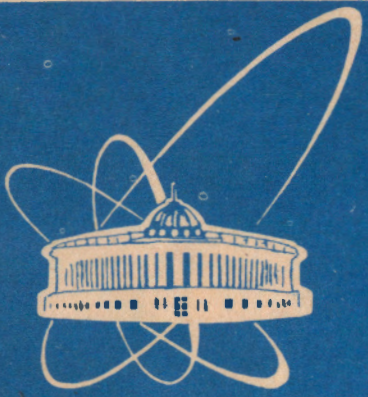

ОБЪЕДИНЕННЫЙ
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PASSAGE OF ATOMS FORMED BY π^+ AND π^- MESONS THROUGH A MATTER

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Прохождение атомов, образованных π^+ и π^- мезонами, через вещество

Рассматривается движение в веществе релятивистских атомов, образованных π^+ и π^- мезонами. Получены точные аналитические выражения формфакторов водородоподобных атомов для дискретных переходов в форме удобной для численных расчетов. Вычислены полные сечения и сечения переходов между дискретными уровнями $\pi^+\pi^-$ атома при взаимодействии с веществом. Эволюция заселенности атомных уровней описывается кинетическими уравнениями. Метод вычислений позволяет получить как заселенности дискретных уровней, так и вероятность перехода в непрерывный спектр (вероятность ионизации). Рассмотренный метод позволил получить первую экспериментальную оценку времени жизни $\pi^+\pi^-$ атома.

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Passage of Atoms Formed by π^+ and π^- Mesons Through a Matter

Relativistic motion of atoms formed by π^+ and π^- mesons in a matter is considered. Exact analytic formulas of discrete-discrete transition form factors of hydrogenlike atoms were obtained in a form convenient for numerical calculations. Total and transition cross sections for interaction of $\pi^+\pi^-$ atoms with a matter were calculated in the Born approximation. Evolution of atomic state populations is treated in terms of kinetic equations. The method of calculation allows to obtain populations of the discrete atomic states as well as probability of transfer to the continuous spectrum (ionization). The considered method has allowed to get the first experimental estimation of the $\pi^+\pi^-$ atom lifetime.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1995

1. Introduction

In the paper [1] production of the Coulomb bound states of various elementary particles (elementary atoms) was predicted for high energy processes. Up till now $\pi^+\pi^-$ atom ($A_{2\pi}$) has been observed in p Ta interaction at 70 GeV [2]. $A_{2\pi}$ decays dominantly through the charge-exchange process $\pi^+\pi^- \to \pi^0\pi^0$ (annihilates). An accurate measurement of the $A_{2\pi}$ lifetime τ will allow to check the high precision prediction of the Chiral Perturbation Theory and therefore a low energy limit of QCD [3].

Interaction of $\pi^+\pi^-$ atoms with ordinary atoms is an essential part of that experiment, as observation of $\pi^+\pi^-$ atoms bases on the atom breakup (ionization) while passing through the target where they are produced [1]. In the experiment [2] yield of $\pi^+\pi^-$ pairs from the $A_{2\pi}$ breakup was measured. To obtain the $\pi^+\pi^-$ atom lifetime one should calculate the breakup probability as a function of the atom lifetime with high accuracy, that is, describe a passage of multilevel atomic system through a matter.

2. Interaction of $A_{2\pi}$ with atoms

Being produced in hadron-nucleus interaction, $A_{2\pi}$ moves through a material of the target and interacts, dominantly, via electric field with target atoms (Coulomb interaction). Cross sections of these processes depend on charge as Z^2 so that the interaction of $A_{2\pi}$ with atom electrons is Z times smaller than with the nucleus. For tantalum the interaction with atom electrons (so called incoherent scattering) increases the cross section about 1.5% only [4]. As shown in [5, 6] one should also consider the interaction of $A_{2\pi}$ with magnetic fields that arise due to the Lorentz transformation. However, for example, for the interaction of relativistic $A_{2\pi}$ with Pb the total cross section of the magnetic interaction is only 0.4% [6] of the electric one and so is not considered here.

For the description of the Coulomb interaction of $A_{2\pi}$ with atoms the first Born approximation, which considers only single photon exchange (see for example [7]), was used. Another method, the so called Coulomb-modified Glauber approximation, allows to consider all multiphoton exchanges [8]. The accuracy of this method is estimated to go as $(Z^{1/3}\alpha)^2$ and for Ta it is better than 0.2%. It has been shown [8] that all the cross sections calculated at this approach are smaller than the ones corresponding to the Born approximation. For the total cross section of $A_{2\pi}$ in 1S state with Ta this difference does not exceed 7%.

In the first Born approximation the transition and total cross sections of atom-atom interactions are expressed via atom form factors [7]:

$$\sigma^{n_2 l_2 m_2}_{n_1 l_1 m_1} = \frac{1}{4\pi\beta^2}\int_0^\infty |V(q)|^2 \, |F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(q/2) - F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(-q/2)|^2 \, dq^2 \quad (1)$$

$$\sigma^{tot}_{n_1 l_1 m_1} = \frac{1}{4\pi\beta^2}\int_0^\infty |V(q)|^2 \, \left(2 - 2F^{n_1 l_1 m_1}_{n_1 l_1 m_1}(q)\right) \, dq^2 . \quad (2)$$

Here β is the velocity of $A_{2\pi}$ ($\hbar = c = 1$); n_1, l_1, m_1 and n_2, l_2, m_2 denote the quantum number of the initial and final states of $A_{2\pi}$; $F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(q)$ is the transition form factor of $A_{2\pi}$; $V(q)$ is the potential energy of interaction with target atoms; q is a transfer momentum.

The Moliére parametrization of the Thomas-Fermi potential (T.F.M.) [9] was used for the potential energy $V(q)$.

$$V(q) = 4\pi Z\alpha \left(\frac{0.35}{q^2+\beta_1^2} + \frac{0.55}{q^2+\beta_2^2} + \frac{0.10}{q^2+\beta_3^2}\right) \quad (3)$$

$$\beta_1 = \frac{0.3Z^{1/3}}{0.885a_0}, \quad \beta_2 = 4\beta_1, \quad \beta_3 = 5\beta_2 \quad \text{and} \quad a_0 = 0.529\cdot 10^{-8} \text{ cm}$$

A more accurate representation of $V(q)$ can be achieved with the use of the self-consistent field method of Hartree-Fock [10, 11]. Calculations performed for interaction of $A_{2\pi}$ with various materials using these two methods [4] show that the uncertainty in the cross sections calculated with the T.F.M. parametrization is about 1% for the $A_{2\pi}$ ground state and slightly more for exited states.

For the $A_{2\pi}$ form factors $F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(q)$ there were employed exact analytic expressions discussed below.

Thus the transition and total cross sections for interaction of $A_{2\pi}$ with target atoms have been calculated with an uncertainty not greater than 7%. Using the more accurate methods mentioned above shall allow to calculate the cross sections within accuracy better than 0.5%.

In Table 1 some illustrative values of the interaction cross sections with Ta are given for nS states. Total cross sections increase with increasing n and consequently the size of the excited atoms. By interactions the exited $A_{2\pi}$ mainly transfers to any other exited states and predominantly to states with greater quantum numbers. The probability of excitation also increases with increasing n. So to obtain the probability of the $A_{2\pi}$

Table 1: Total cross sections ($\sigma^{tot}_{n\mathrm{S}}$) of $A_{2\pi}$ nS state interaction with Ta are shown for different principal quantum number n. The ratio $\sum_{n_2 \le 10} \sigma^{f}_{n\mathrm{S}}/\sigma^{tot}_{n\mathrm{S}}$ gives a probability of transfer from nS state to any other discrete state f with the principal quantum number $n_2 \le 10$. The ratio $\sum_{n \le n_2 \le 10} \sigma^{f}_{n\mathrm{S}}/\sigma^{tot}_{n\mathrm{S}}$ gives a probability of transfer to any discrete state f with the principal quantum number n_2 obeying $n \le n_2 \le 10$ that is the probability of excitation.

n	$\sigma^{tot}_{n\mathrm{S}}$, cm^2	$\sum_{n_2 \le 10} \sigma^{f}_{n\mathrm{S}}/\sigma^{tot}_{n\mathrm{S}}$	$\sum_{n \le n_2 \le 10} \sigma^{f}_{n\mathrm{S}}/\sigma^{tot}_{n\mathrm{S}}$
1	$3.468\cdot 10^{-20}$	0.619	0.619
2	$3.128\cdot 10^{-19}$	0.887	0.887
3	$1.038\cdot 10^{-18}$	0.940	0.935
4	$2.240\cdot 10^{-18}$	0.957	0.946
5	$3.812\cdot 10^{-18}$	0.960	0.944
6	$5.597\cdot 10^{-18}$	0.958	0.937
7	$7.448\cdot 10^{-18}$	0.952	0.926

breakup in the target one need to take into account an evolution of the atom state population during $A_{2\pi}$ passing through the target.

3. Form factors of hydrogenlike atoms

Transition form factors of hydrogenlike atoms are important for a wide range of applications. Explicit expressions of the form factors

$$F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(\vec{q}) = \int \psi^*_{n_2 l_2 m_2}(\vec{r}) e^{i\vec{q}\vec{r}} \psi_{n_1 l_1 m_1}(\vec{r}) \, d\vec{r} \quad (4)$$

for various initial $(n_1 l_1 m_1)$ and final states $(n_2 l_2 m_2)$ have been obtained in numerous papers (see for example [12, 13, 14, 15]). A general solution of the problem for arbitrary $(n_1 l_1 m_1)$ and $(n_2 l_2 m_2)$ has been found in a paper [16] using a group-theoretical method. However the complicated analytic structure of this result makes it unsuitable for practical usage because of long numerical calculations.

Therefore, another approach to calculation of the form factors (4) for arbitrary discrete-discrete transition is developed. Main formulas are given in this section and some details enter in Appendixes A and B.

After integration (4) over angular variables using standard relations of the angular momentum theory (see for example [17]) one has

$$F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(\vec{q}) = N \sum_{s=0}^{s_m} A_s I_s \,, \tag{5}$$

where

$$N = \frac{(2a)^{l_1+1}(2b)^{l_2+1}}{n_1+n_2}\left[\lambda_1\lambda_2 \frac{\Gamma(n_{r1}+1)}{\Gamma(n_{r1}+\lambda_1+1)}\frac{\Gamma(n_{r2}+1)}{\Gamma(n_{r2}+\lambda_2+1)}\right]^{1/2}, \tag{6}$$

$$a = \frac{n_2}{n_1+n_2}, \quad b = \frac{n_1}{n_1+n_2}, \quad a+b=1\,,$$

$$\lambda_1 = 2l_1+1\,, \quad \lambda_2 = 2l_2+1\,, \quad n_{r1} = n_1 - l_1 - 1\,, \quad n_{r2} = n_2 - l_2 - 1\,,$$

$$A_s = i^{l_1-l_2}\sqrt{\frac{4\pi}{2l+1}}\,\langle l_1, l_2, m_1, -m_2 | l, m\rangle\,\langle l_1, l_2, 0, 0 | l, 0\rangle\, Y_{lm}\left(\frac{\vec{q}}{|\vec{q}|}\right), \tag{7}$$

$$L = l_1 + l_2\,, \quad l = L - 2s\,, \quad m = m_1 - m_2\,, \quad s_m = \min(l_1, l_2)\,,$$

and

$$I_s = \int_0^\infty x^{L+2}\, j_{L-2s}(\Delta x)\, e^{-x} L^{\lambda_1}_{n_{r1}}(2ax)\, L^{\lambda_2}_{n_{r2}}(2bx)\, dx\,, \tag{8}$$

$$x = \frac{r\,(n_1+n_2)}{r_B\, n_1 n_2}, \quad \Delta = \frac{q\, r_B\, n_1 n_2}{n_1+n_2}\,.$$

Here r is a distance between particles in the atom, $r_B = 1/\alpha\mu$ is the Bohr radius of the atom (μ is the reduced atomic mass, α is the fine structure constant), j_l is the spherical Bessel function and L^α_n is the Laguerre polynomial.

The spherical Bessel function j_{L-2s} in (8) can be decomposed into a sum (see for example [18]):

$$j_{L-2s}(\Delta x) = \sum_{p=0}^{s} B_{ps} \left(\frac{2}{\Delta x}\right)^p j_{L-p}(\Delta x) \tag{9}$$

$$B_{ps} = (-1)^{s-p}\,\Gamma(p+1)\binom{s}{p}\binom{L-s+1/2}{p} \tag{10}$$

The product of the Laguerre polynomials in (8) can be expanded in a sum also:

$$L^{\lambda_1}_{n_{r1}}(2ax)\, L^{\lambda_2}_{n_{r2}}(2bx) = \sum_{k=0}^{k_m} H_k\, L^{\lambda_1+\lambda_2}_k(2x) \tag{11}$$

$$H_k = (-1)^{k+m}\binom{m+n}{m}\binom{m+n}{k}^{-1} Q^{(n_{r2}, n_{r1})}_k\, Q^{(n_{r1}+\lambda_1, n_{r2}+\lambda_2)}_{\overline{k}} \tag{12}$$

where

$$k_m = n_{r1} + n_{r2}\,, \quad \overline{k} = n_{r1} + n_{r2} - k\,, \quad Q^{(\mu,\nu)}_n = P^{(\mu-n,\,\nu-n)}_n(a-b)\,.$$

Here $P^{(\alpha,\beta)}_n(z)$ are the Jacobi polynomials. The derivation of (12) is given in the Appendix A. Functions $Q^{(\mu,\nu)}_n$ can be easy calculated using recurrence relations

$$(n+1)Q^{(\mu,\nu)}_{n+1} - [(\mu-n)b - (\nu-n)a]\,Q^{(\mu,\nu)}_n + (\mu+\nu-n+1)ab\,Q^{(\mu,\nu)}_{n-1} = 0\,, \tag{13}$$

where $a+b=1$ and starting values $Q^{(\mu,\nu)}_{-1} = 0$ and $Q^{(\mu,\nu)}_0 = 1$.

After substitution of (9) and (11) in (8) the calculation is reduced to the integral having the form:

$$I^{(L,p)}_k(\Delta) = \int_0^\infty x^{L-p+2}\, j_{L-p}(\Delta x)\, e^{-x} L^{2L+2}_k(2x)\, dx =$$

$$= 2\Gamma(L-p+2)\,(2\Delta)^{L-p}\, w^{L-p+2}\left(C^{(L+2,p)}_k(z) + C^{(L+2,p)}_{k-1}(z)\right). \tag{14}$$

Here $w = 1/(1+\Delta^2)$, $z = 1-2w$ and

$$C^{(\lambda,p)}_k(z) = \frac{\Gamma(k+2\lambda)}{\Gamma(k+1)\Gamma(2\lambda)}\,{}_3F_2\left(\begin{matrix}-k, k+2\lambda, \lambda-p\\ \lambda+1/2, \lambda\end{matrix}\middle|\,\frac{1-z}{2}\right)$$

is the generalized Gegenbauer polynomial which obey the following recurrence relations (see [19])

$$(k+2)C^{(\lambda,p)}_{k+2}(z) - [k+1+2p+2z(k+\lambda-p+1)]\,C^{(\lambda,p)}_{k+1}(z) + \tag{15}$$

$$+ [k+2\lambda-2p+2z(k+\lambda+p)]\,C^{(\lambda,p)}_k(z) - (k+2\lambda+1)C^{(\lambda,p)}_{k-1}(z) = 0\,,$$

with starting values $C^{(\lambda,p)}_{-2}(z) = C^{(\lambda,p)}_{-1}(z) = 0$ and $C^{(\lambda,p)}_0(z) = 1$. The derivation of (14) is given in the Appendix B.

Finally for the transition form factors one has

$$F^{n_2 l_2 m_2}_{n_1 l_1 m_1}(\vec{q}) = N \sum_{k=0}^{k_m} H_k \sum_{p=0}^{p_m} D_p\, \Delta^{L-2p}\, w^{L-p+2}\left(C^{(L+2,p)}_k(z) + C^{(L+2,p)}_{k-1}(z)\right), \tag{16}$$

where $p_m = s_m$ and

$$D_p = 2^{L+1}\Gamma(L-p+2)\sum_{s=p}^{s_m} B_{ps}\, A_s\,.$$

The relation (16) is valid for any choice of the quantization axis. For our case it is suitable to choose this axis along a momentum of the atom in the lab frame. Then the transfer momentum $\vec{q}$ is virtually normal to the quantization axis [5] and A_s in (7) becomes

$$A_s = \frac{2^{-l}\sqrt{\Gamma(l+m+1)\,\Gamma(l-m+1)}}{\Gamma\left(\frac{l+m}{2}+1\right)\,\Gamma\left(\frac{l-m}{2}+1\right)} \times \qquad (17)$$

$$\times \cos\left[\frac{\pi}{2}(l_1-m_1-l_2+m_2)\right] e^{im\phi_q}\,\langle l_1,l_2,m_1,-m_2|l,m\rangle\,\langle l_1,l_2,0,0|l,0\rangle\,,$$

where ϕ_q is the azimuth angle of the transfer momentum $\vec{q}$ in the selected frame.

It is worth to note that usage of the recurrence relations (13) and (15) is essential for rate and accuracy of calculations.

4. $A_{2\pi}$ passage through the target

Using the calculated transition (excitation and deexcitation) and the total cross sections one can describe the evolution of the atom state populations while passing through the target, taking also into account the $A_{2\pi}$ annihilation. Yield of $\pi^+\pi^-$ pairs from the $A_{2\pi}$ breakup inside the target (named further the probability of the $A_{2\pi}$ breakup) is calculated basing on these populations.

The quantization axis along the atom momentum, used for the form factors calculation, conserves for all subsequent collisions as the atom momentum is much greater than the transfer one. This allows to describe the evolution of the atom state population in a simple way.

With the chosen quantization axis only atomic transitions that conserve Z-parity $P^Z = (-1)^{l-m}$ are permitted [8] (l and m are the orbital and magnetic quantum numbers). The $A_{2\pi}$ atoms are produced in nS states, so they have positive Z-parity. Thus only states with positive Z-parity have nonzero populations and only these states are considered further.

Population of discrete states is described by a set of differential kinetic equations:

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{\infty} a_{ij}p_j(s)\,. \qquad (18)$$

Here $p_i(s)$ is the probability of $A_{2\pi}$ to be in the state i after travelling a distance s in the matter, a_{ij} is the matrix of inverse lengths of the transition from the state j to the state i. For $i \neq j$ a_{ij} is written

$$a_{ij} = \frac{\sigma_j^i \rho N_0}{A}\,. \qquad (19)$$

Here ρ and A are the density and atomic weight of the target, respectively, N_0 is the Avogadro number. For $i = j$ a_{ii} describes the total decrease of the population of the state i:

$$a_{ii} = -\frac{\sigma_i^{tot}\rho N_0}{A} - \begin{cases} M_A/p_A c\tau_n & \text{for } n\text{S states} \\ 0 & \text{otherwise} \end{cases} \qquad (20)$$

Here the first term is related to the population decrease due to interactions with the target and the second one is related to annihilation (decay). (Elastic scattering is forbidden in the first Born approximation.) M_A and p_A are atom mass and momentum correspondingly. $\tau_n = \tau n^3$ is the lifetime of $A_{2\pi}$ nS states. A decrease of population of any other states due to annihilation has been neglected as well as a decrease of population of all states due to radiation transition because the lifetime for these processes is much greater than the time of flight through the target [1].

The initial condition for the set of equations (18) is given by the probability of $A_{2\pi}$ production with different quantum numbers which proportional to n^{-3} for nS states and zero for all other [1]. The lifetime and momentum of $A_{2\pi}$ are parameters of equations (18). Its solution is reduced to calculation of eigenvalues and eigenvectors of matrix a_{ij} and then to decomposition of initial conditions over the set of eigenvectors.

Since $A_{2\pi}$ can get excited or deexcited at the interaction, the matrix a_{ij} does not show a triangular form like it is in a case of a chain of radiation decays which is described by the analogous set of equations. Therefore an exact solution for any state may only be obtained as a solution of the infinite set of equations. Hence one has to limit the set of equations including only states with the principal quantum number $n \leq n_{\max}$.

The transition cross sections obey the equality $\sigma_i^f = \sigma_f^i$ and so the matrix a_{ij} is symmetric. The cross sections depend on the magnetic quantum numbers of initial and final states m_1 and m_2 as $|m_2 - m_1|$. Moreover,

the initial condition for the equations includes the fact that only nS states (with $m = 0$) are not empty. It can be concluded that states with the opposite sign magnetic quantum numbers have equal populations. This allows to exclude the states with negative magnetic quantum numbers from the equation set and thus to reduce a dimension of the set. For $n_{\max} = 7$ the dimension of the matrix a_{ij} reduces from 84 to 50. However the resulting matrix a_{ij} becomes asymmetric.

The solution of equations (18) has been found for all states with $n \leq n_{\max} = 7$. The uncertainty in solution of equation (18) caused by this limit has an influence on states only with n close to $n_{\max}$, as atoms transfer mainly to states with the nearest quantum number.

Thus populations of all states with $n \leq n_{\max}$ as a function of the path s in the target have been calculated. This function takes into account excitation and deexcitation by interactions with target atoms and $A_{2\pi}$ annihilation. Points of $A_{2\pi}$ production are distributed uniformly over the target thickness. So one has to use the average values over this distribution for all discussed probabilities.

From calculations one can conclude that the sum of population of discrete atomic states P_{dsc} with $n \leq n_{\max}$ is known with precision much better than 1% and the summed population P_n of atomic states with fixed n has an asymptotical behavior $an^{-3} + bn^{-5} + \ldots$ at high n. Using this fact, estimation has been obtained for the sum of populations of all other atomic states P_{tail} those were not included in the equation set ("tail" with $n > 7$). An uncertainty in this estimation (about 20%) has little influence on the final results because of the smallness of the "tail" (see (22)).

Atoms annihilate mainly from 1S state [1]. The population of the first few states is known with high accuracy so the probability of $A_{2\pi}$ annihilation P_{anh} is calculated with the same accuracy.

Thus one has calculated the probability of $A_{2\pi}$ stays in discrete state or annihilates while passing through the target. The remainder is the probability of the $A_{2\pi}$ breakup P_{br}:

$$P_{\text{br}} = 1 - P_{\text{dsc}} - P_{\text{tail}} - P_{\text{anh}}. \qquad (21)$$

The following values of probabilities were obtained with $\tau = 3.7 \cdot 10^{-15}$ s, the average momentum of $A_{2\pi}$ in the experiment $\langle p_A \rangle = 2.9$ GeV/c and the target thickness of 8 μm Ta,

$$P_{\text{dsc}} = 0.421, \quad P_{\text{tail}} = 0.006, \quad P_{\text{anh}} = 0.159 \quad \text{and} \quad P_{\text{br}} = 0.414 \qquad (22)$$

To illustrate the validity of the limit $n_{\max}$ used in these calculations, the value of P_{br} obtained with $n_{\max} = 5$ differs from the above one obtained with $n_{\max} = 7$ by 0.36%. From an analysis of the "tail" one can conclude that the accuracy the P_{br} calculation procedure is estimated not worse than 0.5%. This is much better than that of the precision of the cross sections. Thus the accuracy of P_{br} is limited by the precision of the cross section calculations and is estimated at 7%.

The approach discussed here neglects two effects. The first one is the formation time of atom states. This time is small in comparison with the time between subsequent collisions. Therefore, the formation time has been neglected. This assumption is valid for levels with principal quantum numbers $n \leq 7$. Secondly one neglects possible interference effects between atom states with equal n and m and with orbital quantum numbers l that differ by $\Delta l = 2$. This interference occurs due to the Coulomb degeneration of atom states. Its influence cannot be significant because the interference is permitted only for states with $n \geq 3$ having small populations. These two effects could be considered if one describes the evolution of atom states by quantum mechanical equations in terms of density matrix elements.

The $A_{2\pi}$ breakup probability has been calculated as a function of various parameters to choose conditions for observation of the atoms in the experiment [2]. Some relations are shown in Fig. 1.

In Fig. 1c the $A_{2\pi}$ breakup probability averaged over the spectrum of atoms, observed in the experiment [2], is shown as a function of the lifetime. Using this function and the measured value of $P_{\text{br}} = 0.40 \pm 0.09$ [2] one obtains estimation of the $A_{2\pi}$ lifetime in the ground state

$$\tau = (2.9^{+\infty}_{-2.1}) \cdot 10^{-15}\,\text{s} \qquad (23)$$

or limitation for the lifetime at 90% confidence level

$$\tau > 0.6 \cdot 10^{-15}\text{s}. \qquad (24)$$

This result agrees with the theoretical value of $\tau = (3.7 \pm 0.3) \cdot 10^{-15}$ s predicted by the Chiral Perturbation Theory.

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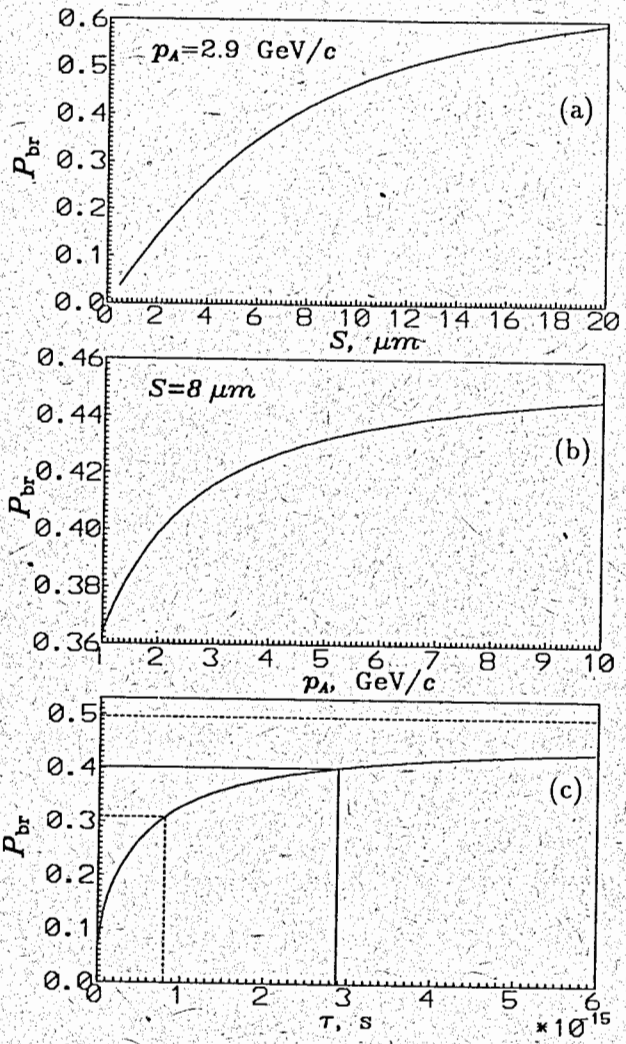

Figure 1: Probability of the $A_{2\pi}$ breakup in the target as a function: (a) of the target thickness for average momentum of $A_{2\pi}$ $\langle p_A \rangle = 2.9$ GeV/c; (b) of the atom momentum for the target thickness $S = 8$ μm; (c) of the $A_{2\pi}$ lifetime, averaged over the spectrum of observed atoms.

Appendix A

Coefficients H_k of the expansion

$$L_n^\alpha(ax)\, L_m^\beta(bx) = \sum_{k=0}^{n+m} H_k\, L_k^\gamma(x)\,; \tag{25}$$

for a and $b > 0$, are given by expression

$$H_k = \frac{\Gamma(k+1)}{\Gamma(k+\gamma+1)} \int_0^\infty x^\gamma\, e^{-x}\, L_n^\alpha(ax)\, L_m^\beta(bx)\, L_k^\gamma(x)\, dx\,. \tag{26}$$

The Laguerre polynomials can be written as

$$\begin{aligned} L_n^\alpha(ax) &= \hat{D}_t^{(n)} \left[(1+t)^{\alpha+n}\, e^{atx}\right] \\ L_m^\beta(bx) &= \hat{D}_u^{(m)} \left[(1+u)^{\beta+m}\, e^{bux}\right] \end{aligned} \tag{27}$$

where

$$\hat{D}_z^{(s)} f(z) = \frac{1}{\Gamma(s+1)} \left.\frac{d^s f(z)}{dz^s}\right|_{z=0}$$

and

$$L_k^\gamma(x) = \frac{x^{-\gamma}\, e^x}{\Gamma(k+1)} \frac{d^k}{dx^k} \left(x^{\gamma+k}\, e^{-x}\right).$$

After not complicated integration over x and differentiation with respect to parameters t and u one has

$$\begin{aligned} H_k &= \hat{D}_t^{(n)} \hat{D}_u^{(m)} \left[\frac{(at+bu)^k (1+t)^{\alpha+n} (1+u)^{\beta+m}}{(1+at+bu)^{k+\gamma+1}}\right] = \\ &= \sum_p \binom{k}{p} \binom{\rho+\overline{n}-1}{\overline{n}} \binom{\sigma+\overline{m}-1}{\overline{m}} a^p b^{k-p} F_2(\omega, -\overline{n}, -\overline{m};\, \rho, \sigma;\, a, b)\,, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \overline{n} &= n-p\,,\ \ \overline{m} = m+p-k\,,\ \ \rho = \alpha+p+1\,, \\ \sigma &= \beta+k-p+1\,,\ \ \omega = k+\gamma+1\,. \end{aligned} \tag{29}$$

Here F_2 is the Appell function [20]

$$F_2(\omega, -\overline{n}, -\overline{m};\, \rho, \sigma;\, a, b) = \sum_{ij} \frac{(\omega)_{i+j}\, (-\overline{n})_i\, (-\overline{m})_j\, a^i b^j}{(\rho)_i\, (\sigma)_j\, \Gamma(i+1)\, \Gamma(j+1)}\,. \tag{30}$$

If parameters a, b, α, β and γ obey relations

$$a + b = 1 \quad \text{and} \quad \gamma = \alpha + \beta, \tag{31}$$

then the analytic structure of (28) becomes much-simplified. Using a transformation formula for the functions F_2 [20] and assuming that

$$a < b \quad \text{and} \quad a + b = 1$$

one has

$$\begin{aligned} & F_2(\omega, -\overline{n}, -\overline{m};\, \rho, \sigma;\, a, b) = \\ & = (1-a)^{-\omega} F_2\left(\omega, \rho - \overline{n}, -\overline{m};\, \rho, \sigma;\, \frac{a}{a-1}, \frac{b}{1-a}\right) = \\ & = b^{-\omega} F_2\left(\omega, \rho - \overline{n}, -\overline{m};\, \rho, \sigma;\, -\frac{a}{b}, 1\right) = \\ & = (-1)^{-\overline{m}} b^{-\omega} \frac{\Gamma(\sigma)\Gamma(\omega - \sigma + 1)}{\Gamma(\sigma + \overline{m})\Gamma(\omega - \sigma - \overline{m} + 1)}\, {}_3F_2\left(\begin{matrix}\omega, \rho + \overline{n}, 1 + \omega - \sigma \\ 1 + \omega - \sigma - \overline{m}, \rho\end{matrix}\middle| -\frac{a}{b}\right). \end{aligned} \tag{32}$$

The relation $\gamma = \alpha + \beta$ leads to a change in the arguments of the function ${}_3F_2$: $1 + \omega - \sigma = \rho$. So the following transformations can be done.

$$\begin{aligned} & {}_3F_2\left(\begin{matrix}\omega, \rho + \overline{n}, 1 + \omega - \sigma \\ 1 + \omega - \sigma - \overline{m}, \rho\end{matrix}\middle| -\frac{a}{b}\right) = {}_3F_2\left(\begin{matrix}\omega, \rho + \overline{n}, \rho \\ \rho - \overline{m}, \rho\end{matrix}\middle| -\frac{a}{b}\right) \equiv \\ & \equiv {}_2F_1(\omega, \rho + \overline{n}; \rho - \overline{m}; -a/b) = \\ & = b^{\omega + \overline{n} + \overline{m}}\, {}_2F_1(\overline{m} - \overline{n}, \rho - \overline{m} - \omega; \rho - \overline{m}; -a/b) \end{aligned} \tag{33}$$

Substituting (29) one can see that no arguments of the function ${}_2F_1$ in (33) depend on the sum index p in (28). It allows to sum (28) over p and leads to the final result for the expansion coefficients H_k

$$H_k = (-1)^{k+m} \binom{m+n}{n} \binom{m+n}{k}^{-1} P_k^{(m-k,\, n-k)}(b-a)\, P_{\overline{k}}^{(m+\alpha-\overline{k},\, n+\beta-\overline{k})}(b-a). \tag{34}$$

Here $\overline{k} = m + n - k$ and $P_s^{(\mu,\nu)}$ is the Jacobi polynomial.

Appendix B

To calculate the integral

$$I_k^{(L,p)} = \int_0^\infty x^{L-p+2}\, j_{L-p}(\Delta x)\, e^{-x} L_k^{2L+2}(2x)\, dx \tag{35}$$

it is convenient to use the representation of the Laguerre polynomials in the form

$$L_k^{2L+2}(2x) = \hat{D}_t^{(k)} \left[(1-t)^{-2L-3} e^{\frac{-2x}{(1-t)}}\right], \tag{36}$$

and the tabular integral [20]

$$\int_0^\infty x^{\mu+2}\, j_\mu(\Delta x)\, e^{-\alpha x}\, dx = \frac{2\alpha(2\Delta)^\mu \Gamma(\mu+2)}{(\Delta^2 + \alpha^2)^{\mu+2}}, \tag{37}$$

here

$$\mu = L - p \quad \text{and} \quad \alpha = \frac{1+t}{1-t}.$$

Introducing variables

$$w = \frac{1}{1+\Delta^2}, \quad u = 1 - \alpha^2 = -\frac{4t}{(1-t)^2} \quad \text{and} \quad z = 1 - 2w \tag{38}$$

one obtains

$$I_k^{(L,p)}(\Delta) = 2\Gamma(L-p+2)(2\Delta)^{L-p} w^{L-p+2} \left(C_k^{(L+2,p)}(z) + C_{k-1}^{(L+2,p)}(z)\right). \tag{39}$$

Here $C_k^{(L+2,p)}(z)$ is defined as

$$C_k^{(\lambda,p)}(z) = \hat{D}_t^{(k)} \left[\frac{1}{(1-t)^{2\lambda}(1-uw)^{\lambda-p}}\right], \tag{40}$$

where $\lambda = L + 2$.

Using the identity

$$\frac{1}{(1-uw)^{\lambda-p}} \equiv \frac{1}{B(\lambda-p, p)} \int_0^1 \frac{v^{\lambda-p-1}(1-p)^{p-1}\, dv}{(1-uvw)^\lambda}, \tag{41}$$

the definition of the Gegenbauer polynomial for $\overline{z} = 1 - 2vw$

$$\hat{D}_t^{(k)}\left[\frac{1}{(1-2t\overline{z}+t^2)^{\lambda}}\right] = C_k^{\lambda}(\overline{z}) = \\ = \frac{\Gamma(k+2\lambda)}{\Gamma(k+1)\,\Gamma(2\lambda)}\,{}_2F_1\left(-k, k+2\lambda; \lambda+\frac{1}{2}; \frac{1-\overline{z}}{2}\right) \qquad (42)$$

and the representation of the ${}_3F_2$ function in the form [19]

$$\frac{1}{B(\mu,\,\nu-\mu)}\int_0^1 v^{\mu-1}(1-v)^{\nu-\mu-1}\,{}_2F_1(\alpha,\beta;\gamma;vw)\,dv = {}_3F_2\left(\begin{matrix}\alpha,\beta,\mu\\ \gamma,\nu\end{matrix}\middle| w\right), \qquad (43)$$

finally one has for $C_k^{(\lambda,p)}$ in (40)

$$C_k^{(\lambda,p)}(z) = \frac{\Gamma(k+2\lambda)}{\Gamma(k+1)\Gamma(2\lambda)}\,{}_3F_2\left(\begin{matrix}-k, k+2\lambda, \lambda-p\\ \lambda+1/2, \lambda\end{matrix}\middle| \frac{1-z}{2}\right) \qquad (44)$$

The polynomials $C_k^{(\lambda,p)}$ obey the recurrence relations (15).

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