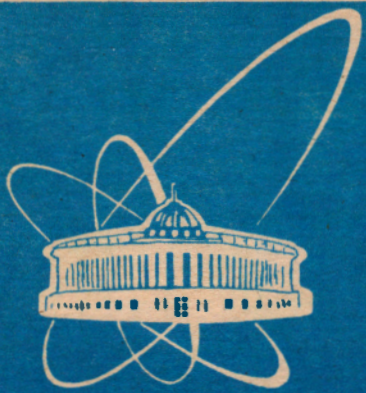

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QUADRUPOLE AND MONOPOLE LARGE AMPLITUDE VIBRATIONS

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1 Introduction

The method for the description of nuclear collective motion using the Wigner function moments (WFM) has been suggested in 1981 [1]. It was applied successively to study isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with different realistic forces [2]. Their energies, excitation probabilities and widths were described. However, all the calculations were performed in the small amplitude approximation.

In this paper the simple model of a harmonic oscillator with separable quadrupole-quadrupole residual interaction is used to demonstrate the possibilities of the WFM method in the description of large amplitude motion. The interest in the investigations of collective motions going beyond the usual RPA (small amplitude approach) has been renewed by the experimental discovery of high-energy structures in heavy ion grazing collisions and their interpretation in terms of multiphonon excitations of giant quadrupole resonances [3]. In the past the problem of large amplitude collective vibrations has been treated along various lines. The ones most known are the boson expansion method, an extended review of which can be found in [4], the generator coordinate method [5] and the time-dependent Hartree-Fock (TDHF) method [5] together with its adiabatic version ATDHF [6, 7].

The practical value of the latter two methods depends in essence on the possibility of selecting a small number of collective degrees of freedom coupled weakly with all other remaining degrees of freedom. Selection of proper collective variables is certainly a great problem and requires some physical intuition. Naturally, such kind of problem exists in our approach also. It can be formulated as follows. We derive (in the TDHF frame) the dynamical equations for the phase space cartesian tensors of different ranks. In the general case of an arbitrary interaction the equations for tensors of all ranks are coupled and the problem arises how to separate the dynamics of the lower rank tensors from that of the higher rank tensors. The only exception is the case of the harmonic oscillator, where the equations for tensors of different ranks are independent. Hence, in a general case the degree of the mutual influence of the dynamics of different rank tensors is determined by the difference of the realistic mean field V_A from the harmonic oscillator potential V_H. Expending the difference $V_A(\mathbf{r}) - V_H(\mathbf{r})$ in a Taylor series with respect to the coordinates

one can develope a procedure of taking successively into account the influence of higher rank tensors on the dynamics of lower rank tensors. This program was realized in [2, 8] in a small amplitude approximation.

The first step of such a program for large amplitude motion is presented in this paper. We use the method of Wigner function moments in the frame of TDHF theory with the simple Hamiltonian of separable forces to derive a set of nonlinear dynamical equations for the quadrupole and monopole moments of nucleus. This model is attractive, because it allows to write down exact equations, which can be solved exactly. Furthermore, it can be generalized such that it becomes rather realistic.

2 Equations of motion

2.1 Formulation of the method

The basis of our method of describing collective nuclear dynamics is the equation for the one-body density matrix $\rho(\mathbf{r}_1, \mathbf{r}_2, t) = < \mathbf{r}_1|\hat{\rho}(t)|\mathbf{r}_2 >$:

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H}, \hat{\rho}\right] \tag{1}$$

where $\hat{H}$ is the one-body Hamiltonian depending implicitly on the density matrix. Equation (1) with a precise definition of the one-body Hamiltonian appears in the Hartree-Fock theory; it is currently used also within the so called energy-functional approach leaving more possibilities for the choice of the one-body Hamiltonian and in addition giving some grounds to believe equation (1) to be rather general [9].

It is convenient to reformulate equation (1) by introducing the Wigner transformations of the density matrix [10]

$$f(\mathbf{r}, \mathbf{p}, t) = \int ds\ \exp(-i\mathbf{p}\cdot\mathbf{s}/\hbar)\rho(\mathbf{r}+\frac{\mathbf{s}}{2}, \mathbf{r}-\frac{\mathbf{s}}{2}, t). \tag{2}$$

and of the Hamiltonian

$$H^W(\mathbf{r}, \mathbf{p}) = \int ds\ \exp(-i\mathbf{p}\cdot\mathbf{s}/\hbar)(\mathbf{r}+\frac{\mathbf{s}}{2}\left|\hat{H}\right|\mathbf{r}-\frac{\mathbf{s}}{2}). \tag{3}$$

Using (2,3) one arrives at [11]

$$\frac{\partial f}{\partial t} = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}(\nabla_{\mathbf{r}}^H\cdot\nabla_{\mathbf{p}}^f - \nabla_{\mathbf{p}}^H\cdot\nabla_{\mathbf{r}}^f)\right)H^W f, \tag{4}$$

where the upper index of nabla shows the function which this operator acts on. If the Hamiltonian is a sum of a kinetic energy and a local potential $V(\mathbf{r})$, its Wigner transformation is just the classical version of the same Hamiltonian:

$$H^W = p^2/2m + V(\mathbf{r}). \tag{5}$$

Then equation (4) becomes:

$$\frac{\partial f}{\partial t} + \frac{1}{m}\mathbf{p}\cdot\nabla_{\mathbf{r}} f = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\nabla_{\mathbf{r}}^V\cdot\nabla_{\mathbf{p}}^f\right)Vf. \tag{6}$$

The generalization for non-local potentials is straightforward and can be found in [12].

2.2 Wigner function moments

Now we apply the method of Wigner function moments to derive the closed system of the dynamical equations for cartesian tensors of second rank. This method was suggested in [1, 13] and is described in detail in reference [2]. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [14]. It is shown in [2], that integrating equation (6) over the phase space $\{\mathbf{p}, \mathbf{r}\}$ with the weights $x_{i_1}x_{i_2}\ldots x_{i_k}p_{i_{k+1}}\ldots p_{i_{n-1}}p_{i_n}$, where k runs from 0 to n, one can obtain a closed finite subsystem of dynamical equations for multipole moments and other integral characteristics of a nucleus. Taking different values of n (0, 1, 2, 3 and so on) one gets subsystems for different multipoles.

Here we will consider the case $n = 2$. Integrating equation (6) over the phase space $\{\mathbf{r}, \mathbf{p}\}$ with the weights x_ix_j, p_ix_j, p_ip_j we get:

$$\int d\mathbf{r}\ x_ix_j\frac{\partial n(\mathbf{r},t)}{\partial t} + \int d\mathbf{r}\ x_ix_j\frac{\partial(n(\mathbf{r},t)u_s(\mathbf{r},t))}{\partial x_s} = 0, \tag{7}$$

$$m\int d\mathbf{r}\ x_j\frac{\partial}{\partial t}(n(\mathbf{r},t)u_i(\mathbf{r},t)) + \int d\mathbf{r}\ n(\mathbf{r},t)x_j\frac{\partial V}{\partial x_i} + \int d\mathbf{r}\ x_j\frac{\partial}{\partial x_s}\mathcal{A}_{si}(\mathbf{r},t) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}\int d\mathbf{r}\ \mathcal{A}_{ij}(\mathbf{r},t) + \int d\mathbf{r}\ n(\mathbf{r},t)\left[u_i(\mathbf{r},t)\frac{\partial V}{\partial x_j}\right]_{ij} + \int d\mathbf{r}\ \frac{\partial}{\partial x_s}\mathcal{A}_{sij}(\mathbf{r},t) = 0, \tag{9}$$

where $[\ldots]_{ij}$ means that the quantity in brackets is symmetrized with respect to the indices i and j ($[a_ib_j]_{ij} = a_ib_j + a_jb_i$) and summation over repeated indices is assumed. Here we have introduced the notations:

$$n(\mathbf{r},t) \;=\; 4\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f(\mathbf{r},\mathbf{p},t),$$

$$mn(\mathbf{r},t)u_i(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} p_i f(\mathbf{r},\mathbf{p},t),$$
$$\mathcal{A}_{i_1 i_2 \ldots i_k}(\mathbf{r},t) = m^{1-k} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} p_{i_1} p_{i_2} \cdots p_{i_k} f(\mathbf{r},\mathbf{p},t).$$

By definition $n(\mathbf{r},t)$ is the nucleon density, $\mathbf{u}(\mathbf{r},t)$ is the mean velocity of nucleons, $\mathcal{A}_{ij}(\mathbf{r},t)$ is the kinetic energy tensor (or pressure tensor). Integrating by parts the last terms in (7)-(9) and introducing the notations

$$J_{ij}(t) = \int d\mathbf{r}\, x_i x_j n(\mathbf{r},t)$$

for the inertia tensor,

$$L_{i,j}(t) = \int d\mathbf{r}\, x_i u_j(\mathbf{r},t) n(\mathbf{r},t)$$

for the mixed momentum-position tensor and

$$\Pi_{ij}(t) = \int d\mathbf{r}\, \mathcal{A}_{ij}(\mathbf{r},t)$$

for the integral kinetic energy tensor we have:

$$\frac{d}{dt} J_{ij}(t) - \frac{1}{m}(L_{i,j} + L_{j,i}) = 0, \tag{10}$$
$$\frac{d}{dt} L_{i,j} + \int d\mathbf{r}\, x_i n(\mathbf{r},t) \frac{\partial V}{\partial x_j} - \Pi_{ij}(t) = 0, \tag{11}$$
$$\frac{d}{dt} \Pi_{ij}(t) + \int d\mathbf{r}\, n(\mathbf{r},t) \left[u_i(\mathbf{r},t) \frac{\partial V}{\partial x_j} \right]_{ij} = 0. \tag{12}$$

The last integral of the equation (9) with the third rank tensor $\mathcal{A}_{sij}$ has disappeared due to the evident boundary condition $\mathcal{A}_{sij}(\mathbf{r},t) \to 0$ at $\mathbf{r} \to \infty$, which follows from the boundary condition for the Wigner function $f(\mathbf{r},\mathbf{p},t) \to 0$ at $\mathbf{r} \to \infty$. So, one want to say that we have derived the system of three dynamical equations for three collective variables $J_{ij}(t), L_{i,j}(t)$ and $\Pi_{ij}(t)$. To make this statement true one must represent the integrals containing derivatives of the potential $V(\mathbf{r})$ in terms of these three variables. This problem can be solved exactly only in the case of a harmonic oscillator potential (that is the subject of this paper). For a realistic potential some approximations are needed.

We suggest the following procedure. Considering the harmonic oscillator potential V_H as the zero approximation to the realistic potential V_R we expand the difference $V_R - V_H$ in a Taylor series and truncate it on the term proportional to $\mathbf{r}^n$. The integration of

the potential will generate the tensors of different ranks from 1 up to n. Hence, to have the closed system of equations we are forced to write down the subsystems of dynamical equations for tensors of all ranks from 1 up to n, these subsystems being coupled. The more terms of the Taylor series are taken into account, the higher rank tensors must be included in the calculations. So, the compulsory minimal rank of tensors is determined by the measure of deviation of the realistic potential from the harmonic oscillator one. The desired maximum rank is determined by the physics of the phenomenon studied: the more detailed information is required, the higher rank tensors must be involved in the consideration.

The equations (10) and (12) are evidently symmetrical with respect to indexes i, j whereas eq. (11) has no definite symmetry. We can construct easily the symmetrical and antisymmetrical equations by evident combinations of (11) with different indices:

$$\frac{d}{dt}(L_{i,j} + L_{j,i}) + \int d\mathbf{r}\, n(\mathbf{r},t) \left[x_j \frac{\partial V}{\partial x_i} \right]_{ij} - 2\Pi_{ij}(t) = 0, \tag{13}$$
$$\frac{d}{dt}(L_{i,j} - L_{j,i}) = \int d\mathbf{r}\, n(\mathbf{r},t) \left\{ x_i \frac{\partial V}{\partial x_j} - x_j \frac{\partial V}{\partial x_i} \right\}. \tag{14}$$

By definition the left-hand side of the equation (14) is the angular momentum:

$$M_{i,j} = L_{i,j} - L_{j,i} = m \int d\mathbf{r}\, n(\mathbf{r},t) \left\{ x_j u_i(\mathbf{r},t) - x_i u_j(\mathbf{r},t) \right\}.$$

When $V(\mathbf{r},t)$ is a self-consistent potential, the right-hand side of (14) is equal to zero and this equation express the angular momentum conservation law.

The model potential we will consider here is a harmonic oscillator with a quadrupole-quadrupole residual interaction. The corresponding mean field can be written as

$$V(\mathbf{r},t) = \frac{1}{2} m\omega^2 r^2 + \lambda \sum_{\mu=-2}^{2} Q_{2\mu}(t) q^{\dagger}_{2\mu}(\mathbf{r}), \tag{15}$$

where the quadrupole moment $Q_{2\mu}(t)$ is defined by the relation

$$Q_{2\mu}(t) = \mathrm{tr}\, q_{2\mu}\, \rho = 4 \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} q_{2\mu}(\mathbf{r}) f(\mathbf{r},\mathbf{p},t)$$

with $q_{2\mu} = 4\sqrt{\pi/5}\, r^2 Y_{2\mu}$ and

$$Q_{20} q^{\dagger}_{20} = (J_{11} + J_{22} - 2J_{33})(x_1^2 + x_2^2 - 2x_3^2),$$
$$Q_{21} q^{\dagger}_{21} + Q_{2-1} q^{\dagger}_{2-1} = 12(J_{13} x_1 x_3 + J_{23} x_2 x_3),$$
$$Q_{22} q^{\dagger}_{22} + Q_{2-2} q^{\dagger}_{2-2} = 3(J_{11} - J_{22})(x_1^2 - x_2^2) + 12 J_{12} x_1 x_2,$$

For this potential

$$\frac{1}{\lambda}\int d\mathbf{r}\, n(\mathbf{r},t)\left\{x_j\frac{\partial V}{\partial x_i}-x_i\frac{\partial V}{\partial x_j}\right\}=2Q_{20}(\delta_{i1}-\delta_{j1}+\delta_{i2}-\delta_{j2}-2\delta_{i3}+2\delta_{j3})J_{ij}$$
$$+12J_{13}(J_{1j}\delta_{i3}-J_{1i}\delta_{j3}+J_{3j}\delta_{i1}-J_{3i}\delta_{j1})+12J_{23}(J_{2j}\delta_{i3}-J_{2i}\delta_{j3}+J_{3j}\delta_{i2}-J_{3i}\delta_{j2})$$
$$+6(J_{11}-J_{22})(\delta_{i1}-\delta_{j1}-\delta_{i2}+\delta_{j2})J_{ij}+12J_{12}(J_{1j}\delta_{i2}-J_{1i}\delta_{j2}+J_{2j}\delta_{i1}-J_{2i}\delta_{j1}).$$

With the help of this expression it is easy to show that $\dot{M}_{1,2}=\dot{M}_{1,3}=\dot{M}_{2,3}=0$, where dot means the time derivative. Hence our model conserves the angular momentum.

Let us further note an interesting observation. With the model potential (15) our system of the dynamical equations (10), (12),(13) for the variables

$$\int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}x_ix_jf(\mathbf{r},\mathbf{p},t),\int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}p_ip_jf(\mathbf{r},\mathbf{p},t),\int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[x_ip_j\right]_{ij}f(\mathbf{r},\mathbf{p},t)$$

becomes identical to the system (65) derived in the paper of Schuck [15] for the variances-covariances

$$D(x_ix_j)=<(\hat{x}_i-x_i^c)(\hat{x}_j-x_j^c)>,\ D(p_ip_j)=<(\hat{p}_i-p_i^c)(\hat{p}_j-p_j^c)>,$$
$$2D(x_ip_j)=2D(p_jx_i)=<\left[(\hat{x}_i-x_i^c)(\hat{p}_j-p_j^c)\right]_{ij}>,$$

where $\hat{x}_i,\hat{p}_i$ are quantum operators and x_i^c,p_i^c are their classical counterparts. The identity of these systems is not very surprising because, being written for principally the same variables, they don't contain any approximations in the case of the model potential (15).

To simplify the following formulae we will use eq. (10) to get rid off the symmetrical combination $(L_{i,j}+L_{j,i})$ everywhere. We introduce also the notation

$$J_-=J_{11}-J_{22}$$

for the measure of a non-axiality of a nucleus. Putting now the expression (15) for the potential into the equations (13), (12) we get finally the following system of the equations of motion for the collective variables J_{ij} and Π_{ij}:

$$\frac{m}{2}\ddot{J}_{11}+m\omega^2J_{11}+2\lambda\left\{6J_{13}^2+6J_{12}^2+3J_-J_{11}+J_{11}Q_{20}\right\}-\Pi_{11}=0,$$
$$\frac{m}{2}\ddot{J}_{22}+m\omega^2J_{22}+2\lambda\left\{6J_{23}^2+6J_{12}^2-3J_-J_{22}+J_{22}Q_{20}\right\}-\Pi_{22}=0,$$
$$\frac{m}{2}\ddot{J}_{33}+m\omega^2J_{33}+2\lambda\left\{6J_{13}^2+6J_{23}^2-2J_{33}Q_{20}\right\}-\Pi_{33}=0,$$
$$\frac{m}{2}\ddot{J}_{12}+m\omega^2J_{12}+2\lambda\left\{J_{12}(4J_{22}+4J_{11}-2J_{33})+6J_{13}J_{23}\right\}-\Pi_{12}=0,$$

$$\frac{m}{2}\ddot{J}_{13}+m\omega^2J_{13}+2\lambda\left\{J_{13}(4J_{33}+4J_{11}-2J_{22})+6J_{12}J_{23}\right\}-\Pi_{13}=0,$$
$$\frac{m}{2}\ddot{J}_{23}+m\omega^2J_{23}+2\lambda\left\{J_{23}(4J_{33}+4J_{22}-2J_{11})+6J_{12}J_{13}\right\}-\Pi_{23}=0,$$
$$\dot{\Pi}_{11}+m\omega^2\dot{J}_{11}+2\lambda\Big\{6J_{12}(\dot{J}_{12}+M_{1,2})+6J_{13}(\dot{J}_{13}+M_{1,3})$$
$$+3J_-\dot{J}_{11}+Q_{20}\dot{J}_{11}\Big\}=0,$$
$$\dot{\Pi}_{22}+m\omega^2\dot{J}_{22}+2\lambda\Big\{6J_{12}(\dot{J}_{12}-M_{1,2})+6J_{23}(\dot{J}_{23}+M_{2,3})$$
$$-3J_-\dot{J}_{22}+Q_{20}\dot{J}_{22}\Big\}=0,$$
$$\dot{\Pi}_{33}+m\omega^2\dot{J}_{33}+2\lambda\Big\{6J_{13}(\dot{J}_{13}-M_{1,3})+6J_{23}(\dot{J}_{23}-M_{2,3})-2Q_{20}\dot{J}_{33}\Big\}=0,$$
$$\dot{\Pi}_{12}+m\omega^2\dot{J}_{12}+2\lambda\Big\{3J_{12}(\dot{J}_{11}+\dot{J}_{22})+3J_{13}(\dot{J}_{23}+M_{2,3})$$
$$+3J_{23}(\dot{J}_{13}+M_{1,3})-3J_-M_{1,2}+Q_{20}\dot{J}_{12}\Big\}=0,$$
$$\dot{\Pi}_{13}+m\omega^2\dot{J}_{13}+\lambda\Big\{6J_{12}(\dot{J}_{23}-M_{2,3})+6J_{13}(\dot{J}_{33}+\dot{J}_{11})+6J_{23}(\dot{J}_{12}+M_{1,2})$$
$$+3J_-(\dot{J}_{13}-M_{1,3})-Q_{20}(\dot{J}_{13}+3M_{1,3})\Big\}=0,$$
$$\dot{\Pi}_{23}+m\omega^2\dot{J}_{23}+\lambda\Big\{6J_{12}(\dot{J}_{13}-M_{1,3})+6J_{13}(\dot{J}_{12}-M_{1,2})+6J_{23}(\dot{J}_{33}+\dot{J}_{22})$$
$$-3J_-(\dot{J}_{23}-M_{2,3})-Q_{20}(\dot{J}_{23}+3M_{2,3})\Big\}=0, \qquad (16)$$

Here we did not write out the time dependence of tensors for simplicity.

It is known [16] that a cartesian tensor of second rank may be represented by a sum of irreducible tensors - one zero rank tensor (monopole moment), one first rank pseudotensor (angular momentum) and one second rank tensor (quadrupole moment):

$$T_{ij}\Rightarrow\mathcal{T}_{00}+\mathcal{T}_{1\mu}+\mathcal{T}_{2\mu}.$$

Taking the evident combinations of eqs. (16) we can rewrite this system in terms of the irreducible tensors

$$\begin{aligned}
Q_{00} &= J_{11}+J_{22}+J_{33}\\
Q_{20} &= J_{11}+J_{22}-2J_{33}\\
Q_{2\pm1} &= \mp\sqrt{6}(J_{13}\pm iJ_{23})\\
Q_{2\pm2} &= \sqrt{3/2}(J_{11}-J_{22}\pm 2iJ_{12})\\
\mathcal{K}_{00} &= \Pi_{11}+\Pi_{22}+\Pi_{33}\\
\mathcal{K}_{20} &= \Pi_{11}+\Pi_{22}-2\Pi_{33}\\
\mathcal{K}_{2\pm1} &= \mp\sqrt{6}(\Pi_{13}\pm i\Pi_{23})\\
\mathcal{K}_{2\pm2} &= \sqrt{3/2}(\Pi_{11}-\Pi_{22}\pm 2i\Pi_{12})
\end{aligned}$$

Due to the conservation of the angular momentum the variables $M_{i,j}$ do not depend on time and are determined by the initial conditions. We take $M_{i,j} = 0$.

Generally speaking all equations (16) are coupled. However, by a proper choice of the initial conditions the system (16) can be reduced to three cases, which correspond (in the limit of a small amplitude approximation) to the components $\mu = 0, 2, 1$ of the quadrupole moment and are known as β-mode, γ-mode and the transverse-shear mode (we will call it α-mode).

2.2.1 β-mode

It is easy to notice that one of the possible solutions of the system (16) is:

$$J_-(t) = 0,\ J_{12}(t) = J_{13}(t) = J_{23}(t) = 0,$$

$$\Pi_{11}(t) - \Pi_{22}(t) = 0,\ \Pi_{12}(t) = \Pi_{13}(t) = \Pi_{23}(t) = 0$$

with all remaining variables different from zero. This solution conserves the axial and triplanar symmetry of a nucleus. In this case the system (16) is transformed into

$$\begin{aligned}
&\frac{m}{2}\ddot{Q}_{00} + m\omega^2\dot{Q}_{00} + 2\lambda Q_{20}^2 - \mathcal{K}_{00} = 0,\\
&\frac{m}{2}\ddot{Q}_{20} + m\omega^2 Q_{20} + 2\lambda Q_{20}(2Q_{00} - Q_{20}) - \mathcal{K}_{20} = 0,\\
&\dot{\mathcal{K}}_{00} + m\omega^2\dot{Q}_{00} + 2\lambda Q_{20}\dot{Q}_{20} = 0,\\
&\dot{\mathcal{K}}_{20} + m\omega^2\dot{Q}_{20} + 2\lambda Q_{20}(2\dot{Q}_{00} - \dot{Q}_{20}) = 0.
\end{aligned} \tag{17}$$

The third equation of this system gives the integral of motion

$$\mathcal{K}_{00} + m\omega^2 Q_{00} + \lambda Q_{20}^2 = \text{const} \tag{18}$$

whose physical meaning is the Hartree-Fock energy corresponding to the Hamiltonian (5), when the potential (15) contains only the $\mu = 0$ term. It is easy to see that (17) has the particular solution $Q_{20} = \mathcal{K}_{20} = 0$ corresponding to the simple harmonic oscillator and describing a pure monopole vibrations with the frequency $\Omega = 2\omega$.

2.2.2 γ-mode

Taking

$$J_{12}(t) = J_{13}(t) = J_{23}(t) = 0, \quad \Pi_{12}(t) = \Pi_{13}(t) = \Pi_{23}(t) = 0$$

we find a second solution of the system (16), which conserves the triplanar symmetry of a nucleus but destroys its axial symmetry. The corresponding set of equations is

$$\begin{aligned}
&\frac{m}{2}\ddot{Q}_{00} + m\omega^2 Q_{00} + 2\lambda(Q_{20}^2 + 3J_-^2) - \mathcal{K}_{00} = 0,\\
&\frac{m}{2}\ddot{Q}_{20} + m\omega^2 Q_{20} + 2\lambda\left\{Q_{20}(2Q_{00} - Q_{20}) + 3J_-^2\right\} - \mathcal{K}_{20} = 0,\\
&\frac{m}{2}\ddot{J}_- + m\omega^2 J_- + 4\lambda J_-(Q_{00} + Q_{20}) - \Pi_- = 0,\\
&\dot{\mathcal{K}}_{00} + m\omega^2\dot{Q}_{00} + 2\lambda(Q_{20}\dot{Q}_{20} + 3J_-\dot{J}_-) = 0,\\
&\dot{\mathcal{K}}_{20} + m\omega^2\dot{Q}_{20} + 2\lambda\left\{Q_{20}(2\dot{Q}_{00} - \dot{Q}_{20}) + 3J_-\dot{J}_-\right\} = 0.\\
&\dot{\Pi}_- + m\omega^2\dot{J}_- + 2\lambda\left\{J_-(2\dot{Q}_{00} + \dot{Q}_{20}) + Q_{20}\dot{J}_-\right\} = 0,
\end{aligned} \tag{19}$$

where $J_- = J_{11} - J_{22} = (Q_{22} + Q_{2-2})/\sqrt{6}$ and $\Pi_- = \Pi_{11} - \Pi_{22} = (\mathcal{K}_{22} + \mathcal{K}_{2-2})/\sqrt{6}$. The fourth equation gives the integral of motion:

$$\mathcal{K}_{00} + m\omega^2 Q_{00} + \lambda(Q_{20}^2 + 3J_-^2) = \text{const}, \tag{20}$$

whose physical meaning is the same as that of (18) but for the case when the potential (15) contains two terms: with $\mu = 0$ and $\mu = 2$.

Analyzing this system of equations one can find three particular solutions. For the first one $J_-(t) = \Pi_-(t) = 0$ and the system (19) is reduced to the system (17). The other two solutions have $J_-(t) = \pm Q_{20}(t)$, $\Pi_-(t) = \pm\mathcal{K}_{20}(t)$. The equality $J_- = Q_{20}$ means that $J_{22} = J_{33}$ and the equality $J_- = -Q_{20}$ leads to $J_{11} = J_{33}$. Hence, these particular solutions describe vibrations conserving the axial symmetry along the first and the second axes correspondingly. The same kind of motion (with the third symmetry axes) is described by the system (17). From the physical point of view all three axes are equivalent, so the corresponding systems of equations must coincide. Really, taking in (19) $J_- = \pm Q_{20}$, $\Pi_- = \pm\mathcal{K}_{20}$ and changing the variables $2Q_{20}$, $2\mathcal{K}_{20}$ by $-Q_{20}, -\mathcal{K}_{20}$ one reduces this system to (17).

Formally there exists one more solution of the systems (17) and (19) with $Q_{20}(t) = Q_{00}(t)$, $\mathcal{K}_{20}(t) = \mathcal{K}_{00}(t)$. However, it has not much physical meaning because the equality $Q_{20} = Q_{00}$ means that $J_{33} = 0$, i.e. we are dealing with a two-dimensional object.

2.2.3 α-mode

The most complicated solution is found, when

$$J_{12}(t) = J_{23}(t) = 0, \quad \Pi_{12}(t) = \Pi_{23}(t) = 0,$$

with all remaining variables different from zero. It destroys the triplanar and axial symmetry conserving only the symmetry with respect to the reflection in the plane $x_2 = 0$. The corresponding system of equations is

$$\begin{aligned}
&\frac{m}{2}\ddot{Q}_{00} + m\omega^2 Q_{00} + 2\lambda\,(Q_{20}^2 + 12J_{13}^2 + 3J_-^2) - \mathcal{K}_{00} = 0,\\
&\frac{m}{2}\ddot{Q}_{20} + m\omega^2 Q_{20} + 2\lambda\,\{Q_{20}(2\,Q_{00} - Q_{20}) - 6J_{13}^2 + 3J_-^2\} - \mathcal{K}_{20} = 0,\\
&\frac{m}{2}\ddot{J}_- + m\omega^2 J_- + 4\lambda\,\{J_-(Q_{00} + Q_{20}) + 3J_{13}^2\} - \Pi_- = 0,\\
&\frac{m}{2}\ddot{J}_{13} + m\omega^2 J_{13} + 2\lambda\,J_{13}(2\,Q_{00} - Q_{20} + 3J_-) - \Pi_{13} = 0,\\
&\dot{\mathcal{K}}_{00} + m\omega^2\dot{Q}_{00} + 2\lambda\,(Q_{20}\dot{Q}_{20} + 3J_-\dot{J}_- + 12J_{13}\dot{J}_{13}) = 0,\\
&\dot{\mathcal{K}}_{20} + m\omega^2\dot{Q}_{20} + 2\lambda\,\left\{Q_{20}(2\,\dot{Q}_{00} - \dot{Q}_{20}) + 3J_-\dot{J}_- - 6J_{13}\dot{J}_{13}\right\} = 0.\\
&\dot{\Pi}_- + m\omega^2\dot{J}_- + 2\lambda\,\left\{J_-(2\,\dot{Q}_{00} + \dot{Q}_{20}) + Q_{20}\dot{J}_- + 6J_{13}\dot{J}_{13}\right\} = 0,\\
&\dot{\Pi}_{13} + m\omega^2\dot{J}_{13} + \lambda\,\left\{J_{13}(3\dot{J}_- - \dot{Q}_{20} + 4\,\dot{Q}_{00}) + \dot{J}_{13}(3J_- - Q_{20})\right\} = 0, \qquad (21)
\end{aligned}$$

where $J_{13} = (Q_{2-1} - Q_{21})/\sqrt{24}$ and $\Pi_{13} = (\mathcal{K}_{2-1} - \mathcal{K}_{21})/\sqrt{24}$. The fifth equation gives the integral of motion:

$$\mathcal{K}_{00} + m\omega^2 Q_{00} + \lambda\,(Q_{20}^2 + 3J_-^2 + 12J_{13}^2) = \text{const}. \qquad (22)$$

Its physical meaning is the same as that of (18) and (20) but for the case when the potential (15) contains all three terms: with $\mu = 0$, $\mu = 1$ and $\mu = 2$.

From the mathematical point of view one nontrivial particular solution exists here: $J_-(t) = -Q_{20}(t), \Pi_-(t) = -\mathcal{K}_{20}(t)$. However, we know that the equality $J_- = -Q_{20}$ leads to $J_{11} = J_{33}$. From the physical point of view the inevitable consequence of the last equality is $J_{13} = 0$, i.e. we return to the β-mode.

3 Analysis of the equations of motion

3.1 Stationary solution

Investigating the stationary solutions of the systems (17, 19,21) we can draw some conclusions about the equilibrium shape of nuclei. Let us investigate the most complicated system which is (21).

By definition the variables of the stationary solution (or equilibrium state) do not depend on time. Supposing the time derivatives in (21) equal to zero one gets four

relations:

$$m\omega^2 Q_{00} + 2\lambda\,(Q_{20}^2 + 12J_{13}^2 + 3J_-^2) = \mathcal{K}_{00}, \qquad (23)$$

$$m\omega^2 Q_{20} + 2\lambda\,\{Q_{20}(2\,Q_{00} - Q_{20}) - 6J_{13}^2 + 3J_-^2\} = \mathcal{K}_{20}, \qquad (24)$$

$$m\omega^2 J_- + 4\lambda\,\{J_-(Q_{00} + Q_{20}) + 3J_{13}^2\} = \Pi_-, \qquad (25)$$

$$m\omega^2 J_{13} + 2\lambda\,J_{13}(2\,Q_{00} - Q_{20} + 3J_-) = \Pi_{13}, \qquad (26)$$

We will call them equations of equilibrium. The first equation is known as the virial theorem saying, that the average potential energy of the system is equal to its average kinetic energy if the potential has a quadratic dependence on the coordinates [17]. All the remaining equations give relations between parameters of nuclear deformation in coordinate space and corresponding ones of Fermi Surface (FS) deformation (momentum space). Very interesting conclusion can be derived from these relations. It turns out that it is impossible to have a static quadrupole deformation ($Q_{20} \neq 0,\ J_{13} \neq 0,\ J_- \neq 0$) without FS deformation ($\mathcal{K}_{20} \neq 0,\ \Pi_{13} \neq 0,\ \Pi_- \neq 0$) and vice versa [18]. To show it we transform equations (24-26) using the self-consistent value of the force constant [15, 19]

$$\lambda = \lambda_{\text{Bohr}} = \frac{-m\omega^2}{4\,A < r^2 >}. \qquad (27)$$

Taking into account the relation $Q_{00} = A < r^2 >$ we can rewrite (27) in the form $m\omega^2 + 4\lambda Q_{00} = 0$, which allows one to simplify the equations (24-26) very much:

$$2\lambda\,(3J_-^2 - Q_{20}^2 - 6J_{13}^2) = \mathcal{K}_{20}, \qquad (28)$$

$$4\lambda(J_-Q_{20} + 3J_{13}^2) = \Pi_-, \qquad (29)$$

$$2\lambda\,J_{13}(3J_- - Q_{20}) = \Pi_{13}, \qquad (30)$$

Let us suppose $\Pi_{13} = \Pi_- = \mathcal{K}_{20} = 0$. As a consequence we have from (30): $3J_- = Q_{20}$. Using this result in (29) we arrive to the relation $Q_{20}^2 + 9J_{13}^2 = 0$ which can be satisfied only by $J_{13} = Q_{20} = J_- = 0$. So, the Bohr self-consistency condition - the shape of the potential well follows the shape of the density - can be reformulated as: any change of the density shape leads inevitably to the change of FS shape. We do not say "one shape follows another" because they can be deformed "in phase" or "out of phase". One can show it analyzing eqs. (28-30). Let us consider axially symmetrical nucleus with $J_{13} = J_- = 0$. In this case eq. (30) gives $\Pi_{13} = 0$ and eq. (29) gives $\Pi_- = 0$ (i.e.

$\Pi_{11} = \Pi_{22}$). Remembering that $\lambda < 0$ we find from (28) that $\mathcal{K}_{20} > 0$ (i.e. $\Pi_{11} > \Pi_{33}$) independently of the sign of Q_{20}. So, a nucleus can have prolate or oblate density shape but its FS always will be oblate. This statement does not contradict to the well known fact that FS deformation in adiabatic processes is small [5]. Formula (28) demonstrates very well that FS deformation, being a second order effect with respect to a density deformation, really must be small.

3.2 Small amplitude approximation

Let us consider the system (21) in the small amplitude approximation. Taking the variations $Q_{\lambda 0}(t) = Q_{\lambda 0}(0) + \delta Q_{\lambda 0}(t)$, $J_{-}(t) = J_{-}(0) + \delta J_{-}(t)$, $J_{13}(t) = J_{13}(0) + \delta J_{13}(t)$, $\mathcal{K}_{\lambda 0}(t) = \mathcal{K}_{\lambda 0}(0) + \delta \mathcal{K}_{\lambda 0}(t)$, $\Pi_{-}(t) = \Pi_{-}(0) + \delta \Pi_{-}(t)$, $\Pi_{13}(t) = \Pi_{13}(0) + \delta \Pi_{13}(t)$ and neglecting the terms quadratic in δ, one obtains four independent systems. One of them is the system for the monopole tensors

$$\begin{aligned} &\frac{m}{2}\delta\ddot{Q}_{00} + m\omega^2 \delta Q_{00} - \delta \mathcal{K}_{00} = 0, \\ &\delta\dot{\mathcal{K}}_{00} + m\omega^2 \delta\dot{Q}_{00} = 0 \end{aligned} \tag{31}$$

and the remaining ones are the systems for the components of the quadrupole tensors with $\mu = 0, 1, 2$. All three have the same structure. For example:

$$\begin{aligned} &\frac{m}{2}\delta\ddot{Q}_{20} + (m\omega^2 + 4\,\lambda\, Q_{00}(0))\,\delta Q_{20} - \delta \mathcal{K}_{20} = 0, \\ &\delta\dot{\mathcal{K}}_{20} + m\omega^2 \delta\dot{Q}_{20} = 0. \end{aligned} \tag{32}$$

We consider spherical ground state nuclei in this paper, so we put everywhere $Q_{20}(0) = 0$, $J_{13}(0) = 0$, $J_{-}(0) = 0$. Supposing the time dependence $e^{i\Omega t}$ for all variables one can easily find the corresponding collective eigenfrequencies. The first system gives the frequency of monopole vibrations:

$$\Omega_0 = 2\omega. \tag{33}$$

The systems describing quadrupole vibrations give

$$\Omega_2 = 2\sqrt{\omega^2 + \frac{2\lambda}{m} Q_{00}(0)}. \tag{34}$$

Using here the expression (27) for the force constant one obtains the well known [15, 19] result for the quadrupole eigenfrequency:

$$\Omega_2 = \sqrt{2}\omega. \tag{35}$$

The energies $E_0 = \hbar\Omega_0$ and $E_2 = \hbar\Omega_2$ are in qualitative agreement with the experimental values of the energies of the monopole (GMR) and quadrupole (GQR) giant resonances (for $\hbar\omega = 41 A^{-1/3}$ MeV).

So, in the small amplitude approximation our model gives only two frequences for spherical ground state nuclei, which can be interpreted as giant 0^+ and degenerate 2^+ resonances. This is true also for the calculations with more realistic interactions [12].

3.3 Numerical solution and Fourier analysis

A principally different situation is observed in the general case, when the systems (17,19,21) are solved without any restriction. We solve them numerically with the help of Runge-Kutta procedure. Most of the calculations are done for ^{208}Pb.

The solutions depend strongly on the Initial Conditions (IC). By definition $Q_{2\mu}(0) = \mathcal{K}_{2\mu}(0) = 0$. For the monopole moment we take $Q_{00}(0) = \frac{3}{5}R_0^2 A$, with $R_0 = 1.18A^{1/3}$. The initial value of $\mathcal{K}_{00}(0)$ is fixed by the relation (23). The time derivatives $\dot{Q}_{\lambda\mu}(0)$ are arbitrary.

The most detailed analysis has been done for the β-mode. The typical time-dependence of the function $Q_{20}(t)$ is demonstrated in fig.1. As one can see, it oscillates quite irregularly. The maximum period of oscillations τ_2 (when the curve begins to repeat itself) depends very much on IC. For this figure $\tau_2 = 457.4$ MeV^{-1} ($\tau = t/\hbar$). The pictures for other cases are more or less similar. Having the periods of oscillations one can perform the Fourier analysis of the curves and represent all the functions by a series

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty}(a_i \cos\omega_i t + b_i \sin\omega_i t).$$

The results of such calculations are demonstrated in the tables 1,2, where the eigenfrequencies $\hbar\omega_i$ and the corresponding coefficients a_i and b_i of the functions Q_{20} and Q_{00} are shown for two variants of IC. Let us analyse the first table in detail. As one can see there are about 30 eigenfrequencies having the diapason of amplitudes a_i, b_i from 10^{-2} to 10^3, half of them having this diapason from 1 to 10^3. All these frequencies correspond to transitions between various levels E_ν of the nucleus, i.e. they can be represented as differences $\hbar\omega_{\mu\nu} = E_\mu - E_\nu$. So, it is necessary to do some combinatorial analysis to find the eigenvalues E_ν. Of course the energies of GQR and GMR can be recognized

immediately without any combinatorics. They are very close to their values in the small amplitude approximation: E_2 ($\hbar\omega_8$) became 9.54 MeV instead of 9.78 MeV and E_0 ($\hbar\omega_{12}$) became 13.28 MeV instead of 13.84 MeV. So, we confirmed the well known fact, that giant resonances are described very well in the small amplitude approximation.

It is very interesting to discover the multiphonon states. One can find two- and three-phonon states, corresponding to GQR. Their energies are $\hbar\omega_{18} = 2 \cdot E_2 = 19.07$ MeV and $\hbar\omega_{24} = 3 \cdot E_2 = 28.61$ MeV. There is two-phonon state corresponding to GMR with the energy $\hbar\omega_{23} = 2 \cdot E_0 = 26.56$ MeV. One two-phonon state consists of the quadrupole and monopole phonons (its energy $\hbar\omega_{21} = E_2 + E_0 = 22.81$ MeV). It is not so difficult to show, that all remaining $\hbar\omega_i$ are just the combinations of two basic energies: E_2 and E_0. The results of the combinatorial analysis are shown in the third columns of the tables.

The comparison of the tables 1 and 2 shows that the strengths of all the states are very sensitive to IC, what is evident. Not so evident is the appreciable dependence of the energies on IC. From the mathematical point of view this result is absolutely correct - any textbook illustrates such dependence by an example of a nonlinear pendulum [20], [17] (see alse the section 4 of the paper). But what does mean this result from the physical point of view? We interpret it as a manifestation of a nucleus dynamical deformation. Really, due to a large amplitude of vibrations one has the grounds to treat a vibrating nucleus as a deformed one, because the most part of time it has a rather large deformation (dynamic deformation). And the dependence of energies on deformation is known very well. To know the order of magnitude of the deformation attained during the vibrations one needs the expression for the quadrupole moment Q_{20} in terms of the deformation parameter β. We derive it in the approximation of the sharp edge of a nucleus. By definition

$$Q_{20}(\beta) = n_0(\beta) \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{R(\theta,\phi)} dr\, r^2 (x_1^2 + x_2^2 - 2x_3^2). \qquad (36)$$

Here $R(\theta,\phi) = R_0(1 + \beta Y_{20}(\theta,\phi))$, $x_1^2 + x_2^2 - 2x_3^2 = -4\sqrt{\frac{\pi}{5}} r^2 Y_{20}(\theta,\phi)$ and the density $n_0(\beta)$ is defined as

$$n_0(\beta) = A \left\{ \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{R(\theta,\phi)} dr\, r^2 \right\}^{-1}.$$

Performing simple but tedious calculations we get:

$$Q_{20}(\beta) = -4\sqrt{\frac{\pi}{5}} n_0(\beta) R_0^5 \left\{ \beta + \frac{4}{7}\sqrt{\frac{5}{4\pi}}\beta^2 + \frac{15}{14\pi}\beta^3 + \frac{100}{77}\sqrt{\frac{5}{(4\pi)^3}}\beta^4 + \frac{25 \cdot 53}{77 \cdot 13 (4\pi)^2}\beta^5 \right\}, \qquad (37)$$

$$n_0(\beta) = 3\frac{A}{R_0^3} \left\{ 4\pi + 3\beta^2 + \frac{2}{7}\sqrt{\frac{5}{4\pi}}\beta^3 \right\}^{-1}.$$

The calculations show that the limit of maximum possible amplitudes is achieved at $\dot{Q}_{00}(0) = 0$, $\dot{Q}_{20}(0)) \simeq 28000$(in Mev fm$^2/\hbar$ units): the maximum positive value of Q_{20} is ~ 2700 and the maximum negative value is ~ 3800. With the help of (37) we find, that the vibrations with the maximum amplitude correspond to the change of β from ~ 0.42 to ~ -0.42. So, the shape of the nucleus changes during the vibrations from oblate to prolate. Further increasing of $\dot{Q}_{20}(0)$ leads to instability: the amplitudes begin to grow indefinitely. The amplitudes presented on fig.1 correspond to $\beta \simeq 0.25$.

The limit of small amplitudes is achieved practically for $\dot{Q}_{00}(0) = 0$, $\dot{Q}_{20}(0)) \simeq 1000$, when only GQR and GMR have appreciable amplitudes and their energies are exactly equal to $\sqrt{2}\hbar\omega$ and $2\hbar\omega$.

As it was expected the solution of the system (19) gives three basic energies. For example, with IC $\dot{Q}_{00} = 6100$, $\dot{Q}_{20} = 10^4$, $\dot{J}_- = 10^3$ we have: $E_M = 13.66$ Mev (GMR), $E_B = 9.5$ Mev (β-mode of GQR) and $E_G = 9.75$ Mev (γ-mode). The difference $\Delta E = E_G - E_B = 0.25$ Mev represents the splitting of GQR in the spherical nucleus due to the large amplitude vibrations which create the dynamical deformation.

3.4 Radiation probability

The radiation probability can be calculated with the help of the classical formula for the intensity of the quadrupole radiation [21]:

$$\mathrm{Int} = \frac{1}{180\, c^5} \sum_{k,l=1}^{3} \dddot{D}_{k,l}^2, \qquad (38)$$

where $D_{k,l} = eZ/A(3J_{kl} - \delta_{kl}\sum_{s=1}^{3} J_{ss})$. Using the relations $6J_{11} = 2Q_{00} + Q_{20} + 3J_-$, $6J_{22} = 2Q_{00} + Q_{20} - 3J_-$, $3J_{33} = Q_{00} - Q_{20}$ one can rewrite this expression in terms of our variables.

For example, in the case of the α-mode we have:

$$\mathrm{Int} = \left(\frac{eZ}{A}\right)^2 \frac{1}{120\,c^5}(\dddot{Q}_{20}^2 + 3\,\dddot{J}_-^2 + 12\,\dddot{J}_{13}^2). \tag{39}$$

The formulae for γ- and β-modes are obtained by neglecting here J_{13} and J_{13}, J_- correspondingly. For the sake of simplicity the following formulae will be written for the β-mode only. Taking $J_{13} = J_- = 0$ and inserting into (39) the Fourier expansion for Q_{20} we get after averaging over the greatest period of oscillations:

$$\overline{\mathrm{Int}} = \left(\frac{eZ}{A}\right)^2 \frac{1}{120\,c^5} \sum_{i=1}^{\infty} \omega_i^6 \frac{a_i^2 + b_i^2}{2} \equiv \sum_{i=1}^{\infty} \overline{\mathrm{Int}}_i\,. \tag{40}$$

Dividing $\overline{\mathrm{Int}}_i$ by $\hbar\omega_i$ we obtain the radiation probability W_i. Taking into account the relation between W_i and the reduced probability [5] we find:

$$B(E2)_i = \left(\frac{eZ}{A}\right)^2 \frac{5}{64\pi}\,(a_i^2 + b_i^2) = \left(\frac{Z}{A}\right)^2 \frac{125}{144} \frac{a_i^2 + b_i^2}{R^4}\,B_{\mathrm{W}}, \tag{41}$$

where B_{W} is the Weisskopf unit. The generalization for γ- and α-modes is elementary.

Using here the values of a_i and b_i from the table 1 we can calculate the $B(E2)$-factors for GQR and multiphonon states in the case of β-excitation:

$$B(E2, \mathrm{GQR}) = 176.3\,B_{\mathrm{W}},$$
$$B(E2, 2\times\mathrm{GQR}) = 0.29\,B_{\mathrm{W}} = 1.7\cdot 10^{-3} B(E2, \mathrm{GQR}).$$

Taking the results from the table 2 we get:

$$B(E2, \mathrm{GQR}) = 259.2\,B_{\mathrm{W}},$$
$$B(E2, 2\times\mathrm{GQR}) = 0.63\,B_{\mathrm{W}} = 2.4\cdot 10^{-3} B(E2, \mathrm{GQR}).$$

So, the $B(E2)$-factor of the two-phonon GQR is approximately three orders of magnitude less than that of the usual one-phonon GQR. This, however, depends appreciably on the initial conditions. The $B(E2)$-factor for the three-phonon state is six orders of magnitude less than that of the GQR.

The authors of the paper [22] calculated the deexcitation probabilities of the one- and two-phonon GQR by a microscopic approach with Skyrme forces. The calculations were done for ^{40}Ca only. Their results are:

$$W(\mathrm{GQR}) = 0.6\cdot 10^{16}\,s^{-1},$$
$$W(2\times\mathrm{GQR}) = 0.26\cdot 10^{15}\,s^{-1} \simeq \frac{1}{23}\,W(\mathrm{GQR}).$$

We also did the calculations for ^{40}Ca. Choosing the initial conditions $\dot{Q}_{20} = 500$, $\dot{Q}_{00} = 0$ to reproduce (approximately) their result for W(GQR) we have:

$$W(\mathrm{GQR}) = 0.9\cdot 10^{16}\,s^{-1},\ W(2\times\mathrm{GQR}) = 0.26\cdot 10^{14}\,s^{-1}.$$

As it is seen our result for $W(2\times GQR)$ is an order of magnitude less than that of the paper [22] so it would be interesting to repeat their calculations for the harmonic oscillator with Q-Q residual interaction. To have an idea on the upper bound for the discussed quantities we have performed the calculations with the initial conditions $\dot{Q}_{20} = 2800$, $\dot{Q}_{00} = 0$ corresponding to the vibrations with the maximum possible amplitudes:

$$W(\mathrm{GQR}) = 14\cdot 10^{16}\,s^{-1},$$
$$W(2\times\mathrm{GQR}) = 10.3\cdot 10^{15}\,s^{-1} \simeq \frac{1}{14}\,W(\mathrm{GQR}).$$

The radiation probabilities here are an order of magnitude larger than the ones of [22] though the ratio $W(2)/W(1)$ is very close to their result.

4 One-dimensional model

4.1 Equations of motion

Solving nonlinear equations of motion one expects (in accordance with the quantum mechanical results) to find anharmonicity effects. We have observed already the main effect of the anharmonicity - the two-phonon states with the energies equal exactly the double of the one-phonon ones. However, this result is in contradiction with the practice of quantum mechanical calculations, where one has usually some deviation from precise doubling, the deviation being a measure of the anharmonicity of the spectrum.

For understanding this problem it will be useful to consider here the classical and quantum mechanical aspects of the exactly soluble one-dimensional model [23] of a harmonic oscillator with a monopole-monopole residual interaction

$$\frac{1}{2}\kappa \sum_{i\neq j}^{A} F(x_i)F(x_j),$$

where $F(x_i) = x_i^2 - x_0^2/A$. Its solution was found previously by Reinhardt and Schulz [24] choosing a rather different derivation. With the help of our method the solution becomes very simple.

The average field of the model (in the notations of [24]) is

$$V(x,t) = \frac{1}{2}m\omega_0^2 x^2 + \kappa(< x^2 > - x_0^2)(x^2 - x_0^2/A), \tag{42}$$

where in correspondence with our notations $\omega_0 = \omega$, $< x^2 >= J_{11}(t)$, $x_0^2 = J_{11}(0)$. Following the rules described in the section 2.2 one can derive the system of equations

$$\begin{aligned} m\ddot{J} + 2J\left[m\omega^2 + 2\kappa(J - J_0)\right] - 2\Pi = 0, \\ \dot{\Pi} + \dot{J}\left[m\omega^2 + 2\kappa(J - J_0)\right] = 0 \end{aligned} \tag{43}$$

with $J = J_{11}(t)$, $J_0 = J_{11}(0)$, $\Pi = \Pi_{11}$. The second equation of this system gives the integral of motion

$$\Pi + m\omega^2 J + \kappa J^2 - 2\kappa J_0 J = c_0, \tag{44}$$

where c_0 is an arbitrary constant.Its value can be fixed by the conditions of equilibrium. In the state of equilibrium $J = J_0$, $\Pi = \Pi_0$ and one has from (43) and (44):

$$\begin{aligned} 2m\omega^2 J_0 - 2\Pi_0 = 0, \\ \Pi_0 + m\omega^2 J_0 - \kappa J_0^2 = c_0. \end{aligned} \tag{45}$$

Combining these two equations one finds:

$$c_0 = 2m\omega^2 J_0 - \kappa J_0^2 \tag{46}$$

and as a result

$$\Pi = m\omega^2(2J_0 - J) - \kappa(J - J_0)^2. \tag{47}$$

Using (47) and introducing the new variable $y = J - J_0$ one reduces the system (43) to the single equation

$$m\ddot{y} + a\,y + b\,y^2 = 0, \tag{48}$$

with $a = 4m\omega^2(1 + \bar{\kappa})$, $b = 6\kappa$, $\bar{\kappa} = \kappa\frac{J_0}{m\omega^2}$.

The authors of [24] have studied the collective variable $r(t)$ which is connected with our variable $y(t)$ by the relation $y = J_0(r^2 - 1)$ (formula (3.28) of [24]). Their dynamical equation for $r(t)$ reads (formulae (3.17), (3.29)):

$$\ddot{r} - \frac{\omega^2}{r^3} + \omega^2\left[r + 2\bar{\kappa}(r^3 - r)\right] = 0. \tag{49}$$

Multiplying (49) by $\dot{r}$ one gets the integral of motion

$$\dot{r}^2 + \omega^2\left[\frac{1}{r^2} + r^2 + \bar{\kappa}(r^2 - 1)^2\right] = c_2, \tag{50}$$

which allows to prove the equivalence of the equations (48) and (49). Really, putting $y = J_0(r^2 - 1)$ and $\ddot{y} = 2J_0(r\ddot{r} + \dot{r}^2)$ into (48) and eliminating the term proportional to $\dot{r}^2$ with the help of the relation (50) one gets:

$$r\ddot{r} - \omega^2\left[\frac{1}{r^2} + r^2 + \bar{\kappa}(r^2 - 1)^2\right] + c_2 + \frac{a}{2m}(r^2 - 1) + \frac{b}{2m}J_0(r^2 - 1)^2 = 0. \tag{51}$$

This equation becomes equivalent to equation (49) if to take $c_2 = 2\omega^2$. With such a value of c_2 the integral of motion (50) will coincide with that of Reinhardt and Schulz (formula (3.30)) only in the case when $E_{\rm HF} = E_0$. By the way, this requirement follows naturally from their condition of self-consistency (see the bottom of the section 3.2 in their paper).

4.2 Analysis of the solution

Neglecting the last term of the equation (48) one finds its solution in the small amplitude approximation. The corresponding eigenfrequency

$$\tilde{\Omega} = 2\omega\sqrt{1 + \bar{\kappa}} \tag{52}$$

reproduces the RPA result.

Searching for the exact solution we multiply (48) by $\dot{y}$ to transform it to

$$\frac{d}{dt}\left\{\frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{b}{3}y^3 + \frac{a}{2}y^2\right\} = 0, \tag{53}$$

demonstrating the existence of the integral of motion (analogous to (50))

$$\frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{b}{3}y^3 + \frac{a}{2}y^2 = c_1, \tag{54}$$

which express the energy conservation. The constant c_1 is determined by initial conditions. Having in mind, that $y(0) = 0$, one finds $c_1 = \frac{m}{2}(\dot{y}(0))^2$. The solution of the equation (54) can be expressed in terms of the Jacobian elliptic function [25]. For $\bar{\kappa} > 0$ (this case was studied in [24]) we have exactly the same result as in [24]:

$$y(t) = \eta_1 + (\eta_2 - \eta_1)\mathrm{sn}^2(\bar{\omega}t). \tag{55}$$

Here $\bar{\omega} = \omega\sqrt{\bar{\kappa}(\eta_3 - \eta_1)/J_0}$, η_i are the roots of the polynomial

$$P(y) = y^3 + \frac{3a}{2b}y^2 - \frac{3c_1}{b} \tag{56}$$

and $\eta_1 < \eta_2 < \eta_3$.The function $\mathrm{sn}(\phi)$ is a periodical one with the period $\Delta\phi = 4\mathbf{K}$ where

$$\mathbf{K} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \tag{57}$$

is the complete elliptic integral of the first kind with $k^2 = \frac{\eta_2 - \eta_1}{\eta_3 - \eta_1}$. There is an analytical expression for the Fourier expansion of this function [26]:

$$\mathrm{sn}\bar{\omega}t = \frac{2\pi}{k\mathbf{K}}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1 - q^{2n-1}}\sin(2n-1)\frac{\bar{\omega}\pi}{2\mathbf{K}}t.$$

Here $q = \exp(-\pi\mathbf{K}'/\mathbf{K})$, $\mathbf{K}'(k) = \mathbf{K}(k')$, $k' = \sqrt{1-k^2}$. This formula contains only frequencies proportional to odd numbers of the basic frequency $\Omega = \frac{\bar{\omega}\pi}{2\mathbf{K}}$ (by the way, its dependence on initial conditions follows from the dependence of the roots of the polynomial (56) on c_1, which is determined by $\dot{y}(0)$). It is evident, that sn^2 will contain frequencies $n\Omega$ with even n only. So, the Fourier expansion of the function $y(t)$ will contain only one basic frequency 2Ω and its satellites 4Ω, 6Ω and so on. Numerically the frequency $2\Omega = \frac{\bar{\omega}\pi}{\mathbf{K}}$ is very close to the result of the solution of the harmonic problem (52). So, the effect of including the anharmonic term $by^2(t)$ into equation (48) is a small change of the basic frequency $\tilde{\Omega} \to 2\Omega$ and the appearence of satellites $n2\Omega$, which are interpreted (using the quantum mechanical language) as the levels of multiphonon states. The equidistance of such spectrum is evident. It is clear that using the Fourier expansion one can not obtain another result.

In the case of our two dimensional problem of coupled dynamics of monopole and quadrupole moments it is natural to expect the existence of two (for β-mode) basic frequencies Ω_1, Ω_2 and their satellites. Due to coupling there must be a lot of linear combinations of these frequencies: $n_1\Omega_1 \pm n_2\Omega_2$. For the $\gamma(\alpha)$ -mode there must be three (four) basic frequences with corresponding satellites. As we have seen, such a picture really takes place.

4.3 Quantization

The quantization of the one dimensional model (equation (48)) is achieved immediately if we remember that the integral of motion c_1 (equation (54)) is the energy of vibrations. Choosing $q = y$ and $p = m\dot{y}$ as the canonically conjugated variables one can write the Hamiltonian in the form

$$H = \frac{1}{2m}p^2 + V(q) \tag{58}$$

with $V(q) = \frac{a}{2}q^2 + \frac{b}{3}q^3$. It is easy to see that equation (48) coincides with one of the Hamilton equations of motion $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$, that confirms the correctness of our choice of canonical variables. The quantum Hamiltonian is obtained by putting into the expression (58) the operators $\hat{p} = -i\hbar\frac{d}{dq}$ and $\hat{q} = q$ instead of p and q. The potential energy $V(q)$ is anharmonic so one can hope to observe the anharmonic effects in the spectrum. The analysis of the spectrum can be done with the help of Bohr-Sommerfeld quantization rule:

$$\int_{q_1}^{q_2} P(q)\,dq = \pi\hbar(n + \frac{1}{2}), \tag{59}$$

where $P(q) = \sqrt{2m(E-V)}$, q_1 and q_2 are the classical turning points. We calculate the integral (59) numerically. The shape of the potential well depends strongly on $\bar{\kappa}$. Three regions of $\bar{\kappa}$ values must be considered separately [24]: $\bar{\kappa} > 0$, $-1 < \bar{\kappa} < 0$ and $\bar{\kappa} < -1$. The potential wells corresponding to each region are shown in the fig.2.

Let us consider the first case: $\bar{\kappa} > 0$. The anharmonicity must be maximum when the barrier height V_B is minimum. The next formula is true for the $\bar{\kappa}$ dependence of V_B:

$$V_B(\bar{\kappa}) = \frac{8}{27}J_0^2 m\omega^2(1+\bar{\kappa})^3/\bar{\kappa}^2. \tag{60}$$

It is easy to see that V_B has its minimum at $\bar{\kappa} = 2$ and $V_B \to \infty$ when $\bar{\kappa} \to \infty$ or $\bar{\kappa} \to 0$. Hence the anharmonicity is maximum at $\bar{\kappa} = 2$. The calculations with $\bar{\kappa} = 2$ show that the levels E_n are equidistant with good accuracy up to very large n. For example the difference $E_1 - E_0$ coincides exactly with $E_{RPA} = 23.971$ Mev and $E_{100} - E_0 = 2397.072$ Mev $\simeq 100E_{RPA}$. So it is not surprising that the authors of [24] had not found any traces of anharmonicity at $n = 2$. Very small anharmonicity can be noticed at $n \approx 1000$. The difference $E_{1001} - E_{1000} = 24.160$ Mev slightly differs from E_{RPA}, demonstrating the existence of the anharmonicity $Anh(n = 1001) = (E_{1001} - E_{1000} - E_{RPA})/E_{RPA} \approx 0.8\%$.

Table 1 Fourier coefficients and energies for $\dot{Q}_{00}(0) = 0$, $\dot{Q}_{20}(0) = 18000$

	$\hbar\omega_i$		Q_{00}		Q_{20}	
i	MeV		b_i	a_i	b_i	a_i
0	0.00		0.00	202.47	0.00	-481.45
1	1.69	3M-4Q	-0.00	0.02	0.24	-0.84
2	2.05	3Q-2M	3.10	-0.39	-15.32	1.72
3	3.74	M-Q	-6.68	-0.91	196.11	32.15
4	5.80	2Q-M	-0.28	27.83	3.46	-68.38
5	7.48	2M-2Q	1.04	-3.68	-7.16	21.24
6	7.85	5Q-3M	-0.16	0.03	0.09	-0.00
7	9.17	5M-6Q	0.05	0.02	-0.01	0.02
8	9.54	Q	-83.83	-12.37	1727.16	373.68
9	9.90	8Q-5M	-0.06	0.01	0.00	0.03
10	11.22	3M-3Q	0.05	0.06	2.37	1.25
11	11.59	4Q-2M	-0.12	4.83	-0.68	6.70
12	13.28	M	64.72	-222.54	25.33	-64.26
13	13.64	7Q-4M	0.00	-0.20	0.01	-0.00
14	14.97	4M-4Q	-0.04	-0.02	0.01	-0.02
15	15.33	3Q-M	27.42	4.29	-9.05	-2.44
16	17.02	2M-Q	-2.48	-1.15	4.27	2.55
17	17.39	6Q-3M	0.01	0.02	0.01	-0.05
18	19.07	2Q	-30.12	99.72	29.42	-64.80
19	20.76	3M-2Q	-0.19	0.32	0.05	-0.06
20	21.13	5Q-2M	-0.60	-0.09	0.28	0.09
21	22.81	Q+M	4.52	2.08	6.07	4.05
22	24.87	4Q-M	0.16	-0.50	-0.86	1.67
23	26.56	2M	-0.11	0.18	0.42	-0.45
24	28.61	3Q	2.08	0.98	-4.33	-3.22
25	30.66	6Q-2M	-0.01	0.02	0.02	-0.04
26	32.35	2Q+M	-0.05	0.08	0.04	-0.04

Table 2 Fourier coefficients and energies for $\dot{Q}_{00}(0) = 0$, $\dot{Q}_{20}(0) = 25000$

	$\hbar\omega_i$		Q_{00}		Q_{20}	
i	MeV		b_i	a_i	b_i	a_i
0	0.00		0.00	378.83	0.00	-787.69
1	0.22	7Q-5M	1.32	-0.08	-21.26	1.31
2	1.70	3M-4Q	-0.17	2.14	7.09	-88.25
3	1.92	3Q-2M	59.95	1.12	-318.53	-5.95
4	3.39	6M-8Q	-0.26	1.62	3.92	-24.25
5	3.62	M-Q	-67.69	-6.72	785.15	77.94
6	5.53	2Q-M	-16.15	136.67	43.90	-371.48
7	5.75	9Q-6M	-11.29	-0.64	34.09	1.92
8	7.01	7M-9Q	-1.31	-0.35	12.65	3.35
9	7.23	2M-2Q	14.97	-74.64	-74.15	369.79
10	7.45	5Q-3M	-18.08	-2.48	31.82	4.37
11	8.93	5M-6Q	5.38	-18.85	-37.80	132.43
12	9.15	Q	-157.77	-34.71	2092.70	460.45
13	9.37	8Q-5M	0.88	-5.62	-17.96	114.87
14	10.85	3M-3Q	-20.76	-6.35	120.68	36.92
15	11.07	4Q-2M	-21.13	88.14	-43.78	182.65
16	12.76	M	177.67	-544.27	113.80	-348.64
17	12.98	7Q-4M	15.84	4.11	14.76	3.83
18	14.46	4M-4Q	-13.89	33.26	4.01	-9.60
19	14.68	3Q-M	202.16	70.20	-62.13	-21.58
20	16.38	2M-Q	-59.05	-25.98	66.71	29.35
21	16.60	6Q-3M	-5.03	13.65	3.05	-8.28
22	18.08	5M-5Q	-24.21	-13.06	14.38	7.76
23	18.30	2Q	-67.25	145.42	44.50	-96.22
24	18.52	9Q-5M	-15.51	-6.05	10.67	4.16
25	19.99	3M-2Q	-11.69	20.73	5.06	-8.98
26	20.21	5Q-2M	-21.15	-10.26	13.53	6.57
27	21.91	Q+M	31.04	18.28	2.67	1.57
28	23.83	4Q-M	1.21	-1.97	-9.65	15.71
29	25.53	2M	-1.77	2.43	6.18	-8.45

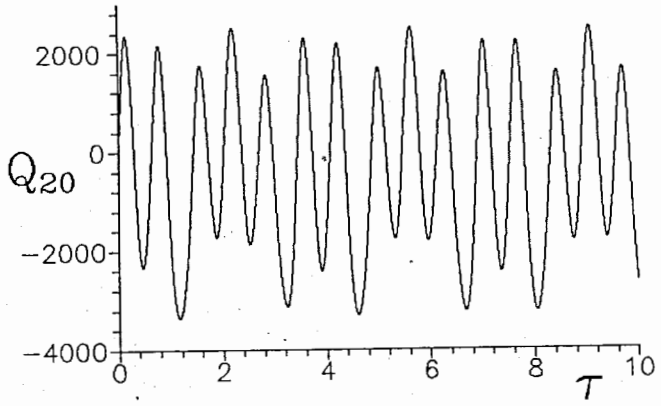

Fig.1. The time dependence (τ=t/h) of the quadrupole moment for the initial conditions $Q_{00}(0)=0$, $Q_{20}(0)=25000$.

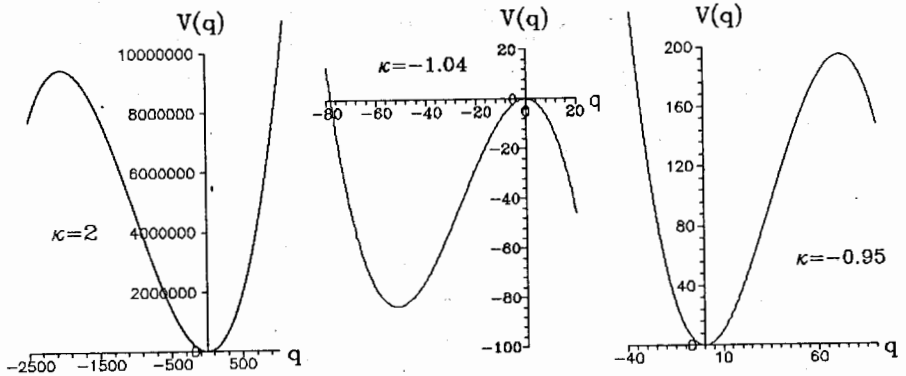

Fig.2. The potential V(q) for different values of the coupling parameter κ.

The second case ($-1 < \bar{\kappa} < 0$) is more interesting, because here the barrier height can be arbitrarily small (when $\bar{\kappa} \to -1$) and consequently the anharmonicity can be arbitrarily large. For example at $\bar{\kappa} = -0.9$ it becomes appreciable already for $n = 100$: $E_1 - E_0 = 4.376$ Mev $= E_{RPA}$, $E_{101} - E_{100} = 4.19$ Mev, $Anh(n = 101) \approx 4.3\%$. For $\bar{\kappa} = -0.99$ there are no bound states.

The third case ($\bar{\kappa} < -1$) is similar to the second one. The anharmonicity, being arbitrary large in the vicinity of $\bar{\kappa} = -1$ decreases quickly when $\bar{\kappa} \to -\infty$ because the depth of the potential well grows together with $|\bar{\kappa}|$ (it is seen from the formula (60) which gives the depth of the well when $\bar{\kappa} < -1$).

By the way, the authors of the paper [24] had not noticed the possibility of the existence of bound states at $\bar{\kappa} < -1$ and had not analyzed this case (their figure 1 for $\bar{\kappa} < -1$ is incorrect).

5 Conclusion

Let us enumerate the main results of the paper. The set of nonlinear dynamical equations for quadrupole Q_{20} and monopole Q_{00} moments of nuclei is derived from the TDHF equation with the help of the method of Wigner function moments. It allows one to describe the large amplitude vibrations of Q_{20} and Q_{00}. Due to the simplicity of the used Hamiltonian all the derivations are performed exactly, without any approximations. These equations are solved numerically for ^{208}Pb and ^{40}Ca. It is found, that the functions $Q_{20}(t)$ and $Q_{00}(t)$ oscillate irregularly. Their Fourier analysis yields the giant quadrupole and monopole resonances and multiphonon states constructed of them. It is shown that the radiation probability of the two-phonon giant quadrupole resonance is one or two orders of magnitude less than that of the one-phonon GQR, being strongly dependent on initial conditions.

The maximum vibration amplitudes amount the value corresponding to the deformation parameter $\beta \simeq 0.42$. The essential features of the large amplitude motion manifest itself by the coupling of GMR and GQR in spherical nuclei, the last one being splitted, and by the dependence of resonance energies on initial conditions which in their turn are determined by the strength of the external field.

The classical and quantum aspects of the analytically solvable one- dimensional model

are studied to show that the anharmonicity of the collective spectrum, being the property of quantum systems, can't be observed in classical ones. The quantization of the two-dimensional model (equations (17)), being more complicate problem, will be done in the next paper.

The theory can be modified to take into account spin and isospin degrees of freedom. Then it will be possible to study spin and isovector collective modes for the case of large amplitude motion. The extension to the description of excitations of higher multipolarities is straightforward.

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Бальбуцев Е.Б., Шук П. E4-96-240

Квадрупольные и монопольные колебания большой амплитуды

Система нелинейных динамических уравнений для квадрупольного и монопольного моментов ядра выводится из уравнения для одночастичной матрицы плотности с помощью метода моментов функции Вигнера. Она позволяет описывать связанные квадрупольные и монопольные колебания большой амплитуды. Эти уравнения с сепарабельными силами решены численно для ^{208}Pb и ^{40}Ca. Хорошо воспроизводятся гигантские квадрупольный и монопольный резонансы. Отличительной чертой движения большой амплитуды является наличие многофононных состояний. Они детально проанализированы. Изучаются классические и квантовые аспекты аналитически решаемой одномерной чисто монопольной модели, чтобы прояснить проблему ангармоничности коллективного спектра.

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Balbutsev E.B., Schuck P. E4-96-240

Quadrupole and Monopole Large Amplitude Vibrations

A set of nonlinear dynamical equations for quadrupole and monopole moments of nuclei is derived from the TDHF equation with the help of the so-called Wigner function moments. It allows the description of coupled large amplitude monopole and quadrupole vibrations. These equations are solved numerically for ^{208}Pb and ^{40}Ca in a model with separable forces. The giant quadrupole and monopole resonances are reproduced very well. However the essential feature of the large amplitude motion is the existence of multiphonon states. They are analyzed in detail. The classical and quantum aspects of the analytically solvable one-dimensional pure monopole model are studied to clarify the problem of the anharmonicity of the collective spectrum.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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