ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E4-95-439

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STABILITY OF THE THREE-BODY COULOMB SYSTEMS WITH $J=1$ IN THE OSCILLATOR REPRESENTATION

Submitted to «Few-Body Systems»

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1995

1 Introduction

The determine stability region of the three-body Coulomb systems with unit charges and arbitrary masses is one of the basic problems in understanding the formation dynamics of few-body Coulomb systems (see, for example, [1], [2]). The dependence of the background energy on the masses of particles has been considered in [3] and numerical calculations have been made in [4] to establish stability boundary for the three-body Coulomb systems with total momentum J=0. More detailed studies on the stability boundaries of the three-body Coulomb system have been performed in ref.[1]. However, for the systems with $J \geq 1$ this type of investigations have not been carried out yet and they were limited only to calculations of bound state of the energy separate molecules in the framework of deferent methods (see,for example, [5]).

The main purpose of these investigations is mainly the construction of highly accurate numerical solutions of the Schrödinger equation for the Coulomb three-body systems with given values of masses and charges. However, the three-body Coulomb systems for arbitrary values of the masses and charges described in the framework of the numerical methods were unsuccessful so far.

The determination of eigenvalues of the Hamiltonian for the three-body Coulomb system by means of numerical methods to a high accuracy is certainly important. However, for understanding the formation dynamics of three-body bound states one needs to study qualitatively the dependence of eigenvalues of the Hamiltonian on the masses and charges of particles. Therefore, the development of analytical tools permitting the study of these dependencies with an accuracy of a few per cent is justified.

The Oscillator Representation Method developed in the papers ([6],[7]) gives the possibility to define the stability region as a function of masses and charges of the particles for the Coulomb three-body systems. In the paper [6], the Oscillator Representation Method based on the ideas and methods of quantum field theory has been proposed to investigate the stability region established for the three-body Coulomb system with the angular momentum J=0.

In this paper, the Oscillator Representation Method will be applied to calculate the energy spectrum of three-body Coulomb systems with the J total angular momentum. By defining the dependence of the bound state energy of the system on the masses of particles, the stability boundaries with respect to the particle masses of the systems are determined. For the systems $(D^+e^-e^+)$, (pB^-e^-), $(A^+A^-e^-)$ and (pe^-C^+), the values of critical masses of the A-, B- ,C- and D-particles are calculated.

This paper is organized as follows. In sections 2 and 3, the method of oscillator representation for the three-body Coulomb systems with the total angular momentum J is formulated. In section 4, the bound state energies of the three-body Coulomb systems with J=0 and J=1 are calculated. In section 5, boundaries with respect to the particle masses of the three-body systems are determined. For the systems $(D^+e^-e^+)$, (pB^-e^-), $(A^+A^-e^-)$ and (pe^-C^+), the values for the critical masses of A-, B-, C- and D-particles are calculated.

2 Three-body Hamiltonian

Formulation of the problem. Let us consider a three-body Coulomb system in 3 dimensions, with the masses m_1, m_2, m_3 and charges Z_1e, Z_2e, $-Z_3e$. As it is well known, a hydrogen mesic molecular ion is a system of three particles: a negatively charged muon μ^- or electron e^-, and two nuclei 1 and 2 of hydrogen isotopes (1, 2=p, d or t). The Hamiltonian for this system has the form:

$$H = \frac{1}{2}\sum_{j=1}^{3}\frac{1}{m_j}\vec{P}_j^2 - \frac{Z_1Z_3e^2}{|\vec{r}_1-\vec{r}_3|} - \frac{Z_2Z_3e^2}{|\vec{r}_2-\vec{r}_3|} + \frac{Z_2Z_1e^2}{|\vec{r}_2-\vec{r}_1|} .$$

After some simplification, this Hamiltonian can be expressed in the center of mass system by (see refs. [6], [7])

$$H = \frac{m_1m_3e^4}{(m_1+m_3)}\cdot\left[\frac{1}{2}\vec{P}_R^2 + \frac{1}{2}\vec{P}_r^2 - \frac{Z_1Z_3}{R} + \frac{Z_2Z_1c}{|\vec{r}+\vec{R}c_1|} - \frac{Z_2Z_3c}{|\vec{r}-\vec{R}c_3|}\right] , \qquad (2.1)$$

with

$$c_1 = \frac{1}{m_1}\cdot\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}} , \qquad c_3 = \frac{1}{m_3}\cdot\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}} ,$$
$$c = c_1 + c_3 .$$

Here $\vec{R}$ ($R = |\vec{R}|$) is the position vector of (nucleus) 1 relative to (muons) 3 and $\vec{r}$ is the position vector of (nucleus) 2 relative to the center of mass system (1,3).

The Schrödinger equation looks like

$$H\Psi = 0 , \qquad (2.2)$$

$$H = \frac{1}{2}\vec{P}_R^2 + \frac{1}{2}\vec{P}_r^2 - \frac{Z_1Z_3}{R} + \frac{Z_2Z_1c}{|\vec{r}+\vec{R}c_1|} - \frac{Z_2Z_3c}{|\vec{r}-\vec{R}c_3|} + \frac{1}{2}U ,$$

where the energy parameter U is introduced

$$E = -\frac{e^4}{2}\cdot\frac{m_1m_3}{m_1+m_3}\cdot U . \qquad (2.3)$$

Our problem is to calculate the energy values E of the three-body Coulomb system. The wave function for the three-body Coulomb system with the total angular momentum J can be written in the standard form

$$\Psi^{\lambda}_{JM} = \sum_{m=0}^{J}\mathcal{D}^{J\lambda}_{mM}(\Phi,\Theta,\phi)\cdot\Psi^{\lambda}_{mJ}(r,R,\theta) , \qquad (2.4)$$

where Φ and Θ are the polar and azimuthal angles for the vector $\vec{R}$ in the space-fixed coordinates, the variables $\vec{r} = (r,\theta,\phi)$ represent spherical coordinates of the nucleus 2, $\mathcal{D}$ are the symmetrical representation of the Wigner D-functions and satisfy the relations

$$\hat{J}^2\mathcal{D}^{J\lambda}_{mM} = J(J+1)\mathcal{D}^{J\lambda}_{mM}$$
$$\hat{J}_z\mathcal{D}^{J\lambda}_{mM} = M\mathcal{D}^{J\lambda}_{mM} = m\mathcal{D}^{J\lambda}_{mM} ,$$

and have the form

$$\mathcal{D}^{J\lambda}_{mM} = \left[\frac{2J+1}{8\pi(1+\delta_{0m})}\right]^{1/2} \qquad (2.5)$$
$$\left[\frac{(-1)^m e^{im\phi}}{\sqrt{2\pi}}D^J_{mM}(\Phi,\Theta,0) + \frac{(-1)^J\lambda e^{-im\phi}}{\sqrt{2\pi}}D^J_{-mM}(\Phi,\Theta,0)\right] .$$

These functions form the complete orthonormal set (standard Wigner D-function D^J_{mM} are defined according to [8]). The parity of states is denoted by λ. The states with the parity $\lambda = (-1)^J$ and $\lambda = -(-1)^J$ are called, respectively, the states with normal and anomalous parity.

Let us consider the kinetic part of the Hamiltonian (2.2)

$$\frac{1}{2}\vec{P}_R^2 + \frac{1}{2}\vec{P}_r^2 = -\frac{1}{2}\left[\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} - \frac{\hat{L}^2}{R^2}\right] - \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{\ell}^2}{r^2}\right] . \qquad (2.6)$$

and introduce the total angular momentum

$$\vec{J} = \vec{L} + \vec{\ell} .$$

The operator $\hat{L}^2$ is defined in the following way:

$$\hat{L}^2 = \hat{J}^2 + \frac{1}{2}\left(\hat{\ell}_+\hat{\ell}_- + \hat{\ell}_-\hat{\ell}_+\right) - \hat{\ell}_z^2 - \hat{B} , \qquad (2.7)$$

where the operator $\hat{B}$ is connected with the Coriolis interactions [9] and has the form

$$\hat{B} = 2(\hat{\ell}\hat{J}) - 2\hat{\ell}_z^2 = \hat{\ell}_+\hat{J}_- + \hat{\ell}_-\hat{J}_+ . \qquad (2.8)$$

Taking into account the following representations for the operators $\hat{J}_\pm$ and $\hat{\ell}_\pm$:

$$\hat{J}_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\Theta} + \frac{i}{\sin\Theta}\frac{\partial}{\partial\Phi} - i\cot\theta\frac{\partial}{\partial\phi}\right] ,$$
$$\hat{\ell}_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]$$

we obtain

$$\hat{J}_\pm\mathcal{D}^{J\lambda}_{m,M} = \sqrt{J(J+1) - m(m\pm1)}\,\mathcal{D}^{J\lambda}_{m\pm1,M} , \qquad (2.9)$$
$$\frac{1}{2}\left(\hat{\ell}_+\hat{\ell}_- + \hat{\ell}_-\hat{\ell}_+\right)\mathcal{D}^{J\lambda}_{m,M} = \left[-\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} + m^2\cot^2\theta\right]\mathcal{D}^{J\lambda}_{m,M} .$$

Let us substitute the representation (2.4) into the Schrödinger equation (2.2), and after some transformations using the orthogonality of the $\mathcal{D}$-functions (2.5), we obtain the system of equations

$$\sum_{m'=0}^{J}\left\{\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} - \frac{J(J+1)-m^2}{R^2}\right) - \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right)\right.\right. \qquad (2.10)$$

$$-\frac{1}{2}\left(\frac{1}{R^2}+\frac{1}{r^2}\right)\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}-m^2\cot^2\theta\right)-\frac{Z_1Z_3}{R}+\frac{1}{2}U$$
$$+\frac{Z_1Z_2c}{\sqrt{r^2+c_1^2R^2+2c_1Rr\cos\theta}}-\frac{Z_2Z_3c}{\sqrt{r^2+c_3^2R^2-2c_3Rr\cos\theta}}\Bigg]\gamma^\lambda_{mJ}\cdot\delta_{m,m'}$$
$$-\frac{1}{2R^2}\Bigg[\sqrt{(J-m+1)(J+m)}\left(\frac{\partial}{\partial\theta}-(m-1)\cot\theta\right)\cdot\delta_{m',m-1}$$
$$+\sqrt{(J+m+1)(J-m)}\left(-\frac{\partial}{\partial\theta}-(m+1)\cot\theta\right)\delta_{m',m+1}\Bigg]G^{J\lambda}_{mm'}\Bigg\}\Psi^\lambda_{Jm'}(R,r,\theta)=0\,,$$

where

$$\gamma^\lambda_{mJ}=\frac{1}{2}\left[1+\delta_{m0}\lambda(-1)^J\right]\,, \tag{2.11}$$
$$G^{J\lambda}_{mm'}=\frac{1}{2}\left[1+(\delta_{m0}+\delta_{m'0})\lambda(-1)^J\right]\left[1+\delta_{m0}\delta_{m'1}+\delta_{m1}\delta_{m'0}\right]\,.$$

Our problem is to determine the eigenvalues and eigenfunctions of the Hamiltonian represented in Eq.(2.10). We shall use the Oscillator Representation Method: First of all, we have to represent this Hamiltonian in the hermitian form. Secondly, we have to change radial variables R and r to get the Gaussian asymptotic behaviour of the wave functions at large distances. For this aim let us make same transformations. The wave functions Ψ^λ_{Jm} depend only on the variables R, r and θ. Let us multiply equation (2.10) by $\sin^2\theta$ and introduce the new variable

$$u=\ln\left(\tan\frac{\theta}{2}\right)$$

so that one gets

$$\sin^2\theta\left[\frac{d^2}{d\theta^2}+\cot\theta\frac{d}{d\theta}\right]=\left(\sin\theta\frac{d}{d\theta}\right)^2=\frac{d^2}{du^2}\,.$$

The Schrödinger equation (2.10) becomes

$$\begin{pmatrix}\hat L_{0J} & \hat T_{01} & \cdots & \hat T_{0J}\\ \hat T^*_{01} & \hat L_{1J} & \cdots & \hat T_{1J}\\ \vdots & \vdots & \vdots & \vdots\\ \hat T^*_{0J} & \hat T^*_{1J} & \cdots & \hat L_{JJ}\end{pmatrix}\begin{pmatrix}\Psi_{0J}\\ \Psi_{1J}\\ \vdots\\ \Psi_{JJ}\end{pmatrix}=0\,, \tag{2.12a}$$

where

$$\hat L_{mJ} = \Bigg[-\frac{1}{2\cosh^2u}\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}-\frac{J(J+1)-2m^2+\nu(\nu+1)}{R^2}\right) \tag{2.12}$$
$$-\frac{1}{2\cosh^2u}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\nu(\nu+1)}{r^2}\right)+W(R,r,u)$$
$$-\frac{1}{2}\left(\frac{1}{R^2}+\frac{1}{r^2}\right)\left(\frac{\partial^2}{\partial u^2}-m^2+\frac{\nu(\nu+1)}{\cosh^2u}\right)\Bigg]\gamma^\lambda_{mJ}$$
$$\hat T_{mJ}+\hat T^*_{mJ}=\Bigg\{-\frac{1}{2R^2}\Bigg[\sqrt{(J-m+1)(J+m)}\left(\frac{\partial}{\partial u}-(m-1)\tanh u\right)\delta_{m',m-1}$$
$$+\sqrt{(J+m+1)(J-m)}\left(-\frac{\partial}{\partial u}-(m+1)\tanh u\right)\cdot\delta_{m',m+1}\Bigg]\Bigg\}\cdot\frac{G^{J\lambda}_{mm'}}{\cosh u}\,.$$

and

$$W(R,r,u) = \frac{1}{\cosh^2u}\Bigg[\frac{Z_1Z_2c}{\sqrt{r^2+c_1^2R^2+2c_1Rr\tanh u}} \tag{2.13}$$
$$-\frac{Z_2Z_3c}{\sqrt{r^2+c_3^2R^2-2c_3Rr\tanh u}}-\frac{Z_1Z_3}{R}+\frac{1}{2}U\Bigg]\,.$$

Here a new parameter ν is introduced and we consider it as a variational parameter. In the case of the two-body systems the parameter ν is the orbital momentum of the system. In the case of the three-body Coulomb systems this parameter ν can be equal to the total angular momentum J with small deflection connected with the axiallity of the system.

Let us modify the variables R and r in equation (2.12) and the wave functions $\Psi^\lambda_{Jm}(R,r;\tanh u)$ to get a modified Schrödinger equation having the solution with the Gaussian asymptotic behaviour (see details in refs. [6], [7]). These substitutions for the Coulomb systems are

$$R=Q^2\,,\qquad r=q^2\,,\qquad \Psi^\lambda_{Jm}(R,r,\tanh u)=Q^{2\nu}q^{2\nu}\cdot\Phi_{mJ}(Q,q,u)\,,$$

which provide the Gaussian asymptotic behaviour of the wave function at large distances. The Schrödinger equation takes the form

$$\mathbf{H}(E_J)\Phi_{mJ}(Q,q,u)=0\,, \tag{2.14}$$

where

$$\mathbf{H}(E_J)=\begin{pmatrix}\hat L_{0J} & \hat T_{01} & \cdots & \hat T_{0J}\\ \hat T^*_{01} & \hat L_{1J} & \cdots & \hat T_{1J}\\ \vdots & \vdots & \vdots & \vdots\\ \hat T^*_{0J} & \hat T^*_{1J} & \cdots & \hat L_{JJ}\end{pmatrix}\,,$$

and

$$\Phi_{mJ}(Q,q,u)=\begin{pmatrix}\Phi_{0J}\\ \Phi_{1J}\\ \vdots\\ \Phi_{JJ}\end{pmatrix}\,.$$

The operators represented in Eq.(2.12) and Eq.(2.13) can be rewritten as

$$\hat L_{mJ} = \Bigg[-\frac{4q^2}{2\cosh^2u}\Delta_Q-\frac{4Q^2}{2\cosh^2u}\Delta_q-\frac{J(J+1)-2m^2}{\cosh^2u}\cdot\frac{16q^2}{Q^2} \tag{2.15}$$

$$+\ W(Q,q,u) - \frac{16}{2}\left(\frac{q^2}{Q^2}+\frac{Q^2}{q^2}\right)\left(\frac{\partial^2}{\partial u^2} - m^2 + \frac{\nu(\nu+1)}{\cosh^2 u}\right)\Bigg]\gamma^{\lambda}_{mJ}\,,$$

$$\hat{T}_{mJ} + \hat{T}^{*}_{mJ} = \Bigg\{ + \frac{8q^2}{2Q^2}\Bigg[\sqrt{(J-m+1)(J+m)}\left(\frac{\partial}{\partial u} - (m-1)\tanh u\right)\delta_{m',m-1}$$

$$+\ \sqrt{(J+m+1)(J-m)}\left(-\frac{\partial}{\partial u} - (m+1)\tanh u\right)\cdot\delta_{m',m+1}\Bigg]\Bigg\}\cdot\frac{G^{J\lambda}_{mm'}}{\cosh u}\,.$$

Here $D = 4 + 4\nu$,

$$W(Q,q,u) = \frac{16q^2Q^2}{\cosh^2(u)}\Bigg[\frac{Z_1Z_2c}{\sqrt{q^4 + c_1^2Q^4 + 2c_1Q^2q^2\tanh u}} - \frac{Z_2Z_3c}{\sqrt{q^4 + c_3^2Q^4 - 2c_3Q^2q^2\tanh u}} - \frac{Z_1Z_3}{Q^2} + \frac{1}{2}U\Bigg]\,, \tag{2.16}$$

$$\Delta_Q = \left(\frac{d^2}{dQ^2} + \frac{D-1}{Q}\cdot\frac{d}{dQ}\right)\,. \tag{2.17}$$

We have the Schrödinger equation (2.14) in the D-dimensional auxiliary space $\mathbf{R}^D$ and look for the solution $\Phi_{mJ}(Q,q,u)$ depending on Q^2, q^2 and u only, i.e., for the ground state in the space $\mathbf{R}^D$. Thus, we can identify the operator (2.17) with the radial part of the Laplacian Δ_Q in the space $\mathbf{R}^D$ and can consider Q_j and q_j as vectors in this space. The Hamiltonian, corresponding to the Schrödinger equation (2.14), is hermitian.

Our problem is formulated in the following way. We have the Hamiltonian $\mathbf{H} = \mathbf{H}(U_J)$, and we have to solve the Schrödinger equation

$$\mathbf{H}(U_J)\Phi_m(Q,q,u) = \varepsilon(U_J)\Phi_{mJ}(Q,q,u), \tag{2.18}$$

i.e. we have to find the ground state energy $\varepsilon(U_J)$. According to (2.14), the desired energy E_J or the energy parameter U_J is determined by the equation

$$\varepsilon(U_J) = 0\,. \tag{2.19}$$

Hamiltonians in the correct form. The correct form of a Hamiltonian is defined (see [10]) as follows. Let the Hamiltonian H be given and the Schrödinger equation

$$H\Psi = \varepsilon\Psi$$

can not be solved analytically. Let us introduce an auxiliary Hamiltonian $H(\xi)$ depending on a set of parameters ξ. This Hamiltonian should be chosen in such a way that, first, it could be close as much as possible to the initial Hamiltonian H and, second, the Schrödinger equation

$$H(\xi)\Psi(\xi) = \varepsilon(\xi)\Psi(\xi) \tag{2.20}$$

could be solved analytically, i.e., the wave function $\Psi(\xi)$ could be found in an explicit form. Now let us do the variational estimation of the initial total Hamiltonian over the wave function $\Psi(\xi)$

$$\varepsilon_0 = \min_{\xi}\left(\Psi(\xi)H\Psi(\xi)\right). \tag{2.21}$$

This minimization gives us the energy ε_0 in the lowest approximation, the values of the parameters $\xi = \xi_0$ and the wave function in the zeroth approximation $\Psi_0 = \Psi(\xi_0)$. Let us represent the initial Hamiltonian in the form

$$\begin{aligned} H &= \mathbf{H}_0 + \mathbf{H}_I + \varepsilon_0, \\ \mathbf{H}_0 &= H(\xi_0) - \varepsilon(\xi_0) =: H(\xi_0) :, \\ \mathbf{H}_I &= H - H(\xi_0) + \varepsilon(\xi_0) - \varepsilon_0 =: H - H(\xi_0) :, \end{aligned} \tag{2.22}$$

where

$$\mathbf{H}_0\Psi_0 = 0\,,$$

and the normal product of operator F with respect to the Hamiltonian $\mathbf{H}_0$ is defined as

$$: F := F - (\Psi_0 F \Psi_0). \tag{2.23}$$

In what follows we shall call the representation (2.22) *the correct form* of the Hamiltonian H with respect to the Hamiltonian $\mathbf{H}_0$.

If the Hamiltonian in the zeroth approximation equals the pure oscillator Hamiltonian in the space R^d, i.e.

$$\mathbf{H}_0 = \frac{1}{2}\left(P_Q^2 + \Omega^2Q^2\right) - \frac{d}{2}\cdot\Omega = \Omega\left(a_j^{+}a_j\right)\,, \quad j = 1,\ldots,d$$

the correct form is equivalent to the *oscillator representation* (see [6]).

The ground state energy ε can be calculated by the perturbation method over the interaction Hamiltonian $\mathbf{H_I}$ and turns out to be

$$\varepsilon = \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \ldots = \varepsilon_0 - \left(\Psi_0\mathbf{H}_I\frac{1}{\mathbf{H}_0}\mathbf{H}_I\Psi_0\right) + \ldots, \tag{2.24}$$

because

$$\mathbf{H}_0\Psi_0 = 0 \quad \text{and} \quad \varepsilon_1 = (\Psi_0\mathbf{H}_I\Psi_0) = 0. \tag{2.25}$$

The accuracy of this approximation can be evaluated as

$$\delta \sim \left|\frac{\varepsilon_2}{\varepsilon_0}\right|. \tag{2.26}$$

The next steps should be done according to the rules formulated above.

3 The correct form of the three–body Hamiltonian with J=1

In the present paper we consider the states $J = 0$ and $J = 1$ with normal parity $\lambda = (-1)^J$. In this case Eq.(2.14) is rewritten as

$$\begin{pmatrix} \hat{L}_{01} & \hat{T}_{01} \\ \hat{T}^*_{01} & \hat{L}_{11} \end{pmatrix} \begin{pmatrix} \Phi_{01} \\ \Phi_{11} \end{pmatrix} = \varepsilon(U) \begin{pmatrix} \Phi_{01} \\ \Phi_{11} \end{pmatrix} , \qquad (3.1a)$$

where

$$\hat{T}_{01} = \frac{8q^2}{Q^2} \left[i \left(\frac{1}{\cosh u} P_u + P_u \frac{1}{\cosh u} \right) + \frac{\sinh u}{\cosh^2 u} \right] , \qquad (3.1)$$

and $P_u = -id/du$. The operators $\hat{L}_{01}$, and $\hat{L}_{11}$ are represented in (2.15) and we rewrite them in the form

$$\begin{aligned} \hat{L}_{m1} &= \frac{1}{2\kappa_Q^m} \left(P_Q^2 + Q^2\Omega^2 - D\Omega \right) + \frac{1}{2\kappa_q^m} \left(P_q^2 + q^2\omega^2 - D\omega \right) \qquad (3.2) \\ &+ \frac{1}{2\kappa_u} \left(P_u^2 + m^2 - \frac{\nu(\nu+1)}{\cosh^2 u} \right) + W(Q,q,u) - \frac{\Omega^2}{2\kappa_Q^m} Q^2 + \frac{D}{2\kappa_Q^m} \Omega \\ &- \frac{\omega^2}{2\kappa_q^m} q^2 + \frac{D}{2\kappa_q^m} \omega + \frac{1}{2} P_Q^2 \cdot \left(\frac{4q^2}{\cosh^2 u} - \frac{1}{\kappa_Q^m} \right) + \frac{1}{2} P_q^2 \cdot \left(\frac{4Q^2}{\cosh^2 u} - \frac{1}{\kappa_q^m} \right) \\ &+ \frac{1}{2} \left(P_u^2 + m^2 - \frac{\nu(\nu+1)}{\cosh^2 u} \right) \left(\frac{16q^2}{Q^2} + \frac{16Q^2}{q^2} - \frac{1}{\kappa_u} \right) + \frac{8q^2}{Q^2} \frac{2}{\cosh^2 u} (1 - m^2). \end{aligned}$$

This Hamiltonian contains six free parameters ν, ω, Ω, κ_Q^m, κ_q^m, and κ_u.

According to (2.22), let us represent the Hamiltonian in the form

$$H = \mathbf{H}_0 + \mathbf{H}_I + \varepsilon_0, \qquad (3.3)$$

$$\mathbf{H}_0 = \begin{pmatrix} H_{00} & 0 \\ 0 & H_{11} \end{pmatrix} ,$$

and choose

$$\begin{aligned} H_{mm} &= \frac{1}{2\kappa_Q^m} \left(P_Q^2 + Q^2\Omega^2 - D\Omega \right) + \frac{1}{2\kappa_q^m} \left(P_q^2 + q^2\omega^2 - D\omega \right) \qquad (3.4) \\ &+ \frac{1}{2\kappa_u} \left(P_u^2 + m^2 - \frac{\nu(\nu+1)}{\cosh^2 u} \right) \quad (m = 0,\ 1) \end{aligned}$$

so that

$$\begin{aligned} \Phi_{m1}(Q,q,u) &= \frac{P_\nu^m(\tanh u)}{\sqrt{2N_m(\nu)}} \cdot \left(\frac{\omega\Omega}{\pi^2} \right)^{D/4} \exp\left\{ -\frac{1}{2}\Omega Q^2 - \frac{1}{2}\omega q^2 \right\}, \qquad (3.5) \\ N_m(\nu) &= \int_{-\infty}^{\infty} \frac{du}{\cosh^2 u} (P_\nu^m(\tanh u))^2 = \int_{-1}^{1} dx : (P_\nu^m(x))^2 , \\ (\Phi,\ \Phi) &= (\Phi_{01},\ \Phi_{01}) + (\Phi_{11},\ \Phi_{11}) = 1. \end{aligned}$$

The oscillator canonical variables $(P_Q,\ Q)$, $(P_q,\ q)$ can be written in the form

$$\begin{aligned} Q_j &= \frac{1}{\sqrt{2\Omega}} \cdot (A_j + A_j^+) , \qquad P_{Q_j} = \sqrt{\frac{\Omega}{2}} \cdot \frac{A_j - A_j^+}{i} , \qquad (3.6) \\ q_j &= \frac{1}{\sqrt{2\omega}} \cdot (a_j + a_j^+) , \qquad P_{q_j} = \sqrt{\frac{\omega}{2}} \cdot \frac{a_j - a_j^+}{i} , \\ [A_i, A_j^+] &= \delta_{ij} , \quad [a_i, a_j^+] = \delta_{ij} , \quad (i,j = 1,2,...D) \end{aligned}$$

and we obtain

$$\begin{aligned} \frac{1}{2\kappa_Q^m} \left(P_Q^2 + \Omega^2 Q^2 - D\Omega \right) &= \frac{\Omega}{\kappa_Q^m} \left(A_j^+ A_j \right) , \\ \frac{1}{2\kappa_q^m} \left(P_q^2 + \omega^2 q^2 - D\omega \right) &= \frac{\omega}{\kappa_q^m} \left(a_j^+ a_j \right) . \end{aligned}$$

The operators A_j, a_j and A_j^+, a_j^+ are annihilation and creation operators. The ground or vacuum oscillator state

$$|0\rangle = \left(\frac{\omega\Omega}{\pi^2} \right)^{D/4} \exp\left\{ -\frac{1}{2}\Omega Q^2 - \frac{1}{2}\omega q^2 \right\}$$

satisfies the conditions

$$\langle 0|0\rangle = 1, \quad A_j|0\rangle = 0 , \quad a_j|0\rangle = 0 , \quad (j = 1,2,...D) .$$

The function $P_\nu^m(z)$ is the solid spherical harmonic satisfying the equation

$$\left(\frac{d^2}{du^2} - m^2 + \frac{\nu(\nu+1)}{\cosh^2 u} \right) P_\nu^m(\tanh u) = 0 , \qquad (3.7)$$

where $m = 0, \pm 1, \pm 2, ...$ are the azimuthal quantum numbers and ν is a parameter which can take any values. According to (2.21) the parameter ν will be determined by the minimization of the energy in the zeroth approximation.

The positive parameters κ_Q^m, κ_q^m and κ_u are defined by the condition that the interaction Hamiltonian does not contain the quadratic terms with $: P_Q^2 :$, $: P_q^2 :$ and $: P_u^2 :$. We have

$$\begin{aligned} : \frac{4q^2}{\cosh^2 u} : &= \frac{4q^2}{\cosh^2 u} - \frac{1}{\kappa_Q}, \qquad \frac{1}{\kappa_Q^m} = \left(\Phi_{m1} \left| \frac{4q^2}{\cosh^2 u} \right| \Phi_{m1} \right) = \frac{D}{\omega} A_m(\nu), \\ : \frac{4Q^2}{\cosh^2 u} : &= \frac{4Q^2}{\cosh^2 u} - \frac{1}{\kappa_q}, \qquad \frac{1}{\kappa_q^m} = \left(\Phi_{m1} \left| \frac{4Q^2}{\cosh^2 u} \right| \Phi_{m1} \right) = \frac{D}{\Omega} A_m(\nu), \\ A_m(\nu) &= \frac{1}{N_m(\nu)} \int_{-\infty}^{\infty} \frac{du}{\cosh^2(u)} (P_\nu^m(\tanh u))^2 = \frac{1}{N_m(\nu)} \int_{-1}^{1} dx \, (P_\nu^m(x))^2 , \\ \frac{1}{\kappa_u} &= \left(\Phi_{m1} \left| \frac{16q^2}{Q^2} + \frac{16Q^2}{q^2} \right| \Phi_{m1} \right) = \frac{8D}{D-2} \cdot \left(\frac{\omega}{\Omega} + \frac{\Omega}{\omega} \right) . \qquad (3.8) \end{aligned}$$

The representation of the functions $A_m(\nu)$ and $N_m(\nu)$ are given in the Appendix.

Let us determine the ground state energy in the zeroth approximation. From (3.1a) we get

$$\varepsilon(U,\Omega,\omega,\nu) = \begin{pmatrix} \Phi_{01} & \Phi_{11} \end{pmatrix} \begin{pmatrix} \hat{L}_{01} & \hat{T}_{01} \\ \hat{T}^*_{01} & \hat{L}_{11} \end{pmatrix} \begin{pmatrix} \Phi_{01} \\ \Phi_{11} \end{pmatrix}$$

The parameters ν, ω, and Ω are determined by the minimum of energy in the zeroth approximation, i.e.

$$\varepsilon_0(U) = \min_{\nu,\omega,\Omega} \varepsilon(U,\Omega,\omega,\nu) = \min_{\nu,\omega,\Omega} \left\{ \frac{1}{4} D^2 \left(A_0(\nu) + \frac{1}{2} A_1(\nu) \right) \cdot \left(\frac{\Omega}{\omega} + \frac{\omega}{\Omega} \right) \right. \tag{3.9}$$

$$+ \left(A_0(\nu) + \frac{1}{2} A_1(\nu) \right) \left[\frac{UD^2}{\omega\Omega} - \frac{4Z_1Z_3D}{\omega} \right] - \frac{8D}{D-2} \frac{\Omega}{\omega} A_0(\nu) \left(2\nu(\nu+1) \sqrt{\frac{N_0}{N_1}} - 1 \right)$$

$$+ \frac{8}{\Gamma^2(D/2)} (\omega\Omega)^{D/2} \cdot \left[\frac{1}{N_0} \int_{-1}^{1} dx \left(P_\nu(x) \right)^2 + \frac{1}{2N_1} \int_{-1}^{1} dx \left(P^1_\nu(x) \right)^2 \right] \int_0^\infty \int_0^\infty dt_1 dt_2$$

$$\left. \cdot (t_1 t_2)^{D/2} \exp\left(-\Omega t_1 - \omega t_2 \right) \left[\frac{Z_1 Z_2 c}{\sqrt{t_2^2 + c_1^2 t_1^2 + 2c_1 t_1 t_2 x}} - \frac{Z_2 Z_3 c}{\sqrt{t_2^2 + c_3^2 t_1^2 - 2c_3 t_1 t_2 x}} \right] \right\}.$$

The interaction Hamiltonian has the form

$$\mathbf{H}_I = \begin{pmatrix} \hat{L}^I_{01} & :\hat{T}^I_{01}: \\ :\hat{T}^*_{01}: & \hat{L}^I_{11} \end{pmatrix}$$

where

$$\hat{L}^I_{m1} = :\left[W(Q,u,E_m) - \frac{\Omega^2}{2\kappa_Q^m} \cdot Q^2 - \frac{\omega^2}{2\kappa_q^m} \cdot q^2 \right]: + \frac{1}{2} : \left(P_Q^2 + \frac{D\Omega}{2} \right) :: : \frac{4q^2}{\cosh^2 u} :$$

$$+ \frac{1}{2} : \left(P_q^2 + \frac{D\omega}{2} \right) :: \frac{4Q^2}{\cosh^2 u} : + \frac{1}{2} : \left(P_u^2 + m^2 - \frac{\nu(\nu+1)}{\cosh^2 u} \right) :$$

$$: \left(\frac{16q^2}{Q^2} + \frac{16Q^2}{q^2} \right) : + (1-m^2) \cdot : \frac{14q^2}{Q^2 \cosh^2 u} : .$$

From Eq.(2.19) and Eq.(3.9) one can see that we have the possibility of defining the energy parameter U as a function of masses and charges of the particles for the Coulomb three-body systems with J.

4 The bound state energy three-body Coulomb system

Let us calculate the binding energy of the three-body Coulomb systems with $J = 0,\ 1$. In the paper [6] we have determined the bound state energy and the stability region for the Coulomb three-body systems with J=0. In that case, the dimensions of the auxiliary space

have been considered as additional variational parameter. In ref.[10] the oscillator representation method is extended to calculation of the energy of the systems described by the Schrödinger equation with axially symmetrical potentials. In the this case, the dimensions of the auxiliary space defined as functions of the parameter are connected with axiallity of the system. The Coulomb three-body systems are also described by the Schrödinger equation with axially symmetrical potentials. Most quantum systems described by the Schrödinger equation with an axially symmetrical potential cannot be solved analytically. Thus, the solution of the Schrödinger equation with sufficiently arbitrary potentials of this type represents the main mathematical problem.

Binding energy of the three–body Coulomb systems with J=0. First of all calculate the ground state energy. According to (3.1a) and (3.9) the ground state energy in the zeroth approximation can be rewritten in the form

$$\varepsilon_0(U) = \min_{\nu,\omega,\Omega} \left(\Phi_{00} \hat{L}_{00} \Phi_{00} \right) \tag{4.1}$$

$$= \min_{\nu,\omega,\Omega} \left\{ A(\nu) \frac{\Omega}{\omega} + B(\nu) \frac{\omega}{\Omega} - \frac{4Z_1Z_3}{\omega} + \frac{UD}{\Omega\omega} + \frac{1}{\Omega} F_0(y) \right\},$$

where $A(\nu) = B(\nu) = D/4 = 1 + \nu$, $y = \omega/\Omega$; the function $F_0(y)$ and calculational detail are represented in Appendix.

The numerical results for the binding energies of the ground state are shown in Table 1.

Table 1. Binding energy (in ev) of the hydrogen isotopes with $J = 0$.

systems	U	ν	Ω	ω	E_0^{or}	E^{ex}
$(pp\mu)$	1.1001	-0.370	0.2241	2.0856	253.10	253.1523
$(dd\mu)$	1.1222	-0.289	0.1497	2.1137	324.97	325.0735
$(tt\mu)$	1.1336	-0.201	0.1204	2.1253	362.23	362.9097
$(pd\mu)$	1.1390	-0.287	0.2722	2.1179	220.73	221.5494
$(pt\mu)$	1.1566	-0.245	0.2929	2.1333	213.22	213.8402
$(dt\mu)$	1.1395	-0.226	0.1669	2.1293	318.955	319.1396
$(dd\overline{p})$	1.0585	-0.243	0.7956	1.8979	965.15	966.691
$(tt\overline{p})$	1.0672	-0.384	0.4580	2.0149	1258.7	1260.787

Binding energy of the three–body Coulomb systems with J=1. The bound state energy of the three-body Coulomb systems with J=1 is defined by

$$\varepsilon_0(U) = \min_{\nu,\omega,\Omega} \left\{ A(\nu) \frac{\Omega}{\omega} + B(\nu) \frac{\omega}{\Omega} - \frac{4Z_1Z_3}{\omega} + \frac{UD}{\Omega\omega} + \frac{1}{\Omega} F_1(y) \right\} \tag{4.2}$$

where $B(\nu) = D/4$

$$A(\nu) = \frac{1}{4} D - \frac{8}{D-2} \cdot \frac{A_0(\nu)}{A_0(\nu) + 0.5 A_1(\nu)} \cdot \left[2\nu(\nu+1) \sqrt{\frac{N_0}{N_1}} - 1 \right].$$

The function $F_1(y)$ and details of the calculation are given in Appendix. Our results are represented in Table 2.

The background energies of the three-body Coulomb systems have been calculated by many authors and via different methods (see, for example, refs.[1],[5] and [11]). The accuracy of these calculations reached a very high level with the perfection of computers. Therefore, these results can be considered as exact ones.

Table 2. Binding energy (in ev) of the hydrogen isotopes with $J = 1$.

systems	U	Ω	$\bar{\omega}$	E_0^{or}	E^{ex}
(ppμ)	1.0424	5.1011	1.2711	107.212	107.2658
(ddμ)	1.0851	7.4742	1.3673	226.642	226.6815
(ttμ)	1.1066	9.4447	1.4469	289.023	289.1419
(pdμ)	1.0918	5.9718	1.3012	97.409	97.4980
(ptμ)	1.1115	6.4149	1.3231	99.119	99.1262
(dtμ)	1.1053	8.3320	1.4956	232.398	232.4714

In Tables 1 and 2 we denoted our results by E_0^{or} and the corresponding exact values by E^{ex}, respectively.

5 The stability

In ref.[6] the stability boundary for the three-body Coulomb systems with J=0 was established. Now we consider the three-body Coulomb systems with the total angular momentum J=1.

In this section the binding of the three-body Coulomb systems with unit charges and various constituent masses for the total momentum J=1 will be discussed. The calculation of the stability region as a function of masses and charges of particles is one of the basic objects in the Coulomb three-body problem (see, for examples, [1], [2]). The dependence of the background energy on masses of particles has been considered in [3] and numerical calculations have been made in [4] to establish the stability boundary. In refs.([1], [3]) only the three-body Coulomb systems with J=0 were considered.

Let us formulate the problem. We consider a three-body Coulomb system $(A^{\pm}, B^{\mp}, C^{\pm})$ with unit charges $\pm, \mp, \pm$ and various masses m_A, m_B, m_C. We shall look for the stability threshold according to the decay

$$(A, B, C) \to (A, B) + C\,, \qquad (5.1)$$

where (A, B) is a two-body atom and C is a free particle.

It is more convenient to use the following variables instead of masses m_j (see [1]):

$$\alpha_j = \frac{1}{m_j}/\left(\frac{1}{m_A}+\frac{1}{m_B}+\frac{1}{m_C}\right), \qquad (j = A, B, C), \qquad (5.2)$$
$$\alpha_A + \alpha_B + \alpha_C = 1\,.$$

Any three-body system can be represented by a point in an equilateral triangle for which the altitude equals 1 so that the identity (5.2) can be interpreted as a sum of altitudes from a point at three sides of this triangle (see picture 1). Let us call it *the stability triangle.*

Our problem is to find on the stability triangle the boundary separating stable and unstable states of three-body systems. The stability of a three-body system (A, B, C) is defined relative to the decay (5.1). Without loss of generality, one can consider that $m_A \geq m_C$. The binding energy can be calculated by the formula

$$\Delta E = -\frac{1}{2}e^4\left(\frac{m_C m_B}{m_C + m_B}U - \frac{m_A m_B}{m_A + m_B}\right), \qquad (5.3)$$

where the parameter U as a function of masses m_j or α_j is determined by (3.9). The condition $\Delta E = 0$ defines the sought boundary. We shall call the masses for which this condition is fulfilled *the critical masses.*

First of all, let us place on the triangle the points corresponding to the well-known systems

$$H_2 = (ppe^-), \quad H^- = (pe^-e^-), \quad (e^+e^-e^-), \quad (pe^-\mu^-), \quad (pe^-e^+).$$

The system $(e^+e^-e^-)$ is studied in ([12],[13]) and it is stable. The molecules $(ppe^-) = H_2^+$ and $(pe^-e^-) = H^-$ are well-known hydrogen ions and they are stable ([12],[13]). The system $(pe^-\mu^-)$ is unstable [14].

In the oscillator representation the critical masses will be calculated according to formula (5.3). We shall proceed in the following way.

The molecules (pe^-e^+) and (pe^-p) are distinguished by the masses of positron and proton. Let us consider the binding energy of the system (pe^-C^+) as a function of the mass m_C of the C-particle in the limits $m_e \leq m_C \leq m_p$. Thus, the mass of the C-particle $m_C = 2.11m_e$ is a critical one for the system (pe^-C^+) with J=1.

The molecules (pe^-e^+) and $(e^+e^-e^+)$ are distinguished by the masses of positron and proton, too. Let us consider the system $(D^+e^-e^+)$. When $m_D = m_p$ the system is unstable and for $m_D = m_e$ it is stable. Decreasing the mass of the D-particle to $m_D = 4.15m_e$ one can get $\Delta E = 0$ and for $m_D \leq 4.15m_e$ this system becomes stable.

For the systems (pe^-C^+) and $(D^+e^-e^+)$ the critical masses of the particles D and C are found to equal $m_C = 2.11m_e$ and $m_D = 4.15m_e$. These systems have been considered in ([1],[15] - [17]) where the restriction on the critical masses is established.

Now let us consider the molecule $(\bar{p}B^-e^-)$ which contains the molecule $(p\mu^-e^-)$ and the hydrogen atom (pe^-e^-). These systems differ by the masses of muon and electron. Our calculation gives the critical mass of the B-particle to be $m_B = 1.49m_e$.

In the system $(A^+A^-e^-)$ which changes from the ion H^- for $m_A = m_e$ to the proton-antiproton ion $(p\bar{p}e)$ for $m_A = m_p$ the critical mass equals $m_A = 2.22m_e$.

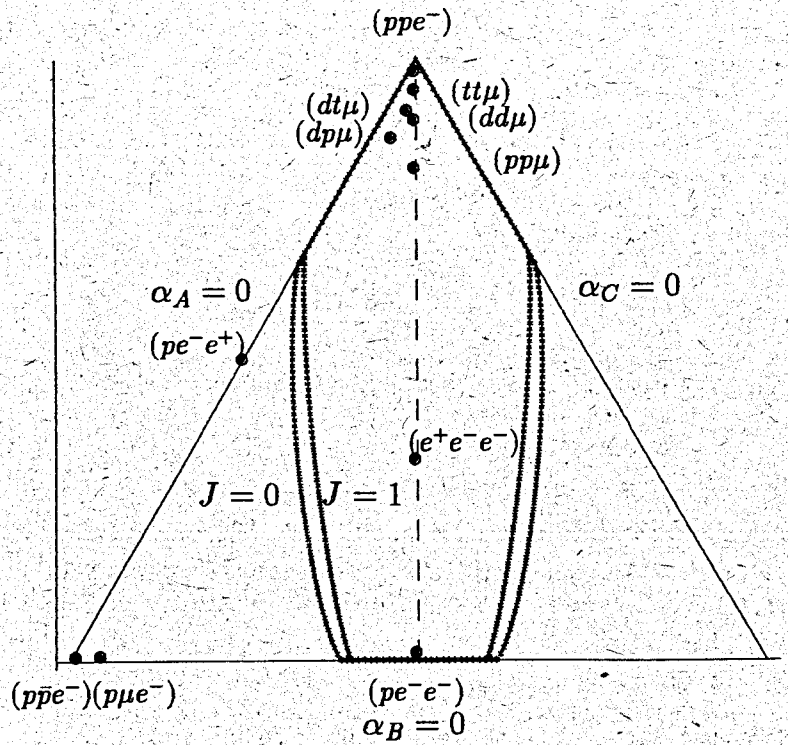

Fig.1. The stability triangle. The stability boundary for the three-body Coulomb systems with J=0,1.

Thus, for the systems (pB^-e^-) and $(A^+A^-e^-)$ the critical masses are $m_B = 1.49m_e$ and $m_A = 2.22m_e$.

Our results (four points) are shown in Table 3 and can be approximated by equation (5.4).

Table 3.Results for stability bound-aries.

α_A	.0003	.1085	.2299	.4032
α_B	0.680	.4444	.2299	.0003

$$\left(\frac{\alpha_A}{.403}\right)^{.92} + \left(\frac{\alpha_B}{.68}\right)^{.92} = 1\,. \tag{5.4}$$

The vertical altitude on which every molecule is stable is the axis of charge symmetry of the three-body system. Therefore, the stability boundaries are symmetrical relative to this altitude. Thus, the inside part of the stability triangle bounded by these lines is the stability region of three-body Coulomb systems with J=1.

ACKNOWLEDGMENTS.

The authors would like to thank Prof. V.B.Belyaev, L.I.Ponomarev and for instructive and enjoyable discussions.

This paper is supported by the Russian Foundation for Fundamental Research (grant N 94-02-03463-a)

Appendix

Let us consider the equation

$$\left(\frac{d^2}{du^2} - m^2 + \frac{\nu(\nu+1)}{\cosh^2 u}\right) P_\nu^m(\tanh u) = 0\,, \tag{A.1}$$

which is defined in the interval $(-\infty < u < \infty)$. Let us introduce the new variable

$$x = \tanh(u)\,,$$

and after some simplifications the equation (A.1) looks like

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2\frac{d}{dx} - \frac{m^2}{1-x^2} - \nu(\nu+1)\right] P_\nu^m(x) = 0\,. \tag{A.2}$$

One can see that $P_\nu^m(x)$ is the solid spherical harmonic, $m = 0, \pm 1, \pm 2, \ldots$ are the azimuthal quantum numbers and ν is a parameter which can take any values.

For the calculations of the bound state energy, i.e., the functions $A_m(\nu)$, $N_m(\nu)$ and F(y) we used the representation for the solid spherical harmonic [18]:

$$P_\nu^m(x) = (-1)^m \cdot \frac{(1-x^2)^{m/2}}{2^m} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(m-\nu+n)\Gamma(1+\nu+m+n)}{\Gamma(m-\nu)\Gamma(\nu-m+1)(m+n)!n!} \left(\frac{1-x}{2}\right)^n . \tag{A.3}$$

Used this representation from (3.5) and (3.8) we get

$$\begin{aligned} N_0(\nu) &= \sum_{k,j=0}^{\infty} I_{kj}(\nu,0) \cdot \frac{2}{\Gamma^2(1+j)\Gamma^2(1+k)} \cdot \frac{1}{1+j+k} \\ N_1(\nu) &= \sum_{k,j=0}^{\infty} I_{kj}(\nu,1) \cdot \frac{2}{j!(1+j)!} \cdot \frac{1}{k!(1+k)!} \cdot \frac{\Gamma(2+j+k)}{\Gamma(4+j+k)}, \\ A_0(\nu) &= \frac{2}{N_0(\nu)} \sum_{k,j=0}^{\infty} I_{kj}(\nu,0) \cdot \frac{1}{\Gamma^2(1+j)\Gamma^2(1+k)} \cdot \frac{1}{1+j+k} \\ A_1(\nu) &= \frac{2}{N_1(\nu)} \sum_{k,j=0}^{\infty} I_{kj}(\nu,1) \cdot \frac{1}{j!\Gamma(2+j)} \cdot \frac{1}{k!\Gamma(2+k)} \cdot \frac{\Gamma(2+j+k)}{\Gamma(4+j+k)}, \end{aligned} \tag{A.4}$$

where

$$I_{jk}(\nu,m) = \frac{\Gamma(m-\nu+k)\Gamma(m-\nu+j)}{\Gamma^2(m-\nu)} \frac{\Gamma(m+1+\nu+k)\Gamma(m+1+\nu+j)}{\Gamma^2(\nu-m+1)}. \tag{A.5}$$

Let us calculate the integral represented in Eqs. (4.1) and (4.2). We used the following relations:

$$\begin{aligned} &\int_0^\infty \int_0^\infty dt_1 dt_2 \, (t_1 \cdot t_2)^{D/2} \cdot \exp\{-\Omega t_1 - \omega t_2\} \cdot \frac{1}{\sqrt{t_2^2 + a^2 t_1^2 + 2a t_1 t_2 x}} \\ &= \frac{\Gamma(1+D)}{\Omega^{1+D}} \cdot a^{D/2} \int_0^\infty dt \frac{t^{D/2}}{(t+ya)^{1+D}} \cdot \frac{1}{\sqrt{1+t^2+2tx}} \end{aligned}$$

and

$$\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{n=0}^{\infty} \begin{cases} t^n P_n(x) & \text{at } |t| \le 1 \\ t^{-1-n} P_n(x) & \text{at } |t| \ge 1 \end{cases},$$

where $y = \omega/\Omega$ and $P_n(x)$ are the Legendre polynomial. Now consider the integral

$$\Omega \cdot F_1(y) = \frac{8 \cdot (\Omega\omega)^{D/2}}{\Gamma^2(D/2)} \left[\frac{1}{N_0(\nu)} \int_{-1}^{1} dx \, (P_\nu(x))^2 + \frac{1}{2N_1(\nu)} \int_{-1}^{1} dx \, (P_\nu^1(x))^2 \right]$$

$$\cdot \int_0^\infty \int_0^\infty dt_1 dt_2 \, (t_1 \cdot t_2)^{D/2} \cdot \exp\{-\Omega t_1 - \omega t_2\} \tag{A.6}$$

$$\cdot \left[\frac{Z_1 Z_2 \cdot c}{\sqrt{t_2^2 + c_1^2 t_1^2 + 2c_1 t_1 t_2 x}} - \frac{Z_2 Z_3 \cdot c}{\sqrt{t_2^2 + c_3^2 t_1^2 - 2c_3 t_1 t_2 x}} \right].$$

Taking into account the relations defined above and after simple transformation, from (A.6) we have

$$\begin{aligned} F_1(y) = \frac{\Gamma(1+D)}{\Gamma^2(D/2)} y^{D/2} c \cdot \sum_{n=0}^{\infty} R_n(\nu) \cdot \Bigg\{ &(-1)^n c_1^{D/2} \int_0^1 dz \cdot z^{D/2+n} \\ &\left[\frac{1}{(z+yc_1)^{D+1}} - \frac{1}{(1+zyc_1)^{D+1}} \right] - c_3^{D/2} \int_0^1 dz \cdot z^{D/2+n} \\ &\left[\frac{1}{(z+yc_3)^{D+1}} + \frac{1}{(1+zyc_3)^{D+1}} \right] \Bigg\}, \end{aligned} \tag{A.7}$$

where

$$R_n(\nu) = \left[\frac{1}{N_0(\nu)} \int_{-1}^{1} dx \, (P_\nu(x))^2 + \frac{1}{2N_0(\nu)} \int_{-1}^{1} dx \, (P_\nu^1(x))^2 \right] P_n(\nu) \tag{A.8}$$

Taking into account the representation (A.1) for the solid spherical harmonic and the relations (A.2-5), from (A.7) we get

$$R_n(\nu) = \left[R_n^{(0)}(\nu) + \frac{1}{2} \cdot R_n^{(1)}(\nu) \right] \cdot \frac{1}{R_0(\nu)}, \tag{A.9}$$

where

$$R_0(\nu) = A_0(\nu) + \frac{1}{2} A_1(\nu)$$

and

$$\begin{aligned} R_n^{(m)}(\nu) = \frac{1}{N_m(\nu)} \cdot \sum_{i=0}^{n} \frac{(-1)^i \cdot (n+i)!}{(n-i)! \cdot (i!)^2} \cdot \sum_{k,j=0}^{\infty} I_{kj}(\nu,m) \cdot \frac{1}{j!(m+j)!k!(m+k)!} \\ \cdot \left[\frac{\Gamma(1+m)\Gamma(m+k+j+i+1)}{\Gamma(2+i+2m+k+j)} + (-1)^n \frac{\Gamma(1+m+i)\Gamma(1+m+k+j)}{\Gamma(2+i+2m+k+j)} \right]. \end{aligned}$$

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Received by Publishing Department
on October 24, 1995.