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THE BIRKHOFF THEOREM AND UNIQUENESS PROBLEM OF SPHERICALLY SYMMETRIC SPACE-TIME MODEL IN GR

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The metrics singularity problem in GR is intimately connected with known Birkhoff theorem. When Birkhoff first formulated his statement, it had not a theorem status and consisted in following [1]:

> The field outside of the spherical distribution of matter is static whether or not the matter is in a static or in a variable state... Thus the Schwarzschild solution is essentially the most general solution of the field equations with spherical symmetry.

The central fact which Birkhoff was interested in was that the gravitational waves have existed or not when spherical distribution of matter has been pulsating. The conclusion obtained by him was negative. Hence we have in GR the same situation that in Newtonian theory of gravitation.

The metrics which Birkhoff meant was g_{DW} :

$$ds^2 = \left(1 - \frac{\beta}{r}\right) dt^2 - \left(1 - \frac{\beta}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \qquad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \beta = 2m$ (in the relativistic units), m - central source mass. This metrics is usually named Schwarzschild metrics, though J.Droste [2] and H.Weyl [3] actually obtained it in 1917. So here it is designated by g_{DW}. The feature of this metrics is the presence of "horizon of events" or singular Schwarzschild sphere. So the space - time topology is non - Euclidean and the "black holes" or collapsed objects are possible in the Universe.

An existence of collapsed objects induces a set of hardly decidable questions connected with physical interpretation of mathematical wording of GR. Hawking and Ellis described in every detail the problems of singularities appearance and collapse in GR [4]. They arrived at a conclusion that it is important to have an exact wording of this theory and to state uniqueness degree of it. They offered a new wording of GR in terms of mathematical space - time models M. It followed by a modification of Birkhoff's statement and an attempt to give it a theorem status. Two another versions of Birkhoff theorem are in [5].

All above mentioned wordings of Birkhoff theorem (including [1]) ascertain that from only spherical symmetry of vacuum metrics it follows that one is static and has Schwarzschildean form (on the uniqueness Schwarzschild geometry see, e.g. [5]). Moreover by that it ensures the uniqueness of spherically symmetrical space - time model in GR. This uniqueness in its turn involves necessity of the existence of black holes as the real physical and astrophysical objects and the non-existence of the astrophysical objects with a naked singularity. However now there are some new astrophysical facts pointing out to a possibility of existence of last type objects. That's why the uniqueness problem of spherically symmetric space - time model in GR is very interesting. The real astrophysical objects are the realization in nature of the theoretical space - time models. So the question of accurate formulation of Birkhoff theorem has to arise.

The Birkhoff theorem wordings in [1] and [5] has lack of accuracy. Firstly these wordings do not point out to the space-time region where Schwarzschildean coordinates can be introduced. Secondly it does not talk precisely in which metric functions differentiable class the problem decision has to be considered. Therefore a simple answer the put question is impossible.

What is the maximum space-time region where the mentioned in the Birkhoff theorem coordinates may be introduced? It's seen from (1) that the metrics g_{DW} is determined in a static map which exists at only R-region: $r > \beta$. Within the sphere $r = \beta$ (i.e. at T-region: $0 < r < \beta$) a time and a space coordinates switch the roles, and the map becomes non-static. An analytic continuation of the metrics (1) to the T-region is realized in the non-static coordinate systems, such as well-known Eddington - Finkelstein and Kruskal - Szekeres coordinates. Thus from the above-mentioned Birkhoff theorem wording it follows that spherically symmetric vacuum gravitational field is static at only region $r > \beta$.

But already in 1916 K.Schwarzschild [6] found the static vacuum metrics g_S in the map covering the whole region of space $0 < r < \infty$.This solution has spherical symmetry but has not "horizon of events". It may be written in the form:

$$ds^2 = \left(1 - \frac{\beta}{R}\right) dt^2 - \left(1 - \frac{\beta}{R}\right)^{-1} (R_1 dr)^2 - R^2 d\Omega^2 \qquad (2)$$

where $R = (r^3 + \beta^3)^{1/3}, R_1 = \frac{dR}{dr}, \beta = 2m = \text{const} > 0$.

The actually Schwarzschildean metrics (2) as satisfy the conditions which are necessary to describe gravitational field of uncharged non-radiating non-rotational point source as Droste-Weyl metrics does. This conditions were formulated in Finkelstein's article [7] as follows:

1. space-time metrics has to be asymptotically Euclidean;
2. this metrics has not to be continued to the line $L(x = y = z = 0)$ corresponding to the central singularity;
3. this metrics has to be invariant under the group of space rotations and reflections as well as the translations of a time coordinate t;
4. a coordinate t has to be global only.

We shall name the space-time V_4 furnished with the metrics (1) Droste-Weyl model (but not Schwarzschild model as it is received in the literature). The spherically symmetric static vacuum solution of the Einstein equations being found by K.Schwarzschild (more exactly the space-time V_4 with the metrics (2)) will be named Schwarzschildean model. In spite of the metrics g_{DW} may be obtained from the metrics g_S by the radial coordinate transformation:$r' = (r^3 + \beta^3)^{1/3}, r > 0, r' > \beta$, these metrics have different determination regions ($M_{DW} : r' > \beta$ and

$M_S : r > 0$). Therefore as Abrams showed in [8] they describe two different non-equivalent space-time models. Schwarzschildean model do not contain "horizon of events" and "black holes".

In the Einstein General Relativity a mathematical space-time model in the Hawking-Ellis sense is a pair (M, g) where M – connected Hausdorff differentiable manifold furnished with a Lorentz metrics g of index 1. Two models (M, g) and $(\tilde{M}, \tilde{g})$ are considered to be equivalent if a diffeomorphism $M \to \tilde{M}$ exists which maps g into $\tilde{g}$ isometrically. Strictly speaking the space-time model in GR is not one pair (M, g) but the whole equivalence class which includes any of pairs $(\tilde{M},\tilde{g})$ being equivalent (M, g). Thus any model is fully defined if some representative (M, g) of the equivalence class and isometric diffeomorphism $(M, g) \to (\tilde{M}, \tilde{g})$ are given.

In [8] L.Abrams obtained the whole equivalence class of the vacuum space-time models meeting Schwarzschildean problem and having the model (M_S, g_S) as their representative. In the quasipolar coordinates this class is formed by the metrics family g_A:

$$ds^2 = \chi dt^2 - \frac{\beta^2 (\chi_1)^2}{\chi(1-\chi)} dr^2 - \frac{\beta^2}{(1-\chi)^2} d\Omega^2, \qquad (3)$$

where β =const> 0, $\chi(r)$ - monotonously increasing C^∞ - differentiable function, which is defined on the interval $(0, +\infty)$ and satisfies the conditions:

$$\chi(0_+) = 0, \quad \chi(r) \sim 1 - \frac{\beta}{r}, \qquad (4)$$

when $r \to \infty$, $\chi_1 = \frac{d\chi}{dr}$.

The radial coordinate transformation $\chi(r) = \tilde{\chi}(\tilde{r})$ gives C^∞- diffeomorphism $(M_S, g_A) \to (M_S, \tilde{g}_A)$, where g_A and $\tilde{g}_A$ - any two metrics of the family (3) (see [8]). The metrics (2) is one of the metrics family (3) if $\chi = 1 - \frac{\beta}{R}, R = (r^3 + \beta^3)^{1/3}$.

Any manifold of Abrams class (M_S, g_A) may be obtained from (M_S, g_S) by C^∞-diffeomorphism which transforms r' as

$$r' = \frac{\beta}{1-\chi}[1 - (1-\chi)^3]^{1/3}, \qquad (5)$$

where $\chi(r)$–the function satisfying the conditions (4) (see [9]). The class (M_S, g_A) does not include the manifold (M_{DW}, g_{DW}), i.e. usual Schwarzschildean geometry, as M_{DW} is only the part of M_S ([8]), and $M_S \to M_{DW}$ mapping is not a diffeomorphism.

So there exist at last two space-time models that meet Schwarzschildean problem for the point source. Furthermore each of these models contains an arbitrariness in the mass parameter β choice. Therefore nothing indicate that spherically symmetric vacuum space-time model in GR is unique. It need be noticed that in the presence of "horizon of events" the external "Schwarzschildean metrics"(i.e.Droste-Weyl metrics (1)) may be obtained also by sewing it together internal Schwarzschildean metrics. The external Schwarzschild solution may be sewed together internal metrics for example for a homogeneous sphere, and it will correspond to not-point source. Then we shall have third type of the spherically symmetric space-time models with the metrics (1).

The answer on the question of the uniqueness of the space-time model depends also on the differentiable class of the metric functions $g_{\mu\nu}(x)$ and of the functions realizing these transformations. It is usually implied (including the proofs of Birkhoff theorem) that they belong to the C^{∞}-class. Really the existence of the coordinates in which the functions $g_{\mu\nu}(x)$ belong to the C^2-class is enough to meet Schwarzschildean problem. Will the new space-time models arise if the requirements on differentiability of the functions $g_{\mu\nu}(x)$ will be weaken? A.Z.Petrov in [10] went into the question of the form of the spherically symmetric vacuum solutions of Einstein equations in the class of the functions differentiability C^1. Hawking and Ellis in [4] went into this question also in the connection with Birkhoff theorem. A.Z.Petrov did not obtain full static solution class in the extended functions class C^1.

The discussion on Petrov's problem and the new attempt to decide it see in [11]. In this article we found that if the differentiable class of the metric functions is extended by the functions of the class C^1 then even in this extended class the static solution will be expressed by the functions of the class non-below C^2. But if the choice of the functions class was limited to the class C^2 under the requirement $0 < r < \infty$ at once, the solutions would be only static and at the manifold M_S would be described by the metrics family g_A. The metrics g_S being regular to the central singularity may be took as a representative of the Abrams equivalence class. The question of the existence of non-static spherically symmetric vacuum solutions of C^1-class may be open only for non-Einsteinean equations. Hawking and Ellis point of view is nonexistence such solutions in GR ([4]).

As shown in [11] if the source may have so small size as one choose then all Birkhoff theorem conditions are satisfied by not only Droste-Weyl model (M_{DW}, g_{DW}) (i.e.the usual Schwarzschildean geometry in which the singular sphere exists) but by another model (M_S, g_S) being obtained by Schwarzschild indeed and being defined on M_S where $0 < r < \infty$ also. The question is: what is the maximum region without the central singularity where the Schwarzschildean problem have a static solution?

The wording of the Birkhoff theorem being given in [1] contains not only no directives to such region but to the function class in which the decision of Schwarzschildean problem is declared unique and having form of the metrics g_{DW} also. At the same time in the accepted proofs of this theorem the transition to curvatures coordinates by the admissible transformations is quietly using, i.e.in fact it is imposed that $C = -r^2$. Here C is the coefficient in front of $d\Omega^2$ in the

spherically symmetric metrics of the general form:

$$ds^2 = Adr^2 + 2Bdrdt + Cd\Omega^2 + Ddt^2, \qquad (6)$$

where A, B, C, D–some functions of r, t.

As A.Z.Petrov noted in [10] this requirement is equivalent to a choice of the differentiable functions class C^2. In [11] shown that lowering the admissible class of the functions differentiability don't extend of the static solutions class. The conclusions follow from the above:

1. the usual wording of the Birkhoff theorem is insufficiently accurate to lead to the unique spherically symmetric static solution;
2. the theoretical proof of the black holes existence in GR based on the uniqueness of the spherically symmetric solutions with a singular sphere can not be considered as a proof ([12]) and it is a directive on the possibility of the such objects existence only.
3. the possibility of existence of new types of astrophysical objects with a naked singularity as a realization of g_S appears.

The main feature of metrics (2) is another space-time model and topology as compared with that of metrics (1). Here the topological structure of the Einstein equations solutions is very important factor for the interpretation of experimental observations in cosmology and astronomy.

So as a result of the above discussion more accurate wording of Birkhoff theorem may be given. It is following ([13]):

> In GR a spherically symmetric vacuum gravitational field of uncharged non-rotating point source being determined by the metric functions of a differentiability class not lower C^2 is static throughout the region $0 < r < \infty$ and is described by Abrams metrics equivalence class g_A, and one of a representative of the class g_A is the metrics g_S.

It is easy to see that the metrics g_{DW} can not be a representative of the class g_A and this fact involves the difference of above-mentioned wording with well known wordings of Birkhoff theorem.

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References

[1] G.D.Birkhoff, Relativity and Modern Physics, Harvard Univ. Press, Cambridge (1923);

[2] J.Droste, J. Proc.K. Ned. Acad. Wet., **19** (1917) 197

[3] H.Weyl, Ann. Phys. (Leipzig), **54** (1917) 117

[4] S.W.Hawking, G.F.R.Ellis, The Large Scale Structure of Space - Time, Cambridge Univ. Press, England (1973)

[5] Ch.Misner, K.Thorne, J.Wheeler, Gravitation, San Francisco, (1973)

[6] K.Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinischen Theorie, Sitzungsber. Akad. Wiss., Berlin, (1916), 189

[7] D.Finkelstein, Phys. Rev., **110** (1958) 965

[8] L.Abrams, Phys. Rev., **20** (1979) 2474

[9] V.D.Zakharov, The Th.of Rep. of YI All-Union Grav.Conf., 3–5 July 1984, Moscow, M., (1984) 62

[10] A.Z.Petrov, Zh.Eksp.Teor.Fiz., **44** (1963) 1525

[11] V.D.Zakharov, N.P.Konopleva, O.V.Savushkin, Dokl.AN, **326** (1992) 1002

[12] L.Abrams, Can.J.Phys. **67**, (1989) 919

[13] V.D.Zakharov, N.P.Konopleva, O.V.Savushkin, The Th.of Rep. of YIII Russian Grav.Conf., 24–28 may 1993, Puschino, M., (1993) 44

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Теорема Биркгофа и проблема единственности сферически симметричной модели пространства-времени в ОТО

Анализируются различные формулировки теоремы Биркгофа, используемые в литературе. Предложена новая формулировка этой теоремы, учитывающая результаты Л.Абрамса, относящиеся к проблеме единственности сферически симметричной модели пространства-времени в ОТО, и точку зрения Хокинга и Эллиса на эту проблему. Результаты, полученные А.З.Петровым в этой области, также обсуждаются.

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Konopleva N.P. E4-95-79

The Birkhoff Theorem and Uniqueness Problem of Spherically Symmetric Space-Time Model in GR

We analyze the different wordings of Birkhoff theorem being used in literature. The new wording of this theorem is given. We take into consideration the results of L.Abrams on the problem of the uniqueness of spherically symmetric space-time model in GR and the Hawking-Ellis point of view at this problem. The A.Z.Petrov results are considered also.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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