x)\, f_{\ell m}(x) + s_{\ell m'}^{\pm}(x)\, g_{\ell m}(x) \Big)\,. \tag{62}$$

Let us analyze the structure of the obtained expansions (60)-(62). The representations (41) and (60) for F_ℓ and u_ℓ^+ are completely identical in structure. They are the products of the factors $qC_\ell(q)$ and $qC_\ell(q)N_\ell^+(q)$ (depending only on q) and the infinite series. The latter contain the integer powers of q^2 and the functions $f_{\ell n}$ and $U_{\ell n}^+$ of x. Therefore, $qC_\ell(q)N_\ell^+(q)$ and $U_{\ell n}^+$ are the analogues of $qC_\ell(q)$ and $f_{\ell n}$, respectively. The representations (39) and (61) for G_ℓ and u_ℓ^- have also the identical functional form. They are the sums of the two terms. The first terms, i.e., $\tilde{G}_\ell$ and its analog $\tilde{u}_\ell^-$, being divided by $C_\ell(q)$ and $C_\ell(q)\, N_\ell^+(q)$, respectively, turn out the entire functions of q^2. The second terms, i.e., $h^c(q)\, F_\ell$ and its analog $\alpha_\ell(q)\, u_\ell^+$, describe the admixture of the regular solutions F_ℓ and u_ℓ^+ into the irregular solutions G_ℓ and u_ℓ^-, respectively.

Thus, we generalized the Lambert and Bessel-Clifford representations in the case, when the total potential $V^c + V$ contains an arbitrary nonzero term V obeying (26). However, the question concerning the approximation $S \approx S^{(M)}$, where S denotes the infinite series u_ℓ^+ and $\tilde{u}_\ell^-$ of (60)-(62) and $S^{(M)}$ is the finite sum of this series, is still open. To answer this question, in (29), we replace each series (41) and (47) for $S = F_\ell$, $\tilde{G}_\ell$ and $S = \tilde{c}_\ell^{\pm}$, $s_\ell^{\pm}$ by the relevant decomposition (27). Employing estimates (44) and (59) in the equalities

$$\begin{aligned} {}^{(M)}u_\ell^{\pm} \; &= \; N_\ell^{\pm}\Big({}^{(M)}F_\ell\, \tilde{c}_\ell^{\pm(M)} + {}^{(M)}\tilde{G}_\ell\, s_\ell^{\pm(M)} + {}^{(M)}c_\ell^{\pm}\, F_\ell + {}^{(M)}s_\ell^{\pm}\, \tilde{G}_\ell \\ &+ \; q^{(1\pm 1)/2} C_\ell^{\pm 1} \sum_{n=M+1}^{2M+2} q^{2n} U_{\ell n}^{\pm} \Big) + \left\{ \begin{array}{c} 0 \\ \alpha_\ell^{(M)} u_\ell^+ \end{array} \right\} \end{aligned}$$

thus obtained, we find that condition (45) is sufficient for both the approximations $u_\ell^+ \approx u_\ell^{+(M)}$ and $\tilde{u}_\ell^- \approx \tilde{u}_\ell^{-(M)}$ with the absolute accuracy

$${}^{(M)}u_\ell^{\pm}(x,q) = O\Big(q^{2M+3}\, C_\ell(q)\Big) \quad \text{and} \quad {}^{(M)}\tilde{u}_\ell^{-}(x,q) = O\Big(q^{2M+2}/C_\ell(q)\Big)\,.$$

As we have shown, the sums $u_\ell^{+(M)}$ and $\tilde{u}_\ell^{-(M)}$ can be found in practice by using the following algorithm: calculating C_ℓ, $f_{\ell n}$, and $g_{\ell n}$ $(n = 0, 1, \ldots, M)$ using (9) and (43); solving the recurrence chain $(n = 0, 1, \ldots, M)$ of $(M+1)$ problems (49)-(51) for $c_{\ell n}^+$, $s_{\ell n}^+$ and then for $c_{\ell n}^-$, $s_{\ell n}^-$; constructing $U_{\ell n}^{\pm}$ using (48); solving problem (35)-(37) for c_ℓ^+, s_ℓ^+ and then for c_ℓ^-, s_ℓ^-; defining $N_\ell^{\pm}$ and α_ℓ by using (31), (33), and (34), and, finally, constructing $u_\ell^{+(M)}$ and $\tilde{u}_\ell^{-M}$ via (60) and (61).

In further, we will clarify how expansions (47) and (60)-(62) earlier proved can be applied.

3. Effective-range function

To present all our formulae in a more compact and uniform way, we introduce two mutually coupled superscripts a and a'. The symbol $A^{ca}(x,q)$ with the superscript ca and $a = s, l, ls$ stands for the considered function $A(x,q)$ describing the scattering for the sum V^{ca} with the same $a = s, l, ls$. The symbols $A^{a'}(q)$ with the superscripts $a' = c, s$, $a = c, l$, and $a' = cl, s$ containing the comma will refer to the contributions $\delta_\ell^{c,s}(k)$, $\delta_\ell^{c,l}(k)$, and $\delta_\ell^{cl,s}(k)$ to the total scattering phase-shift $\delta_\ell^{ca}(k)$, corresponding effective-range functions $K_\ell^{c,s}$, and $K_\ell^{cl,s}$ and the coefficients of their asymptotics that now we want to derive. First, we consider the case, when the potential V obeys (26) and rapidly vanishes at large distances.

3.1 Short-range potential

Let $V = V^s$, $a = cs$, and $a' = c, s$. Using (40) we rewrite the Coulomb-nuclear effective-range function (8) in the more convenient and dimensionless form

$$K_\ell^{a'}(q) \equiv R^{2\ell+1}\, K_\ell^{a'}(E)\,. \tag{63}$$

Let us now prove the asymptotic representation

$$K_\ell^{a'}(q) = \sum_{n=0}^{M} q^{2n}\, K_{\ell n}^{a'} + O(q^{2M+2})\,, \quad M < \infty\,, \quad q \to \infty\,. \tag{64}$$

the formulae for its three first coefficients

$$\begin{aligned} &K_{\ell 0}^{a'} = c_{\ell 0}^{+a}(\infty)/s_{\ell 0}^{+a}(\infty)\,, \; K_{\ell 1}^{a'} = \big(c_{\ell 1}^{+a}(\infty) - K_{\ell 0}^{a'}\, s_{\ell 1}^{+a}(\infty)\big)/s_{\ell 0}^{+a}(\infty)\,, \\ &K_{\ell 2}^{a'} = \big(c_{\ell 2}^{+a}(\infty) - K_{\ell 0}^{a'}\, s_{\ell 2}^{+a}(\infty) - K_{\ell 1}^{a'}\, s_{\ell 1}^{+a}(\infty)\big)/s_{\ell 0}^{+a}(\infty)\,; \end{aligned} \tag{65}$$

and, finally, the relations

$$a_\ell^{a'} = -R^{2\ell+1}/K_{\ell 0}^{a'}\,, \quad r_{0\ell}^{a'} = 2R^{1-2l}\,K_{\ell 1}^{a'}\,, \quad P_\ell^{a'} = -R^{3-2\ell}\,K_{\ell 2}^{a'}/r_{0\ell}^{a'} \tag{66}$$

defining the Coulomb-nuclear scattering parameters ($a' = c, s$) as the simple algebraic combinations (65) of the limit ($x \to \infty$) values $c_{\ell n}^{+a}(\infty)$ and $s_{\ell n}^{+a}(\infty)$, of the functions $c_{\ell n}^{+}$ and $s_{\ell n}^{+}$ obeying problem (49), (50) with $V = V^s$ and $n = 0, 1, 2$.

To begin with we, expand the half-axis $\mathcal{R}^+$ in the point

$$b = b(q) \equiv x_c^p \equiv (\rho_\ell^c/q)^p\,, \quad 2/3 < p < 1\,, \tag{67}$$

on the inner ($x \leq b$) and outer ($x \geq b$) regions. By this definition, b depends on q and $b/x_c \to 0$, while $b \to \infty$ as $q \to 0$. Hence, when q is sufficiently small, relations (45) and (59) are valid in the inner region and the integral (26) satisfies the inequality

$$I_\ell(b, x) < \ln\sqrt{3} \tag{68}$$

in the outer region. Under this condition, perturbation theory of Ref. [1] can be applied to Eqs. (35) at $x \geq b$. Now, we sketch the scheme used for constructing of this theory and recall some key formulae. It is necessary, because to analyze problem (55)-(57) we will follow the analogous scheme.

In [1], using the substitution

$$\begin{aligned} c_\ell^\pm(x,q) &= \exp\big(+B_{\ell 3}(b,x,q)\big)\,y_1^\pm(x,q)\,, \\ s_\ell^\pm(x,q) &= \exp\big(-B_{\ell 3}(b,x,q)\big)\,y_2^\pm(x,q)\,, \end{aligned} \tag{69}$$

$$B_{\ell 3}(b,x,q) \equiv q^{-1}\int_b^x V(t)\,F_\ell(\rho,\eta)\,G_\ell(\rho,\eta)\,dt \tag{70}$$

Eqs.(35) with the boundary conditions shifted to $x = b$ were reduced to more simple equations, which were rewritten as the integral Volterra-type ones [33]. The integral equations were iterated in the outer region. The Klarsfeld inequalities [34]

$$F_\ell^2(\rho,\eta)/q < x(2\pi/(2\ell+1))^{1/2}\,, \tag{71}$$

$$|F_\ell(\rho',\eta)\,G_\ell(\rho,\eta)| < (2\pi x' x/(2\ell+1))^{1/2}\,, \quad \rho' = qx' \leq \rho = qx \tag{72}$$

were used as the key ones to prove the uniform convergence of the iterations under the conditions (26) and (68) and to estimate the convergence rate of these iterations (see bounds (35) in [1]).

Now, we substitute V, I_ℓ, and b given by (1), (26), and (67) to the above bounds and use estimates (59) taken at $x = b$. After the simple transformations, we find that for $x \geq b$ and $q \to 0$

$$\begin{aligned} &|c_\ell^+(x,q) - c_\ell^+(b,q)|\,,\ |s_\ell^\pm(x,q) - s_\ell^\pm(b,q)| \leq O\Big(\exp(-\mu R q^{-2p})\Big)\,, \\ &|c_\ell^-(x,q) - c_\ell^-(b,q)| \leq O\Big(C_\ell^{-2}(q)\exp(-2\mu R q^{-2p})\Big)\,. \end{aligned} \tag{73}$$

Then, we study problems (49)-(51) with the fixed n consequently ($n = 0, 1, \ldots$). We use the scheme which is slightly different from the given above-mentioned. There are only two differences. First, for the substitution we use an analog of the zero-energy limit ($q = 0$) form of (69)

$$\begin{aligned} c_{\ell n}^\pm(x) &= exp\big(+B_{\ell 3}(b,x,0)\big)\,y_{1n}^\pm(x)\,, \\ s_{\ell n}^\pm(x) &= exp\big(-B_{\ell 3}(b,x,0)\big)\,y_{2n}^\pm(x)\,, \end{aligned} \tag{74}$$

where, owing to (41) and (70),

$$B_{\ell 3}(b,x,0) = \int_b^x V(t)\,f_{\ell 0}(t)\,g_{\ell 0}(t)\,dt\,.$$

Second, as the key inequalities, we use the bound relation

$$\int_b^x f_{\ell 0}^2(t)\,|V(t)|\,dt < |V_0 R/4\pi|\,b^{1/2}\exp(-\mu R b + 2b^{1/2})\,,$$

which is valid due to (1) and the asymptotics of $f_{\ell 0}(x)$ at $x \geq b \gg 1$. And, finally, we making in use also the limit form of (72), i.e.,

$$|f_{\ell 0}(x')\,g_{\ell 0}(x)| \leq \big(2\pi x' x/(2\ell+1)\big)^{1/2}\,.$$

By induction, we prove that, for all $x \geq b$ and $n < \infty$, all the functions $y_{in}^\pm$ are finite and, moreover, we find the explicit estimates for the differences $y_{in}^\pm(x) - y_{in}^\pm(b)$. These results allow us to conclude that, for all $x \geq b$ and $M < \infty$, functions (74) are also finite and are so that

$$\left\{ \begin{array}{l} |\tilde{c}_{\ell n}^\pm(x) - \tilde{c}_{\ell n}^\pm(b)| \\ |s_{\ell n}^\pm(x) - s_{\ell n}^\pm(b)| \end{array} \right\} < O\left(q^{-(6M+1)p}\exp(-\mu R q^{-2p} + 2q^{-p})\right)\,. \tag{75}$$

Therefore, for each finite sum $S^{(M)} = \tilde{c}_\ell^{\pm(M)}$, $s_\ell^{\pm(M)}$ of series (47), one has the corresponding asymptotic ($q \to 0$, $x \geq b$) relation:

$$\begin{aligned} |\tilde{c}_\ell^{\pm(M)}(x,q) - \tilde{c}_\ell^{\pm(M)}(b,q)| &= o\left(q^{(2M+2)}\left(qC_\ell^2(q)\right)^{(-1\pm1)/2}\right)\,, \\ |s_\ell^{\pm(M)}(x,q) - s_\ell^{\pm(M)}(b,q)| &= o\left(q^{(2M+2)}\left(qC_\ell^2(q)\right)^{(+1\pm1)/2}\right)\,. \end{aligned} \tag{76}$$

As the next step, we write each residual term ${}^{(M)}S$ of series (47) as the identity

$$
\begin{aligned}
{}^{(M)}S(x,q) \;&\equiv\; \big(S(x,q) - S(b,q)\big) + \big(S(b,q) - S^{(M)}(b,q)\big) \\
&+\; \big(S^{(M)}(b,q) - S^{(M)}(x,q)\big) .
\end{aligned}
$$

Then, we estimate three differences included in the brackets by means of (73), (59), and (76), respectively. As a result, we find that estimates (59) are fulfilled for any $M \le \infty$ and all $x \ge b$.

Now, we substitute c_ℓ^+ as the sum (46) into (32), decompose series (47) for the functions $\tilde{c}_\ell^+$ and s_ℓ^+ as (27) and take $x \to \infty$. As we showed early, in this limit, the functions $c_{\ell n}^+$ and $s_{\ell n}^+$ are restricted and estimates (59) are valid. Therefore, the above limit is transformed to the asymptotic relation

$$
\cot\!\operatorname{an}\delta_\ell^{c,s}(q) = \big(q\,C_\ell^2(q)\big)^{-1}\left(\frac{\sum_{n=0}^{M} q^{2n} c_{\ell n}^+(\infty)}{\sum_{n=0}^{M} q^{2n} s_{\ell n}^+(\infty)} + O(q^{2M+2})\right) - h^c(q)
$$

which ensures the desired formulae (64) and (66). Comparing them with the standard expansion (11), we find relations (66). Thus, the calculating three Coulomb-nuclear scattering parameters $a_\ell^{c,s}$, $r_{0\ell}^{c,s}$, and $P_\ell^{c,s}$ is reduced to the consequent $(n = 0, 1, 2)$ integration of three energy-independent problems (49), (50) with $V = V^s$. As we will show in further, for $V = V^{ls}$, the recipe for calculating the modified scattering parameters is complicated.

3.2. Sum of long- and short-range potentials

Now let $V = V^l + V^s$, where V^s vanishes as (41) and V^ℓ has the long-range asymptotics (3) with $d \ge 3$. Let us prove that the function

$$
K_\ell^{cl,s}(q) = \Big(C_\ell(q) N_\ell^{+cl}(q)\Big)^2 \Big(q \cot\!\operatorname{an}\delta_\ell^{cl,s}(q) + q\alpha_\ell(q)\Big) \tag{77}
$$

is related with the modified effective-range function (19) by (63) and has the asymptotics (64) with coefficients (65), where now $a = cls$, $a' = cl, s$ and $c_{\ell n}^{+cls}(\infty)$, $s_{\ell n}^{+cls}(\infty)$ are some finite values, which will be determined in the proof.

We will solve the initial problem (23)-(26) with $V = V^{ls}$ step by step. An idea of the two-step solution was mentioned in [17] and discussed in a more detail way in [15]. Her we now realize this idea as follows.

As the first step, we find the auxiliary wavefunctions $u_\ell^{\pm cl}$, which are the solutions $u_\ell^\pm$ of problem (23)-(26) in the case of $V = V^l$, $a = cl$, $a' = c, l$. To construct these solutions and their low-energy expansions, we apply the method described in Sect.2.

Using this method, we obtain the representations

$$
\begin{aligned}
&u_\ell^{+cl}(x,q) = N_\ell^{+cl}(q)\, U_\ell^{+cl}(x,q) , \\
&u_\ell^{-cl}(x,q) = \tilde{u}_\ell^-(x,q) + \alpha_\ell^{cl}(q) u_\ell^{+cl}(x,q) , \quad \tilde{u}_\ell^-(x,q) = N_\ell^{-cl}\, U_\ell^{-cl}(x,q) , \\
&U_\ell^{\pm cl}(x,q) = c_\ell^{\pm cl}(x,q)\, F_\ell(\rho,\eta) + s_\ell^{\pm cl}(x,q)\, G_\ell(\rho,\eta) ,
\end{aligned} \tag{78}
$$

where the amplitude functions $c_\ell^{\pm cl}$ and $s_\ell^{\pm cl}$ stand for the solutions $c_\ell^\pm$ and $s_\ell^\pm$ of problems (49)-(51) with $V = V^l$ and the factors $N_\ell^{\pm cl}(q)$ and $\alpha_\ell^{cl}(q)$ are limits (31) of the relevant functions (33) and (34). Let us notice that, according to (25), the Wronskian relation for functions (78) reads as

$$
u_\ell^{-cl}(x,q)\, \partial_x u_\ell^{+cl}(x,q) + u_\ell^{+cl}(x,q)\, \partial_x u_\ell^{-cl}(x,q) = q , \quad x \ge 0 . \tag{79}
$$

Then, we construct the low-energy expansions

$$
\begin{aligned}
u_\ell^{+cl}(x,q) \;&=\; qC_\ell(q)\, N_\ell^{+cl}(q) \sum_{n=0}^{\infty} q^{2n}\, U_{\ell n}^{+cl}(x) \\
\tilde{u}_\ell^{-cl}(x,q) \;&=\; \big(C_\ell(q)\, N_\ell^{+cl}(q)\big)^{-1} \sum_{n=0}^{\infty} q^{2n}\, U_{\ell n}^{-cl}(x) \\
U_{\ell n}^{\pm cl}(x) \;&\equiv\; \sum_{m'+m=n} \Big(c_{\ell m'}^{\pm cl}(x)\, f_{\ell m}(x) + s_{\ell m'}^{\pm cl}(x)\, g_{\ell m}(x)\Big) .
\end{aligned} \tag{80}
$$

in which the functions $c_{\ell n}^{\pm cl}$ and $s_{\ell n}^{\pm cl}$ are defined as the solutions $c_{\ell n}^\pm$ and $s_{\ell n}^\pm$ of problems (49)-(51) with $V = V^l$.

In the second step, we reformulate the initial problem (23)-(25) with $V = V^{ls}$ by using a generalized linear form of the variable phase approach [18] as follows. First, we write the asymptotics (25) for the sought function u_ℓ^{+a}, $a = cls$ in the form corresponding to decomposition (14) of the total scattering phase:

$$
\begin{aligned}
u_\ell^{+cls}(x,q) \;&\sim\; \cos\delta_\ell^{cl,s}(q)\, u_\ell^{+cl}(x,q) + \sin\delta_\ell^{cl,s}(q)\, u_\ell^{-cl}(x,q) \\
&\sim\; \sin\big(\rho - \eta\ln 2\rho + \delta_\ell^{cl}(q) + \delta_\ell^{cl,s}(q)\big) , \quad x \to \infty .
\end{aligned} \tag{81}
$$

Then, we use the found functions u_ℓ^{+cl} and u_ℓ^{-cl} as the basic functions instead of F_ℓ and G_ℓ, respectively. We look for u_ℓ^{+cls} as

$$
\begin{aligned}
u_\ell^{+cls}(x,q) \;&=\; N_\ell^{+cls}(q)\, U_\ell^{+cls}(x,q) , \\
U_\ell^{+cls}(x,q) \;&\equiv\; c_\ell^{+cls}(x,q)\, u_\ell^{+cl}(x,q) + s_\ell^{+cls}(x,q)\, u_\ell^{-cl}(x,q) .
\end{aligned} \tag{82}
$$

where, by definition, the generalized and sought amplitude functions c_ℓ^{+cls} and s_ℓ^{+cls} obey the Lagrange identity

$$\partial_x c_\ell^{+cls}(x,q)\, u_\ell^{+cl}(x,q) + \partial_x s_\ell^{+cls}(x,q)\, u_\ell^{-cl}(x,q) \equiv 0\,. \qquad (83)$$

Substituting (82) into (23)-(25) and taking into consideration (79) and (83), we prove that the above amplitude functions obey the equations

$$\partial_x \left\{ \begin{array}{c} c_\ell^{+cls}(x,q) \\ s_\ell^{+cls}(x,q) \end{array} \right\} = q^{-1}\, V^s(x)\, U_\ell^{+cls}(x,q) \left\{ \begin{array}{c} +u_\ell^{-cl}(x,q) \\ -u_\ell^{+cl}(x,q) \end{array} \right\} \qquad (84)$$

and the simple boundary conditions $c_\ell^{+cls} = 1$, $s_\ell^{+cls} = 0$ at $x = 0$. Comparing the asymptotics of wavefunction (82) as $x \to \infty$ with its required form (81), we find the relation defining the relative phase-shift $\delta_\ell^{cl,s}(q)$:

$$\cot\!an\, \delta_\ell^{cl,s}(q) = \lim_{x\to\infty} c_\ell^{+cls}(x,q)/s_\ell^{+cls}(x,q)\,. \qquad (85)$$

To separate explicitly the nonanalytical terms in the low-energy expansions for the generalized amplitude functions, we look for them as

$$c_\ell^{+cls}(x,q) = \tilde{c}_\ell^{+cls}(x,q) - \alpha_\ell^{cl}(q)\, s_\ell^{+cls}(x,q)\,, \qquad (86)$$

$$\begin{aligned} \tilde{c}_\ell^{+cls}(x,q) &= \sum_{n=0}^{\infty} q^{2n}\, c_{\ell n}^{+cls}(x)\,, \\ s_\ell^{+cls}(x,q) &= q\big(C_\ell(q) N_\ell^{+cl}(q)\big)^2 \sum_{n=0}^{\infty} q^{2n}\, s_{\ell n}^{+cls}(x) \end{aligned} \qquad (87)$$

Substituting (80) and (86), (87) into (84) and separating x from q, we arrive at the recursive ($n = 0, 1, \ldots$) equations

$$\begin{aligned} \partial_x \left\{ \begin{array}{c} c_{\ell n}^{+cls}(x) \\ s_{\ell n}^{+cls}(x) \end{array} \right\} &= V^s(x) \sum_{m'+m=n} U_{\ell m'}^{+cls}(x) \left\{ \begin{array}{c} +U_{\ell m}^{-cl}(x) \\ -U_{\ell m}^{+cl}(x) \end{array} \right\}, \\ U_{\ell n}^{+cls}(x) &\equiv \sum_{m'+m=n} \big(U_{\ell m'}^{+cl}(x)\, c_{\ell m}^{+cls}(x) + U_{\ell m'}^{-cl}(x)\, s_{\ell m}^{+cls}(x) \big) \end{aligned} \qquad (88)$$

for the functions $c_{\ell n}^{+cls}$ and $s_{\ell n}^{+cls}$ and define their appropriate boundary values as

$$c_{\ell n}^{+cls}(0) = \hat{\delta}_{n0}\,, \quad s_{\ell n}^{+cls}(0) = 0\,. \qquad (89)$$

Using the iteration method [23], one can prove that all these functions are limited on $\mathcal{R}^+$, if V^s obeys (41). We omit the proof of this statement, because it is too cumbersome

and rather tedious than interesting. Moreover, this proof actually repeats those given in Subsect.2.3.

To come to our main goal, we insert c_ℓ^{+cls} of (86) into (85) and let $x \to \infty$. The obtained limiting relation is easily transformed into the desired formulae (77). The letter defines the dimensionless effective-range function in terms of the functions $\tilde{c}_\ell^{+cls}$ and s_ℓ^{+cls} taken at $x = \infty$. Expanding series (87) as (27), we prove relations (64) and (65), in which $a = cls$, $a' = cl, s$ and $c_{\ell n}^{+cls}(\infty)$ and $s_{\ell n}^{+cls}(\infty)$ are the limiting values for the solutions of problems (88), (89). Comparing (19) with (77), we find the relations

$$C_\ell^{cl}(\eta) = C_\ell(\eta)\, N_\ell^{+cl}(q)\,, \quad h^{cl}(\eta) = k\, \alpha_\ell^{cl}(q) \qquad (90)$$

and formulae (66) for the modified scattering parameters with superscript $a' = cl, s$.

Now, we can answer the questions that have been unsettled in [17].

First, it should be emphasized that definition (22) suggested in [17] for the modified scattering length is not quite correct. In fact, this definition does not contain the normalization factor caused by the long-range potential in the Coulomb field. According to (64), (66), and (77), the correct definition of the modified scattering length is

$$a_\ell^{cl,s} \equiv -\lim_{k\to 0} \tan \delta_\ell^{cl,s}(k) / \Big[k\big(k^\ell\, C_\ell(\eta)\, N_\ell^{+cl}(k)\, \big)^2 \Big]\,. \qquad (91)$$

This definition differs from the Bencze et al. [17] relation (22) by the existence of the squared normalization factor N_ℓ^{+cl}. Owing to estimates (37) of Ref. [1], this factor is close to unity, when V^l is sufficiently small. Only in this case the relation (22) gives an approximate value of $a_\ell^{cl,s}$.

As we have shown, to calculate C_ℓ^{cl} and h_ℓ^{cl} from (19) using (90), one should solve problem (35), (36) with $V = V^l$ and find limits (31) of the relevant functions (33) and (34). In our method,the calculation of the modified scattering parameters is reduced to the consequent solving of problem (49)-(51) with $V = V^l$, $n = 0, 1, 2$ and problems (88), (89) with $n = 0, 1, 2$. All these problems are the energy-independent linear differential equations with simple boundary conditions. The practical solution of these equations does not met any certain difficulties and, moreover, can be performed with a high accuracy.

4. Concluding remarks

Our main results are the following: the algorithm for constructing the analogies of (61) and (60)-(62) to the Lambert (39) and Bessel-Clifford (41) representations and formulae (77), (90), and (66) for the modified effective-range function (19) and nuclear scattering parameters $a_\ell^{cl,s}$, $r_{0\ell}^{cl,s}$ and $P_\ell^{cl,s}$.

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Низкоэнергетические разложения
для одномерной задачи рассеяния Шредингера

Регулярная и нерегулярная волновые функции рассеяния суперпозицией отталкивающего кулоновского потенциала и потенциала, убывающего быстрее центробежного, представляются бесконечными рядами, в которых импульс рассеяния и расстояние разделены. Построение конечных сумм, равномерно аппроксимирующих эти ряды, при вполне определенных условиях сводится к решению простейших дифференциальных задач.

Выведенные представления волновых функций могут быть использованы для аналитического и численного исследований различных низкоэнергетических асимптотик в задаче столкновения двух частиц.

Особое внимание уделено понятию длины рассеяния и построению модифицированной функции эффективного радиуса для суперпозиций отталкивающего кулоновского, коротко- и дальнодействующего потенциалов.

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Pupyshev V.V. E4-97-125

Low-Energy Expansions
for the One-Dimensional Shrodinger Scattering Problem

The regular and irregular scattering wavefunctions for the superposition of the repulsive Coulomb potential and the central potential that decreases more rapidly than the centrifugal one are represented as an infinite series with the distance and momentum separated. The constructing of the finite sums that uniformly approximate these series under well-defined conditions is reduced to solving the simplest differential problems.

The derived representations of the wavefunctions can be successfully used for analytical and numerical treatments of various low-energy asymptotics in two-body collision problem.

A special attention is paid to the concept of the scattering length and construction of the modified effective-range function for the scattering by the superposition of the repulsive Coulomb and long- and short-range potentials.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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