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CAN PARTICLE-LIKE BREATHER STRUCTURES OCCUR IN SUPERFLUID FILMS?

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Барашенков И.В. E4-95-314
Могут ли частицеподобные дышащие структуры возникать в сверхтекучих пленках?

В рамках уравнения Гросса—Питаевского теории сверхтекучести (нелинейного уравнения Шредингера с отталкиванием на плоскости) исследуется возможность существования долгоживущих дышащих структур, подобных пульсонам уравнения Кляйна—Гордона.

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Barashenkov I.V. E4-95-314
Can Particle-Like Breather Structures Occur in Superfluid Films?

We investigate the possibility that the Gross—Pitayevski equation of superfluidity (alias repulsive nonlinear Schrödinger equation on the plane) admits long-lived breather structures similar to the Klein—Gordon pulsons.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1. Recently there has been an upsurge of interest in long-lived breather-like solitons in multidimensions. It has been argued that these structures may play an important rôle in the dynamics of first and second order phase transitions, in particular in cosmological models [1]- [3]. Multidimensional pulsating structures have been known since mid-seventies; they were first observed [4] within the Klein-Gordon equation with the "ϕ^4" nonlinearity:

$$\phi_{tt} - \Delta\phi + \lambda\phi(\phi^2 - m^2/\lambda) = 0, \tag{1}$$

with $\phi \underset{r\to\infty}{\longrightarrow} m/\sqrt{\lambda}$. A series of more precise and detailed simulations then revealed the extreme longevity of these particle-like solutions which were subsequently coined "pulsons" [5]. Since then pulsons have been discovered within Klein-Gordon equations with other nonlinearities and a great depth of understanding of their properties has become available [6].

Although there is an extensive literature on the pulsons of the Klein-Gordon equation, no work has been done so far on its nonrelativistic counterpart, the Gross-Pitayevski (alias repulsive nonlinear Schrödinger) equation:

$$i\hbar\Psi_t + \frac{\hbar^2}{2m}\Delta\Psi + \mu\Psi - \lambda|\Psi|^2\Psi = 0. \tag{2}$$

Meanwhile, this equation also has a wide range of important applications, and in the first place the Bose condensate, where pulsons (if exist) could be identified with certain physical excitations.

It is the aim of this note to explore the existence of breathing solitons in the repulsive NLS equation (2). Here we consider the two-dimensional case ($\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$), where the underlying physical problem is a thin superfluid film. The 3D case will be addressed in future publications.

Before we proceed to the analysis, it is appropriate to mention that the existence of pulsons within eq.(1) does not yet imply that eq.(2) should necessarily exhibit similar structures. One may of course argue that the nonlinear Schrödinger is the nonrelativistic limit of the Klein-Gordon equation. However, this transition uses the assumption that solutions of (1) are small in amplitude, the requirement which is clearly not met in the case of the pulson boundary conditions, $\phi(r,t) \longrightarrow m/\sqrt{\lambda}$ as $r \to \infty$.

2. Rescaling the coordinates and field,

$$\Psi = \sqrt{\frac{\mu}{\lambda}}\psi, \quad t = \frac{\mu}{\hbar}\tilde{t}, \quad \mathbf{r} = \frac{\hbar}{\sqrt{2m\mu}}\tilde{\mathbf{r}}, \tag{3}$$

eq. (2) is brought into the dimensionless form

$$i\psi_t + \Delta\psi + (1 - |\psi|^2)\psi = 0, \tag{4}$$

where we have omitted tildes. The condensate boundary condition is now $|\psi|^2 \underset{r\to\infty}{\longrightarrow} 1$. The simplest possible approach is as follows. Take a trial function depending on some set of parameters; assume that the dependence on r is known and parameters are functions of time; substitute into the action functional of eq.(4),

$$S=\int_0^T\int dtd^2r\left\{\frac{i}{2}(\psi_t\overline{\psi}-\overline{\psi}_t\psi)-|\nabla\psi|^2-\frac{1}{2}(|\psi|^2-1)^2\right\}; \quad (5)$$

integrate off the spatial dependence and arrive at a finite dimensional dynamical system. This approach is known as the method of collective coordinates, or as the variation-of-action method. (A good exposition is in [7].) Although providing little information on how close the collective coordinate solution is to the real solution, this approach has several obvious advantages. Firstly, it ensures that the resulting finite dimensional evolution is automatically energy-conserving. Secondly, if its results agree with direct numerical simulations, one has an idea of what nonlinear collective modes are responsible for the observed behaviour.

We shall confine ourselves to the case of rotationally-symmetric fields, $\psi(\mathbf{r},t)=\psi(r,t)$. To start with, we mention that one cannot simply borrow the trial function of the relativistic problem [3]: $\psi(r,t)=\varphi(r/a)$, with $\varphi(r)$ given function and $a=a(t)$. Such a simple choice would not do as eq. (5) is linear in ψ_t and $\overline{\psi}_t$, and so the velocity $\dot{a}$ would enter the resulting effective Lagrangian linearly, as $\dot{a}/a^2$. Since this total derivative can be simply discarded, there will be no dynamics. We have therefore to include a variable phase.

Thus our second guess will be to write $\psi=\rho^{1/2}e^{i\theta}$ with

$$\rho=\rho\left(\frac{r}{a(t)}\right), \qquad \theta=\mu(t)f\left(\frac{r}{a(t)}\right) \quad (6)$$

where $f(r)$ and $\rho(r)$ are some localised functions satisfying $\rho_r(0)=f_r(0)=0$, and $\rho\to1$, $f\to0$ as $r\to\infty$. Substituting into (5) and integrating over r, we arrive at a (linear!) dynamical system with the Lagrangian

$$L=-(\dot{\mu}xI_1+\mu^2I_2+xI_3+I_0), \quad (7)$$

where

$$I_0=\int\frac{\rho_\xi^2}{\rho}\xi d\xi, \qquad I_1=\int\left(f(\xi)+\frac{\xi}{2}\frac{df}{d\xi}\right)\rho(\xi)\xi d\xi,$$

$$I_2=\int f_\xi^2\rho(\xi)\xi d\xi, \qquad I_3=\frac{1}{2}\int(\rho(\xi)-1)^2\xi d\xi.$$

The Euler-Lagrange equations

$$I_1\dot{x}=2I_2\mu \quad (8)$$

$$I_1\dot{\mu}+I_3=0 \quad (9)$$

conserve energy,

$$E=I_2\mu^2+I_3x+I_0. \quad (10)$$

Substituting $(I_1/2I_2)\dot{x}$ for μ in eq. (10), we get

$$E=\frac{m}{2}\dot{x}^2+I_3x+I_0 \quad (11)$$

where $m=I_1^2/2I_2>0$. Consequently we have a motion of a classical particle in a linearly growing potential. This motion is periodic: the particle brakes smoothly at the point $x_{\max}=(E-I_0)/I_3$, turns back, bounces off the origin with a finite speed $|\dot{x}|=[4I_2(E-I_0)/I_1^2]^{1/2}$, reaches the turning point $x_{\max}$ again and so on.

The above collective coordinate description furnishes a characteristic size of the pulson, $a_{\max}=\sqrt{x_{\max}}$, and its period:

$$a_{\max}=\sqrt{\frac{E-I_0}{I_3}}, \quad T=\frac{2I_1}{I_3}\sqrt{\frac{E-I_0}{I_2}}. \quad (12)$$

If a pulsating structure is observed in direct numerical simulations of the NLS eq. (4), one will be able to compute the energy of this configuration by formula

$$E=\int\left\{|\nabla\psi|^2+\frac{1}{2}(|\psi|^2-1)^2\right\}rdr, \quad (13)$$

calculate $I_0, ..., I_3$ with some plausible guess for ρ and f, and eventually evaluate the size and period of the pulson by eqs. (12). A good comparison with the numerically observed size and period would imply that the ansatz (6) captures the essentials of dynamics.

3. How far is the dynamics described by this two-dimensional system from the actual behaviour of the pulson in the partial differential equation (4) (if it exists)? A natural test is to check whether the above collective motions conserve the integrals of the nonlinear Schrödinger equation. Apart from the energy eq.(13) the NLS evolutions conserve the number of particles,

$$N=\int(|\psi|^2-1)rdr \quad (14)$$

and the radial component of the linear momentum:

$$P=\frac{i}{2}\int(\overline{\psi}\psi_r-\overline{\psi}_r\psi)rdr. \quad (15)$$

It is straightforward to verify, however, that the ansatz (6) conserves neither N nor P. Indeed, substituting (6) into (14) and (15), one gets $N=a^2(t)\int(\rho-1)\xi d\xi$ and $P=a(t)\mu(t)\int\rho f_\xi\xi d\xi$.

Can we design a trial function which would conserve integrals? We may try to do it by allowing more degrees of freedom in the finite dimensional dynamics. Take, for instance,

$$\rho(r,t)=1-b(t)g\left(\frac{r}{a(t)}\right), \qquad \theta(r,t)=\mu(t)f\left(\frac{r}{a(t)}\right), \quad (16)$$

where $g(\xi)$ and $f(\xi)$ are decaying functions of ξ, with $g'(0) = f'(0) = 0$, and $g(\infty) = f(\infty) = 0$. The corresponding Lagrangian is

$$L = -\dot{\mu}a^2(J_1 - bJ_5) + 2a\dot{a}\mu bJ_6 - a^2b^2J_3 - b^2J_0(b) - \mu^2(J_4 - bJ_2), \tag{17}$$

where

$$J_1 = \int \left(f(\xi) + \frac{\xi}{2}\frac{df}{d\xi}\right)\xi d\xi, \qquad 2J_3 = \int g^2 \xi d\xi,$$

$$2J_6 = -\int f_\xi g \xi^2 d\xi > 0, \qquad 4J_0(b) = \int \frac{g_\xi^2}{1 - bg(\xi)}\xi d\xi,$$

$$J_4 = \int f_\xi^2 \xi d\xi, \qquad J_2 = \int f_\xi^2 g \xi d\xi, \qquad J_5 = \int f g \xi d\xi.$$

Now let us impose the condition that the collective evolutions conserve the number of particles, eq. (14). Substituting (16) into (14) we see that $N = const$ if $a^2 b = const$. Without loss of generality we may identify

$$b = \frac{1}{a^2}$$

and the Lagrangian (17) becomes

$$L = -\dot{\mu}xJ_1 + \dot{x}x^{-1}\mu J_6 - x^{-1}J_3 - x^{-2}J_0(x^{-1}) - \mu^2(J_4 - x^{-1}J_2), \tag{18}$$

where we have introduced $x = a^2$.

The Lagrangian (18) is linear in velocities and so the conserved energy does not depend on $\dot{x}$ or $\dot{\mu}$:

$$E = \mu^2(J_4 - x^{-1}J_2) + V(x) \tag{19}$$

where

$$V(x) = J_0(x^{-1})/x^2 + J_3/x. \tag{20}$$

One of the two equations of motion,

$$\dot{x}(J_1 + J_6/x) = 2\mu(J_4 - J_2/x)$$

can be used to eliminate μ from the Hamiltonian (19) which becomes a function of x and $\dot{x}$ only:

$$E = \frac{m(x)}{2}\dot{x}^2 + V(x). \tag{21}$$

Here m is the variable mass of fictitous particle which moves in the potential $V(x)$:

$$m(x) = \frac{1}{2}\left(\frac{xJ_1 + J_6}{xJ_4 - J_2}\right)^2 (J_4 - \frac{J_2}{x}). \tag{22}$$

Notice that the fact that $\rho = 1 - g(r/a)/a^2$ is positive, implies that a cannot take values smaller than $a_{\min}$: $a^2 > a_{\min}^2 = g(0)$. Accordingly,

$$xJ_4 - J_2 > \int [g(0) - g(\xi)] f_\xi^2 \xi d\xi > 0$$

and the denominator in (22) can never be zero. The mass is bounded and nowhere vanishing; more precisely,

$$0 < \frac{1}{2}\frac{J_1^2}{J_4} < m < \frac{1}{2g(0)}\frac{[J_6 + J_1 g(0)]^2}{J_4 g(0) - J_2}.$$

The potential $V(x)$ is equal to plus infinity for $x < x_{\min} = g(0)$, takes certain finite value $V_0 > 0$ at $x = x_{\min}$ and then monotonously decreases to zero as $x \to \infty$. There can be of course no periodic motions in such a potential. If the particle with energy $E < V_0$ approaches the origin from large values of x, it will brake smoothly at the point x where $E = V(x)$, then reverse and escape to infinity. If $E > V_0$, the particle will bounce off the infinitely high wall at $x = x_{\min}$ and also escape to infinity. The motion is unbounded.

4. Thus it may seem that the requirement of the number of particle conservation prohibits the existence of pulson solutions. We will show, however, that this prohibition, which would be fatal in the case of bell-like solitons, can be easily circumvented in the case of solutions with nonvanishing boundary conditions.

Consider a configuration $\psi_0(r,t)$ which does not conserve the integral (14):

$$\int \{|\psi_0(r,t)|^2 - 1\} r dr = N_0(t) = \tilde{N}_0 + n_0(t)$$

where we have decomposed N_0 into a constant and variable parts. Suppose we add to ψ_0 a "small" function $\delta\psi(r,t)$ such that $\int |\delta\psi|^2 r dr = \epsilon^2 \ll 1$. The energy and action functionals, eqs.(13) and (5) will change by $O(\epsilon^2)$ whereas the integral N will become

$$N = \int \{|\psi_0 + \delta\psi|^2 - 1\} r dr = \tilde{N}_0 + n_0(t) + N_1(t) + \epsilon^2$$

where

$$N_1 = \int (\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) r dr.$$

Can one choose $\delta\psi$ in such a way that the variation of $N_0(t)$ is compensated by $N_1(t)$ and so $N(t) = \tilde{N}_0$, a constant? In the case of fields vanishing at infinity, $\psi_0 \underset{r\to\infty}{\longrightarrow} 0$, this would contradict the continuity of the functional $N_1[\delta\psi]$. Indeed, by the Schwartz inequality,

$$\left|\int (\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) r dr\right| \leq 2\epsilon \left(\int |\psi_0|^2 r dr\right)^{1/2} \tag{23}$$

and so if $\delta\psi$ is small, N_1 will also be small and insufficient to compensate changes in $N_0(t)$. However if $|\psi_0| \to 1$ as $r \to \infty$, the integral $\int |\psi_0|^2 r dr$ diverges, the Schwartz inequality is inapplicable and

N_1 need not to be small anymore. In fact, it can be rigorously proved that $N_1 = N_1[\delta\psi, \delta\overline{\psi}]$ is a *discontinuous* functional in this case [8] and so it may take arbitrary large values even if $||\delta\psi||$ is small. In particular the "small" function $\delta\psi$ can always be chosen in such a way that $N_1[\delta\psi, \delta\overline{\psi}] = -n_0(t)$ and therefore the number of particles in the configuration $\psi_0(r,t) + \delta\psi(r,t)$ is constant, $N = \tilde{N}_0$.

This can be understood in very simple terms. If $\delta\psi$ decays slowly enough as $r \to \infty$, the integral N_1 does not have to be small even if $\int |\delta\psi|^2 dr$ is small. For example if $\delta\psi \sim \epsilon/r^2$, $\int |\delta\psi|^2 dr$ is $\sim \epsilon^2$ whereas $\int(\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) dr$ diverges. Thus we can always "correct" the field configuration [for instance, the one in eq.(6)] by adding a "long tail of small amplitude" such that on one hand, the action of the resulting configuration will be as close to $S[\psi_0]$ as desired, and on the other hand, $\psi_0 + \delta\psi$ will conserve the number of particles. This implies that the fact that our pulson configuration eq.(6) does not conserve the number of particles, does not mean that this cofiguration is far from the actual solution of the Gross-Pitayevski equation (4).

In a similar way we can ensure that the momentum eq. (15) is conserved. Adding $\delta\psi$ to ψ_0, the momentum $P_0(t) = (i/2)\int(\overline{\psi_0}\psi_0' - \psi_0\overline{\psi_0'})dr$ receives an increment

$$P_1(t) = i\int\left(\frac{\partial\psi_0}{\partial r}\delta\overline{\psi} - c.c.\right) r dr + \frac{i}{2}\int(\psi_0 \delta\overline{\psi} - \overline{\psi}_0 \delta\psi) dr.$$

The second integral in the right-hand side is a discontinuous functional which can take arbitrarily large values.

5. Thus the method of collective coordinates does predict the existence of the breather-like ("pulson") solution to the Gross-Pitayevski equation. The pulson can be roughly described as a superposition of two nonlinear modes. One is the configuration (6) with the localisation scale $\sim a_{\max}$. Superimposed over this configuration is a small amplitude "tail" whose characteristic wavelength is much larger than $a_{\max}$. This "tail" appears to be an important component of the pulson; if we attempt to approximate the pulson by a single-scale configuration, like e.g. eq.(16), the resulting evolution turns out to be nonperiodic.

An important argument in favour of the validity of the ansatz (6) is its remarkable structural stability; the periodic motions in the system (7) occur not for a certain specific choice of $\rho(r)$ and $f(r)$ but for a wide class of these functions. We are nevertheless fully aware that the method of collective coordinates is not invulnerable to criticism. The ultimate answer to whether long-lived pulsating structures occur in the Gross-Pitayevski equation can only be given by the direct numerical simulation. This work is in progress.

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