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SPIN STATE DETERMINATION USING STERN—GERLACH DEVICE

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Измерение спинового состояния устройством Штерна—Герлаха

Предлагается использовать известное устройство Штерна—Герлаха для нахождения спинового состояния частицы вместо измерения спиновых наблюдаемых. Показано, что измерение импульсных распределений частицы (до и после действия магнитного поля устройства) позволяет определить начальное спиновое состояние частицы в случае произвольного спина. Установлено, что для этой цели нельзя использовать обычную трактовку эксперимента Штерна—Герлаха, основанную на корреляции спинового и пространственного состояний частицы.

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Spin State Determination Using Stern—Gerlach Device

The well-known Stern—Gerlach device is proposed here for determination of a particle spin state instead of using it for measurement of spin observables. It is shown that measurement of particle momentum distributions (before and after the action of the device magnetic field) allows one to determine the particle initial spin state in the case of an arbitrary spin value. It is demonstrated that one cannot use for this purpose the usual treatment of the Stern—Gerlach experiment based on the entanglement of spin and spatial states.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

In order to determine a physical system state, one needs a physical procedure which allows one to find the density matrix which the system had before this procedure. If the state is pure, then one must determine the system wave function. The state determination differs from the well–known quantum observable measurement. The problem and different approaches to its solution have been reviewed in [(1)]. The paper contains references to Kemble (1937), Gale, Guth and Trammel (1968), Lamb (1969), d'Espagnat (1976). The book [(2)] gives additional references (the problem being called "informationaly complete measurement").

The determination of a particle spin state is the simplest example. In the case of spin one–half one has to find two real parameters which determine the spin wave function or three parameters determining the density matrix. The purpose of this paper is to show how the initial spin state can be found (in the case of an arbitrary spin $s = 1/2, 1, ...$) using the Stern–Gerlach device. The device is well–known as the experimental procedure destined for spin observable measurement, e.g. see [(3,4)].

In order to describe a spin state, one may use instead of the density matrix D the expectation values $\langle s_i s_j \cdots s_k \rangle = Tr(s_i s_j \cdots s_k)D$ of products of the spin vector components $s_i, i = 1, 2, 3$. This way of the spin state description is set forth in section 2 and is used further.

In Section 3 we describe the model of the Stern–Gerlach device which is employed in this paper. Remind that the device allows one to get information on particle spin by measuring the particle momentum or coordinate [3,4]. For our purpose one must be able to measure momentum distributions which the particle had before and after the action of the device magnetic field. It will be shown that calculation of mixed moments of these distributions (the momentum expectation value being the first moment) allows one to find $\langle s_i s_j \cdots s_k \rangle$, i.e. the particle spin state.

The usual treatment of the Stern–Gerlach device is based upon the entanglement of the spin and spatial parts of the particle wave function [3,4]. Busch and Schroeck [5] have shown that this entanglement is only approximate. We show in section 4 that it is the corrections to this entanglement that are of particular importance for obtaining correct values of the moments described above.

Some concluding remarks are given in section 5.

2 Description of the spin states by polarization tensors

In the general case, the particle spin state is described by a $(2s+1)\times(2s+1)$ hermitian density (or statistical) matrix D having the unit trace $TrD = 1$. It is well-known (see e.g. [6]) the the state in the case $s = 1/2$ can equivalently be described by the polarization vector $\vec{T} = Tr\vec{s}D$, i.e. by expectation values of the spin operator $\vec{s} = (s_1, s_2, s_3)$. Indeed, if $\vec{T}$ is known, then $D = 1/2 + 2\vec{T}\vec{s}$. This description can be generalized for the case $s \geq 1$ [7]. But in addition to the expectation values of s_1, s_2, s_3, one needs then expectation values of their products $s_i s_j$, $s_i s_j s_k$ and so on. In the case $s = 1/2$, these products reduce either to the number (a multiple of the unit matrix) or to the operators s_i themselves, e.g. $s_x^2 = (\sigma_x/2)^2 = 1/4$ and $s_1 s_2 = is_3/2 = -s_2 s_1$. More exactly, the spin state can be described besides $\vec{T}$ by expectation values of irreducible rotation group tensors $t_q, q = 2, 3, \ldots 2s$, constructed of the symmetrized products of s_1, s_2, s_3, see e.g. [8], eq.(16) and [7], ch.III.22, example 10b. For example, the rank 2 tensor t_2 depends upon $s_i s_j + s_j s_i$.

The irreducible tensor t_q has $2q+1$ components t_{qM}, $M = -q, -q+1, \cdots + q$ which have expectation values

$$Trt_{qM}D = \sum_{m_1 m_2} \langle m_2 | t_{qM} | m_1 \rangle \langle m_1 | D | m_2 \rangle \tag{1}$$

To calculate (1) one may use the Wigner–Eckart theorem (e.g. see [7] or [9])

$$\langle m_2 | t_{qM} | m_1 \rangle = (s \| t_q \| s)(-1)^{s-m_2} \langle ssm_1 - m_2 | ssqM \rangle \tag{2}$$

where $\langle ssm_1 - m_2 | ssqM \rangle$ is the Clebsch–Gordan coefficient [7]. I assume here that t_{qM} transforms under rotations not as the spin wave function $|qM\rangle$ (or spherical function $Y_{qM}(\Theta, \Phi)$) but as the corresponding bra vector $\langle qM | = |qM\rangle^*$. In particular, t_{1M} are the cyclic (or spherical) components of the spin vector (see [7] ch.III.App D)

$$t_{1,-1} = (s_x - is_y)/\sqrt{2}\,, \quad t_{1,0} = s_z\,, \quad t_{1,+1} = -(s_x + is_y)/\sqrt{2} \tag{3}$$

We see that $Trt_{qM}D$, eq.(1), is proportional to the quantity

$$T_{qM} = \sqrt{2s+1} \sum_{m_1, m_2} (-1)^{s-m_2} \langle ssm_1 - m_2 | ssqM \rangle \langle m_1 | D | m_2 \rangle \tag{4}$$

Using the Clebsch–Gordan coefficient property

$$\sum_{qM} (s_1 s_2 m_1 m_2 | s_1 s_2 qm)(s_1 s_2 m_1' m_2' | s_1 s_2 qM) = \delta_{m_1 m_1'} \delta_{m_2, m_2'}$$

one can verify that

$$\langle m_1 | D | m_2 \rangle = (2s+1)^{-1/2} \sum_{qM} T_{qM} (-1)^{-s+m_2} \langle ssm_1 - m_2 | ssqM \rangle \tag{5}$$

So D can be represented by T_{qM}, $q = 0, 1, \cdots 2s$, $M = -q, -q+1, \cdots + q$. T_{qM} are called the multipolar tensor parameters in [7], other names are also used. I call T_{qM} here the polarization tensors because they are generalizations of the polarization vector, which is the rank 1 tensor. T_{qM} can be constructed from the expectation values $Tr(s_i s_i \cdots s_k)D$. The number of multipliers in the product $s_i s_i \cdots s_k$ need not exceed $2s$ (if follows from eq.(2) that the tensors with $q > 2s$ vanish).

It will be shown in the next section how one can determine expectation values of $s_i s_j \cdots s_k$ measuring the momentum distribution of the particle before and after the action of the Stern–Gerlach magnetic field.

3 Polarization tensor determination by measuring momentum distribution

3.1. I use the model of the Stern–Gerlach device which has been described in [10]. Initially, at $t = 0$ a neutral particle is in a state $\psi_0 = \phi_0 \chi_0$, its spatial part $\phi_0(\vec{x})$ being a wave packet and χ_0 being a spin wave function. After $t = 0$ a magnetic field $\vec{B}(\vec{x})$ is turned on in the region of the packet localization. The field is turned out at $t = \tau$, and then the particle momentum is measured. The model Hamiltonian is

$$H = p^2/2m + g(t)\mu\vec{s}\cdot\vec{B}(\vec{x}) \equiv H_0 + H_I \tag{6}$$

Here $\vec{p} = -i\vec{\nabla}_x$ is the atom momentum operator conjugated to the atom center–of–mass coordinate $\vec{x}$; μ is the particle magnetic moment; $\vec{s}$ is the spin operator. The function $g(t)$ is zero outside the interval $0, \tau$ and equals approximately the unit almost everywhere inside it, so that $\int_o^\tau g(t)dt = \tau$.

The inhomogeneous field $\vec{B}(\vec{x})$ is supposed to be a linear function of $\vec{x}$:

$$\vec{B}(\vec{x}) = \vec{B}_0 + \vec{b}(\vec{x}), \quad b_i(\vec{x}) = \sum_j L_{ij} x_j\,, \quad i, i = 1, 2, 3 \tag{7}$$

$\vec{B}(\vec{x})$ must satisfy the equations div$\vec{B} = 0$ and rot$\vec{B} = 0$ because it is the magnetic field outside generating currents. The most general field [7] satisfying the equations is of the form [10]

$$\vec{B}(\vec{x}) = \vec{B}_0 + \sum_\alpha \beta_\alpha(\vec{x}\cdot\vec{\nu_\alpha})\vec{\nu_\alpha}\,, \quad \beta_1+\beta_2+\beta_3 = 0 \tag{8}$$

where the mutually orthogonal unit vectors ν_α, $\alpha = 1,2,3$, can be called eigenvectors of $\vec{b}(\vec{x})$. Besides them $\vec{b}(\vec{x})$ is specified by the pseudoscalars β_α (which are field gradients along eigenvectors $\vec{\nu}_\alpha$). One can direct the coordinate axes along $\vec{\nu}_\alpha$, so that $(\vec{x}\cdot\vec{\nu}_\alpha) = x_\alpha$ and then

$$B_k(\vec{x}) = B_{0k} + \beta_k x_k \tag{9}$$

The solution of the Schroedinger equation

$$i\partial_t\psi(t) = (H_0+H_I)\psi(t)\,, \quad \psi(t=0) = \psi_0 \tag{10}$$

can be represented as $\psi(t) = \exp(-iH_0t)\psi_I(t)$ where $\psi_I(t)$ is the particle wave function in the interaction picture (for details see [10])

$$\psi_I(t) = U(t)\psi_I(t)\,, \quad \psi_I(0) = \psi_0 \tag{11}$$

$$U(t) = T\exp\Big[-i\int_0^t dt' H_I(t')\Big] \tag{12}$$

$$H_I(t) \equiv \exp(iH_0t)H_I\exp(-iH_0t) = g(t)\mu\vec{s}\vec{B}(t)$$

$$B_k(t) \equiv B_{0k} + \beta_k(x_k + tp_k/m) \tag{13}$$

The calculations of this section can be generalized for the case when the initial particle state is described by the product ρD of the spatial density matrix ρ and spin density matrix D.

The so-called "impulsive approximation" (e.g. see [3,4,5]) is not used here, i.e. we do not neglect H_0 as compared to H_I.

3.2. I am going to show that expectation values of the operators $p_i, p_ip_j, p_ip_jp_k\cdots$ in the state $\psi(t)$ allow one to determine expectation values of the spin operators s_i and their products in the initial state which is equivalent to the initial spin state determination, see sect 2.

The expectation values can be evaluated if the momentum distribution $w(p_1,p_2,p_3)$ is measured: they are equal to $\int d^3p\, p_i w(p_1,p_2,p_3)$ and so on. Quantum mechanical expression $\langle\psi(t)|O|\psi(t)$, $O = p_i,\ p_ip_j\cdots$ for the expectation value can be written as the expectation value of the corresponding Heisenberg operator $O(t)$ in the initial state

$$\begin{aligned}\langle\psi(t)|O|\psi(t) &= \langle e^{-iH_0t}U(t)\psi_0|O|e^{-iH_0t}U(t)|\psi_0\rangle = \\ &= \langle\psi_o|O|(t)|\psi_0\rangle\end{aligned} \tag{14}$$

If O commutes with H_0 (it is the case for momentum operators), one has

$$O(t) = U^+(t)OU(t) \tag{15}$$

In Appendix I present the most simple derivation of the formula which allows one to calculate the r.h.s. of eq.(15) in any order of the interaction H_I

$$\begin{aligned}U^+OU &= O + \\ &+ \sum_{n=1} i^n\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n\Big[H_1(t_n),[H_I(t_{n-1}),[\cdots[H_I(t_1),O]\cdots\Big]\end{aligned} \tag{16}$$

With the help of this formula and usual commutation relations $[s_i,s_j] = i\epsilon_{ijk}s_k$ and $[p_i,f(\vec{x})] = i\partial f(\vec{x})]/\partial x_i$, one obtains the following expression for the Heisenberg operator $p_k(t)$ in terms of the Schroedinger operators p_k, x_k, s_k:

$$\begin{aligned}p_k(t) &= p_k - \mu\beta_k g_1(t)s_k + \mu^2\beta_k\int_0^t dt_1 g(t_1)\int_0^{t_1} dt_2 g(t_2)[\bar{s}\times\bar{B}(t_2)]_k - \\ &- \mu^3\beta_k\int_0^t dt_1 g(t_1)\int_0^{t_1} dt_2 g(t_2)\int_0^t dt_3 g(t_3)\{[[\vec{s}\times\vec{B}(t_3)]\times\vec{B}(t_2)]_k + \\ &+ \frac{t_2-t_3}{m}\sum_{m,n}\epsilon_{kmn}s_ms_n\beta_n^2\} + \cdots\end{aligned} \tag{17}$$

Here $g_1(t) \equiv \int_0^t dt_1 g(t_1)$; $g_1(t>\tau) = \tau$; $[s\times\vec{B}]_k$ denotes the k–th component of the vector product $[\vec{s}\times\vec{B}]$; $[[\vec{s}\times\vec{B}(t_3)]\times\vec{B}(t_2)]$ denotes the double vector product. The interior of the curly brackets in eq.(17) can be represented as the explicitly hermitian operator:

$$\{\cdots\} = \frac{1}{2}\Big[[\vec{s}\times\vec{B}(t_3)]\times\vec{B}(t_2)\Big] + \frac{1}{2}\Big[\vec{B}(t_2)\times[\vec{B}(t_3)\times\vec{s}]\Big]_k + \\ + \frac{t_2-t_3}{2m}\sum_{mn}\beta_n^2(s_ms_n+s_ns_m)$$

3.3. Now one can calculate $\langle p_k\rangle \equiv \langle\phi_0|p_k(t)|\phi_0\rangle$ for $t>\tau$

$$\begin{aligned}\langle p_k\rangle &= \langle p_k\rangle_0 - \mu\beta_k\tau T_k + \\ &+ \mu^2\beta_k\int_o^t dt_1 g(t_1)\int_0^{t_1} dt_2 g(t_2)[\vec{T}\times\langle\phi_0|\vec{B}(t_2)|\phi_0\rangle]_k - \\ &-\mu^3\beta_k\int_0^t dt_1 g(t_1)\int_0^{t_2} dt_2 g(t_2)\int_0^{t_3} dt_3 g(t_3)\{\frac{1}{2}[[\vec{T}\times\langle\phi_0|\vec{B}(t_3)]\times\vec{B}(t_2)]_k|\phi_0\rangle + \\ &+\frac{1}{2}[\langle\phi_0|\vec{B}(t_2)\times[\vec{B}(t_3)|\phi_0\rangle\times\vec{T}]_k + \\ &+\frac{t_2-t_3}{2m}\sum_{mn}\epsilon_{kmn}\beta_n^2\langle s_ms_n+s_ns_m\rangle\} + \cdots\end{aligned} \tag{18}$$

We use the equation $\int_0^t g(t_1)dt_1 = \tau$ valid for $t>\tau$ and the notation

$$\begin{aligned}\langle p_k\rangle_0 &\equiv \langle\phi_0|p_k|\phi_0\rangle,\ T_k \equiv \langle\chi_0|s_k|\chi_0\rangle \\ \langle\phi_0|\vec{B}|\phi_0\rangle &= \int d^3x\,\phi_0^*(\vec{x})\vec{B}\phi_0(\vec{x}) \\ \langle s_ms_n+s_ns_m\rangle &\equiv \langle\chi_0|s_ms_n+s_ns_m|\chi_0\rangle\end{aligned} \tag{19}$$

So the change $\langle p_k\rangle - \langle p_k\rangle_0$ of the momentum expectation value induced by the magnetic field depends upon the polarization vector $\vec{T}$ and polarization tensors of higher ranks if $\beta_k \neq 0$ (one can show that the rank 3 tensor appears only in terms $\sim \mu^5$).

In the case $s = 1/2$ the r.h.s. of eq.(17) depends on the operators s_i only and not on their products. For instance, one has

$$s_m s_n + s_n s_m = \delta_{mn} \cdot \frac{1}{2} \cdot I$$

where I is the 2x2 unit matrix. So $\langle p_k\rangle$ depends on $\vec{T}$ only and eqs.(18) for $k = 1,2,3$ form the system of linear equations which allows one to find $\vec{T}$ if $\langle p_k\rangle - \langle p_k\rangle_0$ are known and $\beta_k \neq 0$. If only terms of the order μ are taken into account in this system (terms $\sim \mu^2, \mu^3 \cdots$ are neglected), one obtains a simple expression for $\vec{T}$

$$T_k = -[\langle p_k\rangle - \langle p_k\rangle_0]/\mu\beta_k\tau,\ k = 1,2,3 \tag{20}$$

3.4.If $s \geq 1$, one needs to determine the polarization tensors of higher ranks $q \geq 2$ and eqs.(18) do not suffice. I suggest to add to eqs.(18) the equations for mixed moments $\langle p_i p_j\rangle \equiv \langle\phi_0|p_i(t)p_j(t)|\phi_0\rangle$, $\langle p_i p_j p_k\rangle$ and so on. Using eq.(17) one obtains

$$\begin{aligned}\langle p_i p_j\rangle &= \langle p_i p_j\rangle_0 - \mu g_1(t)[\beta_i T_i \langle p_j\rangle_0 + \beta_j T_j \langle p_i\rangle_0] + \\ &+ \mu^2 \int_0^t dt_1 g(t_1) \int_0^{t_1} dt_2 g(t_2) \{\beta_i [\vec{T} \times \langle\phi_0|\vec{B}(t_2) p_j|\phi_0\rangle]_i + \\ &+ \beta_j [\vec{T} \times \langle\phi_0|\vec{B}(t_2) p_i|\phi_0\rangle]_j\} + \mu^2 \frac{1}{2} g_1^2(t) \beta_i \beta_j \langle s_i s_j + s_j s_i\rangle + \cdots \end{aligned} \tag{21}$$

The system of 9 linear equations (18) and (21) allows one to determine three components T_k and six components $\langle s_i s_j + s_j s_i\rangle$ in the case $s = 1$. The system assumes the simplest form if one neglects the terms $\sim \mu^3, \mu^4, \cdots$. Then, one may determine at first T_1, T_2, T_3 from eq.(18), insert them into the r.h.s. of eqs.(21) and calculate $\langle s_i s_j + s_j s_i\rangle$ (in order to calculate $\langle\phi_0|\vec{B} p_i|\phi_0\rangle$, one needs the spatial wave function ϕ_0).

In the case $s \geq 3/2$, one must determine the rank 3 polarization tensor. For this purpose, one must add to eqs.(18) and (21) the equations for $\langle p_i p_j p_k\rangle$, $i,j,k = 1,2,3$. The simplest equations result, if one neglects the terms $\sim \mu^4$. I shall not write out these combersome equations which show that $\langle p_i p_j p_k\rangle$ depend upon symmetrized components $\langle s_i s_j s_k\rangle$.

The above presentation makes it clear how to determine polarization tensors fo arbitrary ranks q up to $q = 2s$.

4 Comparison with the usual treatment of the Stern–Gerlach device

The purpose of this section is to show that one obtains incorrect theoretical values for the moments of particle momentum distribution if the so–called entanglement approximation is used.

4.1. Busch and Schroeck [5] have shown how this approximation follows from the model theory of the Stern–Gerlach device if one uses the correct description of the Stern–Gerlach field $\vec{B}(\vec{x})$ (satisfying the equation $\mathrm{div}\vec{B} = 0$ and $\mathrm{rot}\vec{B} = 0$) and employs the "impulsive" approach ignoring the term $p^2/2m$ in eq.(6), see [3,4,5]. I shall obtain approximation equations which are necessary for my purpose by using instead another simplification suggested and discussed in [10], sect 3: the $T\exp$ turns into the usual exponent $\exp[-ig_1(t)\mu\vec{s}\cdot\vec{B}(\vec{x})]$ if one replaces $\vec{B}_k(t)$ in $H_I(t)$, see eq.(13), by $B_k(\vec{x}) = B_{0k} + \beta_k x_k$, omitting the term $t\beta_k p_k/m$.It suffices to deal, in what follows, with the case $s = 1/2$. In this case, the latter exponent can be calculated nonperturbatively: using the equation $(2\vec{s}\vec{h})^2 = h^2$ one obtains for $t > \tau$

$$\exp[-i\tau\mu\vec{s}\cdot\vec{B}(\vec{x})] \equiv \exp[-i2\vec{s}\vec{h}] = \cos h + i2\vec{s}\vec{h}\frac{\sin h}{h} \tag{22}$$

The notation $\vec{h}(\vec{x}) = \frac{1}{2}\mu\tau\vec{B}(\vec{x})$ was introduced. So at $t > \tau$ one obtains for $\psi_I(t)$, eq.(11), the following approximate solution

$$\psi_I(t) \cong \exp(-i2\vec{s}\vec{h})\phi_0\chi_0 = \phi_0(\cos h - i2(\vec{s}\vec{h})\sin h/h)\chi_0 \tag{23}$$

Our approach unlike the "impulsive" one allows one to describe the spatial propagation and smearing of the particle packet during and after the action of the magnetic field due to the equation (see sect 2)

$$\psi(t) = \exp(-iH_0(t)\psi_I(t)$$

Later we deal with the particle momentum distribution and its moments which are the same in the Schroedinger and interaction pictures.

Below let us use the following particular simple choice of the field $\vec{B}(\vec{x})$: $\vec{B}_0$ is parallel to the unit eigenvector $\vec{v}_3$ of the inhomogeneous part $\vec{b}(\vec{x})$; see eq.(8), and β_2 is equal to zero. Then

$$\vec{B}(\vec{x}) = (B_o + \beta_3 x_3)\vec{v}_3 + \beta_1 x_1 \vec{v}_1 \tag{24}$$

$$h(\vec{x}) = \frac{1}{2}\mu\tau[(B_0 + \beta_3 x_3)^2 + \beta_1^2 x_1^2]^{1/2} \tag{25}$$

Note that this choice does not allow to find the polarization vector component T_2 but allows to find T_1 and T_3. This will be enough for our purpose.

In the case $s = 1/2$ one can expand the initial spinor χ_0 in the eigenfunctions $\chi_\pm$ of the operator $\vec{B}_0\cdot\vec{s}$ or $\vec{v}_3\cdot\vec{s} = s_3$[1]:

$$\chi_0 = \begin{pmatrix} \cos\frac{\theta_0}{2} \\ \sin\frac{\theta_0}{2}e^{i\phi_0} \end{pmatrix} = A_+\chi_+ + A_-\chi_-, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$

[1]Note that in [10], sect 3, another expansion in eigenfunctions $\chi_\pm(\vec{x})$ of $(\vec{B}(\vec{x})\vec{s})$ has been used. The expansion does not suit for the ensuing investigation of the corrections to the convectional "entangled" approximation, see eq.(34) below. Note also that the parametrization of χ_0 used in eq.(26) corresponds to the related polarization vector $\vec{T}$ having the spherical angles θ_0, ϕ_0.

$$A_+ = \langle\chi_+|\chi_0\rangle = \cos\frac{\theta_0}{2} \tag{27}$$

Then, eq.(23) can be rewritten as

$$\begin{aligned}\psi_I(t>\tau) &= \phi_0\{A_+\cos h - iA_+\frac{\sin h}{h}h_3 - iA_-\frac{\sin h}{h}h_1\}\chi_+ + \\ &+ \phi_0\{A_-\cos h + iA_-\frac{\sin h}{h}h_3 - iA_+\frac{\sin h}{h}h_1\}\chi_-\end{aligned} \tag{28}$$

(the equation $(2s_3h_3 + 2s_1h_1)\chi_\pm = \pm\chi_\pm + h_1\chi_\mp$ has been employed). When considering the Stern–Gerlach experiment, one assumes the inequalities

$$B_0 >> \beta_1 x_1, \quad B_0 >> \beta_3 x_3, \quad x_1, x_3 \in \text{ Support } \phi_0(\vec{x}) \tag{29}$$

Then, one has (see eq.(24)):

$$\frac{h_3}{h} \cong 1 - \frac{\beta_1^2 x_1^2}{2B_0^2}; \quad \frac{h_1}{h} \cong \frac{\beta_1 x_1}{B_0}(1 - \frac{\beta_3 x_3}{B_0} + \frac{\beta_1^2 x_1^2}{2B_0^2}) \tag{30}$$

Neglecting the terms of the order $[\beta_x x_k/B_0]^2$ one has

$$\begin{aligned}\psi_I(t>\tau) &= \phi_0\{A_+e^{-ih(x)} - iA_-\frac{\beta_1 x_1}{B_0}\sin h\}\chi_+ + \\ &+ \phi_0\{A_-e^{+ih(x)} - iA_+\frac{\beta_1 x_1}{B_0}\sin h\}\chi_-\end{aligned} \tag{31}$$

The inequalities (29) allow one to write the following approximation for $h(\vec{x})$, eq.(25)

$$h(\vec{x}) = \frac{1}{2}\mu\tau B_0(1 + \beta_3 x_3/B_0 + \beta_1^2 x_1^2/2B_0^2) \tag{32}$$

The term $\beta_3^2 x_3^2/B_0^2$ was neglected, but the term $\beta_1^2 x_1^2/2B_0^2$ must be retained when considering the case of small x_3. We see that $\exp(\pm i\mu\tau\beta_3 x_3/2)$ in eq.(31) are proportional to the plane waves $\exp(\pm i\mu\tau\beta_3 x_3/2)$ but only if x_3 is not small. With this reservation one obtains from (31) (neglecting the terms proportional to $\beta_1 x_1/B_0 << 1$)

$$\psi_I(t>\tau) = \phi_0\{A_+e^{-i\mu\tau B_0/2}e^{-i\vec{q}\vec{x}}\chi_+ + A_-e^{+i\mu\tau B_0/2}e^{+i\vec{q}\vec{x}}\chi_-\} \tag{33}$$

$$\vec{q} \equiv \frac{1}{2}\mu\tau\beta_3\vec{v}_3 \tag{34}$$

Eq.(33) shows that the spin-up spinor χ_+ is entangled with the spatial wave function $\phi_0\exp(-i\vec{q}\vec{x})$ and χ_- is entangled with $\phi_0\exp(+i\vec{q}\vec{x})$. The consequence is that the initial beam separates into two beams, in the up beam the spin state being χ_- and in the down beam the spin state being χ_+ (the corresponding probabilities being equal to $|A_-|^2$ and $|A_+|^3$).

The derivation of eq.(33) given above shows that there are several deviations from this simple picture. Busch and Schroeck [5] have discussed at length these deviations (using the "impulsive" approximation) I can add to this topic that $\exp ih(x)$ cannot be approximated by the plane wave at small x_3. Our purpose here is to discuss the relation of the entanglement approximation given by eq.(33) to the calculation of the momentum expectation values $\langle p_k\rangle$.

4.2. The following simple argument shows that the entanglement approximation gives an incorrent value for $\langle p_k\rangle$. Eq.(33) means that the initial packet ϕ_0 (having the approximate momentum $\langle\vec{p}\rangle_0 \equiv \vec{p}_0$) is divided in the device into two separated packets $\phi_0\exp(\pm i\vec{q}\vec{x})$ so that the probability for the particle to be in the down packet is $|A_+|^2 = \cos^2\theta_0/2$, see eq.(27). Then, the average momentum must be

$$\langle\vec{p}\rangle \cong (A_-|^2(\vec{p}_0 + \vec{q}) + (A_+|^2(\vec{p}_0 - \vec{q}) = \vec{p}_0 - \vec{q}\cos\theta_0\,, \tag{35}$$

The resulting vector $\langle\vec{p}\rangle - \vec{p}_0$ is parallel to $\vec{v}_3$; meanwhile according to eq.(20), it should have also nonzero $\vec{v}_1$ component in the particular case $\beta_2 = 0$, $\beta_1 = -\beta_3 \neq 0$ under consideration, see eq.(24) (remind that eq.(20) is valid in the first order of perturbation theory, so one must suppose here that $\mu B(\vec{x})$ is small).

It is possible to show that the correct value for $\langle p_k\rangle = \langle\psi_I(t)|p_k|\psi_I(t)\rangle$ can be obtained if one uses a more exact eq.(31) for $\psi_I(t)$. The calculation turns out to be much more difficult than that of sect 3, which uses the equation $\langle p_k\rangle = \langle\psi_0|p_k(t)|\psi_0\rangle$. But this calculation shows that the value, $-\mu\tau\beta_3 T_3$ (where $T_3 = \frac{1}{2}\cos\theta_0$) for $\langle p_3\rangle - \langle p_3\rangle_0$ has the origin described above before eq.(35). The terms $A_\pm\sin h\beta_1 x_2/B_0$ from eq. (31) do not contribute to $\langle p_3\rangle - \langle p_3\rangle_0$. But the terms give the correct nonzero value $-\mu\tau\beta_1 T_1$ to the difference $\langle p_1\rangle - \langle p_1\rangle_0$, cf. eq.(20). In this calculation one must consider h to be small and take into account only terms of the first order in β_k.

So one may conclude that the entanglement approximation is inadmissible for the $\langle p_k\rangle$ calculation, i.e. for the determination of the initial spin state.

5 Conclusion

The Stern–Gerlach device is known as a way to measure neutral particle spin observables. It is shown here how to use it for the determination of the particle initial spin state. This allows one to achieve the "determinative" purpose of the spin observable measurement: if one knows the initial spin state, one can calculate probabilities of eigenvalues of any spin observable. One need not to measure for this purpose either the usual spin observables or any its generalizations ("unsharp" observables or POV measures, e.g. see [2,5]). But this is another, "preparatory" purpose of quantum measurement: it allows one to prepare the physical system in a known state. It has been shown in [2,5] that the Stern-Gerlach device can realize this purpose only approximately. So one may conclude that the device suits rather for the determination of the initial spin state than for the spin observable measurement.

Appendix. Derivation of equation (16)

The interaction picture evolution operator $U(t)$, see eq.(12), and its hermitian conjugate $U^+(t)$ satisfy the equations

$$\partial_t U(t) = -iH_I(t)U(t), \quad \partial_t U^+(t) = iU^+(t)H_I(t) \tag{A.1}$$

Following Schwinger's approach [11] let us use the identity

$$U^+(t)OU(t) = O + \int_0^t dt_1 \frac{d}{dt_1}[U^+(t_1)OU(t_1)] \tag{A.2}$$

Eqs.(A.1) give

$$\frac{d}{dt_1}[U^+(t_1)OU(t_1)] = iU^+(t_1)[H_I(t_1), O]U(t_1) \tag{A.3}$$

So one has

$$U^+(t)OU(t) = O + \int_0^t dt_1 iU^+(t_1)[H_I(t_1)O]U(t_1)$$

Further use again the identity analogous to (A.2)

$$U^+(t_1)[H_I(t_1), O]U(t_1) = [H_I(t_1), O] + \int_0^t dt_2 \frac{d}{dt_2}\{U^+(t_2)[H_I(t_1), O]U(t_2)\} =$$

$$= [H_I(t_1), O] + \int_0^{t_1} dt_2 iU^+(t_2)[H_I(t_2), [H_I(t_1), O]]U^+(t_2)$$

The infinite repetition of this procedure gives eq.(16).

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