ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E4-95-313

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CHERN-SIMONS VORTICES IN THE CONDENSATE OF NONRELATIVISTIC BOSONS COUPLED TO A UNIFORM BACKGROUND ELECTRIC CHARGE DENSITY

Submitted to «Physical Review D»

*Introdel SA, P.O.Box 31266, Braamfontein 2012, South Africa

1995

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Вихри Черна—Саймонса в конденсате нерелятивистских бозонов, взаимодействующих с внешним электрическим зарядом постоянной плотности

Известно, что модель Джакива—Пи для самогравитирующего газа нерелятивистских бозонов, взаимодействующих с полем Черна—Саймонса, обладает солитонными решениями, убывающими на бесконечности. В настоящей работе показано, что для того, чтобы рассмотреть случай газа отталкивающихся частиц (когда поле материи стремится к ненулевым значениям на бесконечности), следует добавить, например, внешний электрический заряд. Переформулируя возникающую таким образом модель как гамильтонову систему со связями, можно найти самодуальный предел в чисто Черн—Саймоновском и в смешанном Максвелл—Черн—Саймоновском случаях. Показано, что импульс топологически нетривиальных конфигураций может быть определен лишь как трансляционно-неинвариантная величина, и алгебра спонтанно нарушена: $\{P_x, P_y\} = 2\pi\rho_0 n$.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1995

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Chern—Simons Vortices in the Condensate of Nonrelativistic Bosons Coupled to a Uniform Background Electric Charge Density

Jackiw—Pi's model of the self-gravitating gas of nonrelativistic bosons coupled to the Chern—Simons gauge field is known to exhibit asymptotically vanishing, lump-like soliton solutions. We show that in order to extend this model to include the case of repulsive gases where the matter field approaches nonzero values at infinities, one has to add, for instance, the background electric charge. Reformulating the model arising in this way as a constrained Hamiltonian system allows to find the self-duality limit in the pure Chern—Simons and in the mixed Chern—Simons—Maxwell cases. We prove that the linear momentum of the topologically nontrivial configurations can only be defined as a translationally noninvariant quantity and the algebra is spontaneously broken: $\{P_x, P_y\} = 2\pi\rho_0 n$.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1995

I. INTRODUCTION

Vortices, topologically nontrivial localised structures, lie at the heart of all theories of particles with fractional statistics. It is these collective excitations of the field quanta that are considered as candidates for anyonic objects in quasi-planar condensed matter physics. More precisely, in the case of charged matter interacting with a Maxwell field, the anyon is a bound state of an (electrically neutral) vortex and a field quantum, a "flux" and a "charge". If the gauge field is of the Chern-Simons type, the vortex is no more electrically neutral and behaves as an anyon itself [1].

Both Maxwell and Chern-Simons vortices have been discussed in the literature, for both relativistic and nonrelativistic matter fields. So far the relativistic model has been amenable for a more thorough analysis. It admits a nontrivial ground state over which topological vortices can be superimposed [2–4]. In addition to these asymptotically nonvanishing solutions with quantized flux, the relativistic model exhibits nontopological solitons for which the matter field vanishes at infinity [4]. (We will call such bell-shaped solitons "lumps" in this paper.) The two classes of solutions pertain to different choices of scalar self-interaction, vortices to repulsion, lumps to attraction.

The *nonrelativistic* model, i.e. the gauged nonlinear Schrödinger equation, was found to exhibit only asymptotically vanishing solitons [5,6]. The magnetic flux of these solitons is quantised and for this reason these collective excitations are also referred to as topological vortices. However they are clearly different from, for instance, topological vortices of the relativistic model which have the form of defects interpolating between topologically distinct vacua. Solutions of the latter type are not supported by the "standard" nonrelativistic model.

As in the ordinary, nongauged nonlinear Schrödinger equation, the above-said bell-like solitons arise in the case of the self-attractive boson gas (coupled to the Chern-Simons field.) In the case of *repulsive* interaction the natural ground state should be symmetry-breaking, i.e. correspond to a nonzero charge density at infinity. (We will call this nontrivial vacuum solution *condensate.*) However, the "standard" nonrelativistic model does not admit the condensate solution, no matter what the scalar self-interaction $U(\psi)$ is. If one wants to have a homogeneous condensation of charges, not just the sign of the interaction has to be changed but the very structure of the model.

In our recent Letter [7] we demonstrated that the necessary modification can be attained simply by the revision of the Lagrangian formulation of the nongauged predecessor of the "standard" model. Using the revised Lagrangian as a basis for the gauge theory, we arrived at a new version of the gauged nonlinear Schrödinger equation which is completely compatible with the nonvanishing boundary conditions at infinity. Physically, the resulting modification corresponds to the addition of the background electric charge. In the present paper the new model is analysed in more detail.

Here we reformulate it as a (constrained) Hamiltonian system. On the one hand, the Hamiltonian formulation allows to find self-dual solutions of the model. On the other hand, it makes the analysis of conserved quantities sensible. This paper focuses on the linear momentum of the vortices. We construct the momentum that satisfies all the requisite properties: it is gauge-invariant, generates translations and is compatible with the Hamiltonian structure of the model. We show that such a momentum can only be defined as a translationally noninvariant quantity. Speaking otherwise, the components of the momentum do not commute: $\{P_x, P_y\} = 2\pi\rho_0 n$.

It is appropriate to mention here that there is also another way to incorporate the condensate into the nonrelativistic model. The alternative remedy is to add an external, nondynamical static magnetic field [8]. Although the background electric charge and external static magnetic field correspond to entirely different physical settings, these two systems are mathematically equivalent, provided their Lagrangians contain the Chern-Simons term.

The paper is organized as follows. In sec. II we explain why the "standard" model is not applicable for the description of the asymptotically nonvanishing structures. In sec. III the model with the background charge is introduced and its relation to the CS theory with the external magnetic field delineated. In sec. IV we discuss the Hamiltonian formulation and in sec. VI A analyze asymptotic properties of the new model. Section VI B is devoted to the self-dual solutions which arise both in Chern-Simons and Chern-Simons-Maxwell cases. These include the asymptotically nonvanishing topological vortices and some other solutions. The linear momentum of topologically nontrivial configurations is defined in sec. V. Finally, in sec. VII we make several concluding remarks, in particular about the "spontaneous algebra breaking", the anomalous commutation relation between the components of the momentum.

II. THE "STANDARD" MODEL AND ASYMPTOTICALLY NONVANISHING FIELDS

The gauged nonlinear Schrödinger equation was formulated by Jackiw and Pi [5]-[6]:

$$i\psi_t - eA_0\psi + \frac{1}{2m}\mathbf{D}^2\psi - U'(\rho)\psi = 0, \tag{2.1a}$$

where $\rho = |\psi|^2$, $U'(\rho) = dU/d\rho$, and $\mathbf{D} = \nabla - ie\mathbf{A}$. The gauge field $A^\alpha = (A_0, \mathbf{A})$ satisfies its own equation for which the conserved matter current, $J^\alpha = (J_0, \mathbf{J})$, serves as a source. (Note that there are no *external* gauge fields.) The most general linear equation for A^α in (2+1) dimensions comprises both Maxwell and Chern-Simons terms:

$$\mu\partial_\beta F^{\beta\alpha} + \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = eJ^\alpha. \tag{2.1b}$$

Here $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$; Greek and Latin indices run over 0,1,2 and 1,2, respectively; the metric signature is $(+,-,-)$. The matter current of Jackiw and Pi is

$$J_0 = \rho = |\psi|^2, \tag{2.2a}$$

$$\mathbf{J} = \frac{1}{2mi}(\overline{\psi}\mathbf{D}\psi - \psi\overline{\mathbf{D}\psi}). \tag{2.2b}$$

The parameters μ and κ control the relative contributions of the Maxwell and Chern-Simons terms in the corresponding Lagrangian:

$$\mathcal{L} = \frac{i}{2}(\overline{\psi}D_0\psi - \psi\overline{D_0\psi}) - \frac{1}{2m}\overline{D_k\psi}D_k\psi - \frac{\mu}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}A_\gamma F_{\alpha\beta} - U(\rho). \tag{2.3}$$

In eq. (2.3) $D_0 = \partial_0 + ieA_0$; μ is ≥ 0, and κ will be considered nonnegative as well. The case of negative κ is recovered simply by the parity transformation ($x^1 \rightleftarrows x^2$, $A^1 \rightleftarrows A^2$). Finally, the scalar self-interaction $U(\rho)$ is taken to be completely arbitrary in this section. We demonstrate that, no matter what $U(\rho)$ is, the system (2.1)-(2.3), which we will call the standard model, does not admit the condensate solution.

Componentwise, the gauge field equations (2.1b) can be written as

$$\mu\,\mathrm{div}\,\mathbf{E} - \kappa B = eJ_0, \tag{2.4a}$$

$$-\mu\mathbf{E}_t + \mu\,\mathrm{curl}\,B - \kappa\widetilde{\mathbf{E}} = e\mathbf{J}, \tag{2.4b}$$

where $E^i = F^{i0}$ and $B = F^{21}$ are the electric and magnetic fields respectively. $\widetilde{\mathbf{E}}$ stands for the dual to $\mathbf{E}$: $\widetilde{E}^i = \epsilon_{ij}E^j$.

We will call *condensate* any solution to the equations (2.1a), (2.4) with the following properties: (i) ψ is a time-independent nonsingular matter field distribution with a uniform density, i.e. $\psi = \sqrt{\rho_0}e^{i\chi(\mathbf{x})}$ with $\rho_0 = $ const and $\chi(\mathbf{x})$ single-valued; (ii) the electric and magnetic fields are also static, $\mathbf{E} = \mathbf{E}(\mathbf{x})$, $B = B(\mathbf{x})$, and bounded.

In the pure Maxwell case ($\kappa = 0$), the nonexistence of a solution with the above properties is straightforward and widely known to workers in this field. Eq. (2.4a) is then simply $\mathrm{div}\,\mathbf{E} = $ const and so the electric field has to grow indefinitely. In the pure Chern-Simons case ($\mu = 0$), eq. (2.4a) becomes $B = $ const and the nonexistence of the condensate is not so obvious (but also known to specialists.) Below we prove this fact for the most general situation where both the Maxwell and Chern-Simons terms are present.

Our argument appears somewhat shorter if we choose the real gauge, $\chi(\mathbf{x}) = 0$. The imaginary part of eq. (2.1a) amounts to $\mathrm{div}\,\mathbf{A} = 0$ and hence $\mathbf{A}$ is a curl, $\mathbf{A} = \nabla\times\alpha$. The current (2.2b) is then a curl as well: $\mathbf{J} = -(e\rho_0/m)\nabla\times\alpha$. The requirement $\mathbf{E}_t = B_t = 0$ does not yet ensure $\mathbf{A}_t = 0$ and so the electric field, $\mathbf{E} = -\nabla A_0 - \mathbf{A}_t$, may have, in general, both conservative and solenoidal components. However, taking the divergence of eq. (2.4b) with $\mathbf{E}_t = 0$, we get

$$-\kappa\nabla\cdot\widetilde{\mathbf{E}} = -\kappa\nabla\times\mathbf{E} = 0, \tag{2.5}$$

which means that $\mathbf{E}$ is a pure gradient: $\mathbf{E} = -\nabla A_0$.

Evaluating the curl of both sides of eq. (2.4b) and subtracting eq. (2.4a) with an appropriate coefficient, it is not difficult to obtain a stationary nonhomogeneous Klein-Gordon equation for the magnetic field:

$$-\mu^2 \Delta B + \left(\kappa^2 + \mu \frac{e^2 \rho_0}{m}\right) B = -e\kappa\rho_0. \tag{2.6}$$

The only bounded solution of eq. (2.6) is a constant,

$$B(\mathbf{x}) = B_0 = -\frac{e\kappa\rho_0}{\kappa^2 + \mu e^2 \rho_0 / m}. \tag{2.7}$$

All other solutions grow exponentially as $r \to \infty$ and/or have nonintegrable singularities in a finite part of the (x, y)-plane. Next for constant B eq. (2.4b) becomes $\mathbf{E} = (e^2\rho_0/m\kappa)\nabla\alpha$. Recalling that $\mathbf{E} = -\nabla A_0$, we conclude that $\alpha = -(m\kappa/e^2\rho_0)A_0$ plus, possibly, a function of time. Substituting this into $B = -\Delta\alpha$, we obtain

$$\partial\overline{\partial} A_0 = \frac{e^2\rho_0}{m}\frac{B_0}{\kappa}. \tag{2.8a}$$

Here $\partial = \partial/\partial z$, $\overline{\partial} = \partial/\partial\overline{z}$ with $z = (x + iy)/2$, $\overline{z} = (x - iy)/2$. The real part of the nonlinear Schrödinger equation (2.1a) is another equation for A_0:

$$A_0 + \left(\frac{m\kappa^2}{2e^3\rho_0^2}\right)\partial A_0 \overline{\partial} A_0 = -\frac{1}{e}U'(\rho_0). \tag{2.8b}$$

Up to now we have been implicitly assuming that $\kappa \neq 0$. If $\kappa = 0$, the equation (2.4a) is equivalent to (2.8a) where we should only replace B_0/κ by its limit value:

$$\lim_{\kappa \to 0} \frac{B_0}{\kappa} = -\frac{m}{\mu e}.$$

The general solution of the Poisson equation (2.8a) is

$$A_0 = \frac{e^2\rho_0}{m}\frac{B_0}{\kappa} z\overline{z} + f(z) + \overline{f(z)}, \tag{2.9}$$

where $f(z)$ is an arbitrary analytic function. The nonexistence of the condensate solution is obvious already from eq. (2.9). (A_0 comprises a term quadratic in coordinates and so the electric field grows at least linearly as $r \to \infty$.) However, we can prove even a stronger assertion. We will demonstrate that equations (2.8a) and (2.8b) are, in general, incompatible and, with a single exception, there are even no *polynomially* growing solutions.

Differentiating eq. (2.8b) by z one obtains

$$\left(1 + \frac{B_0\kappa}{2e\rho_0}\right) a = -\left(\frac{m\kappa^2}{2e^3\rho_0^2}\right)\overline{a}\partial a, \tag{2.10}$$

where $a = \partial A_0$. Multiplying (2.10) by its complex conjugate yields

$$|\partial a|^2 = \left(\frac{2e^3\rho_0^2}{m\kappa^2}\right)^2 \left(\frac{m\kappa^2/2 + \mu e^2\rho_0}{m\kappa^2 + \mu e^2\rho_0}\right)^2. \tag{2.11}$$

Here we used the fact that, in view of (2.8a), $a \neq 0$. Thus, $\partial a = \partial^2 A_0 = \partial^2 f(z)$ is an analytic function with constant modulus. By Liouville's theorem, the only such function is a constant.

Having noted that and differentiating eq. (2.10) by z again, one arrives at the relation $e\rho_0/\kappa = -B_0$ which can only be satisfied if $\mu = 0$. This means that a solution to the system (2.8) exists only in the pure Chern-Simons case. However, this solution is quadratic in the coordinates,

$$A_0 = -\frac{e^3\rho_0^2}{m\kappa^2} z\overline{z} + (C_2 z^2 + C_1 z + C_0 + c.c.), \tag{2.12}$$

and so the corresponding electric field grows linearly as x or $y \to \pm\infty$. As we have already mentioned, this solution is not suitable as the condensate.

Thus all constant density solutions of the system (2.1) grow exponentially as $|\mathbf{x}| \to \infty$ and/or have nonintegrable singularities in the finite part of the (x, y) plane. The only exception is the pure Chern-Simons case, $\mu = 0$, where the electric field grows linearly as $|\mathbf{x}| \to \infty$. We have to conclude therefore that the condensate exists for no μ and κ [9].

III. THE MODEL WITH THE BACKROUND CHARGE DENSITY

A. The regularised model

The structure of the necessary modifications that have to be made to the "standard" model (2.3) is suggested [7] by a more basic system, the *nongauged* NLS equation:

$$i\psi_t + \frac{1}{2m}\Delta_D\psi - U'(\rho)\psi = 0. \tag{3.1}$$

Here Δ_D is the D-dimensional Laplacian, $\Delta_D = \partial^2/\partial x_1^2 + ... + \partial^2/\partial x_D^2$. In one dimension the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\psi_t\overline{\psi} - \overline{\psi}_t\psi) - \frac{1}{2m}|\nabla_D\psi|^2 - U(\rho) \tag{3.2}$$

which is normally associated with eq.(3.1) does not *automatically* produce the correct integrals of motion for solutions with $|\psi|^2 \to \rho_0$ at infinity. The number of particles integral takes the form $N = \int \rho dx$ and obviously diverges while the momentum, $P = (i/2)\int(\psi_x\overline{\psi} - \overline{\psi}_x\psi)dx$, is not functionally differentiable and, therefore, is incompatible with the Hamiltonian structure of the model. The finite number of particles,

$$N = \int (\rho - \rho_0)\, dx, \tag{3.3}$$

and the differentiable momentum,

$$P = \frac{i}{2}\int(\psi_x\overline{\psi} - \overline{\psi}_x\psi)\left(1 - \frac{\rho_0}{\rho}\right) dx \tag{3.4}$$

are obtained only by an *a posteriori* regularisation. Proceeding to two dimensions, the situation is no better. The Noetherian momentum resulting from the Lagrangian (3.2), $\mathbf{P} = \int \mathcal{P} d^2x$, with

$$\mathcal{P} = \frac{i}{2}\left(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\right) \tag{3.5}$$

is not only nondifferentiable but also divergent on certain configurations. Assuming, for instance, the vortex boundary conditions $\psi(\mathbf{x}) \to \sqrt{\rho_0}e^{in\theta}$ as $|\mathbf{x}| \to \infty$ we have $\mathcal{P} \to (-n)\rho_0\mathbf{x}/r^2$ and so the integral $\int \mathcal{P} d^2x$ converges only in the sense of the principal value. Another important conserved quantity in two dimensions is the angular momentum:

$$L = \int \mathbf{x} \times \mathcal{P} d^2x. \tag{3.6}$$

As $|\mathbf{x}| \to \infty$, the integrand in (3.6) approaches $n\rho_0$ and so the angular momentum, in general, diverges.

Thus we see that the Lagrangian (3.2) is not compatible with the nonvanishing boundary conditions at infinity. On the one hand this incompatibility manifests itself in the fact that there is no condensate solution in the gauge theory based on this Lagrangian. On the other hand eq. (3.2) gives rise to ill-defined integrals of motion even on the nongauged level. These two problems are obviously not unrelated. As noticed in [7], curing the conserved quantities gives a clue to the condensate problem.

Adding the time derivative of a function to the standard Lagrangian (3.2) does not change its Euler-Lagrange equation (3.1). In one dimension this function can be chosen in such a way that the corresponding integrals of motion *automatically* arise in the regularised form given by (3.3) and (3.4). The appropriate time derivative is $\rho_0(\operatorname{Arg}\psi)_t$, and the regularised Lagrangian is

$$\widetilde{\mathcal{L}} = \frac{i}{2}(\psi_t\overline{\psi} - \overline{\psi}_t\psi)\left(1 - \frac{\rho_0}{\rho}\right) - \frac{|\nabla_D\psi|^2}{2m} - U(\rho) \tag{3.7}$$

with $D = 1$. For $D = 2$ the Lagrangian (3.7) gives rise to the two-dimensional nonlinear Schrödinger equation while the conserved quantities (as evaluated by Noether's theorem) are: $N = \int(\rho - \rho_0)d^2x$;

$$P = \int \mathcal{P} d^2x, \tag{3.8a}$$

with

$$\mathcal{P} = \frac{i}{2}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi})\left(1 - \frac{\rho_0}{\rho}\right); \tag{3.8b}$$

and

$$L = \int \mathbf{x} \times \mathcal{P} d^2x. \tag{3.9}$$

It is easy to check that these integrals are convergent for fields with $\rho \to \rho_0$ as $|\mathbf{x}| \to \infty$. Furthermore, for *some* configurations, for instance solitonic bubbles, P eq. (3.8) and L eq. (3.9) are differentiable. (Bubbles are nowhere vanishing nontopological solutions with $\psi(\mathbf{x}) \to \sqrt{\rho_0}$ as $r \to \infty$ [10].) However, the momentum (3.8) is not differentiable at a vortex-like configuration, i.e.

a configuration of the form $\psi(\mathbf{x}) = \psi(r)e^{in\theta}$ with $\psi(r) \to \sqrt{\rho_0}$ as $r \to \infty$, and $\psi(r) \sim r^n$ for $r \sim 0$. The variation of the momentum is

$$\delta P = i\int(\delta\overline{\psi}\nabla\psi - \delta\psi\nabla\overline{\psi})d^2x - \int \nabla[(\rho - \rho_0)\delta \operatorname{Arg}\psi]d^2x. \tag{3.10}$$

As in the one-dimensional situation, the quantity $(\rho - \rho_0)\delta \operatorname{Arg}\psi$ approaches zero at infinity and it is tempting to expect that the second term in (3.10) will vanish after the double integral has been transformed to a line integral over an infinitely remote contour. However, Green's theorem is not applicable for vortices as $\operatorname{Arg}\psi$ is not differentiable at the centre of the vortex. [For instance, if the vortex is placed at the origin, $\operatorname{Arg}\psi = n\theta = n\arctan(y/x)$.] To apply Green's theorem we would have to integrate not only over the infinitely remote contour but also over a small contour encircling the vortex core. The result will generally be nonzero.

Note that this nondifferentiability is of a different nature than the nondifferentiability of the momentum $(i/2)\int(\psi\overline{\psi}_x - \overline{\psi}\psi_x)dx$ in one dimension. The latter arises from the nonvanishing boundary conditions at infinity; that discontinuity is an attribute of the condensate. The expression (3.8) is free from this drawback. It is not surprising therefore that the gauge theory based on the Lagrangian (3.7) is not so hostile to asymptotically nonvanishing fields. Coupling A_μ minimally to eq.(3.7) yields

$$\widetilde{\mathcal{L}} = \frac{i}{2}\left(\overline{\psi}D_0\psi - \psi\overline{D_0\psi}\right)\left(1 - \frac{\rho_0}{\rho}\right) - \frac{1}{2m}\overline{D_k\psi}D_k\psi - \frac{\mu}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}A_\gamma F_{\alpha\beta} - U(\rho). \tag{3.11}$$

Dropping the time derivative $\rho_0\partial_0\operatorname{Arg}\psi$, we obtain the final form of the regularised model [7], [11]:

$$\widetilde{\mathcal{L}} = \frac{i}{2}\left(\overline{\psi}D_0\psi - \psi\overline{D_0\psi}\right) + e\rho_0A_0 - \frac{1}{2m}\overline{D_k\psi}D_k\psi - \frac{\mu}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}A_\gamma F_{\alpha\beta} - U(\rho). \tag{3.12}$$

It is only one term, $e\rho_0A_0$, that makes eq. (3.12) different to the standard model (2.3). However, this term has a transparent physical interpretation (it describes the gauge field coupling to the uniformly charged static background) and produces a drastic change in solutions of the model. The equations of motion look the same, eqs. (2.1), with the vector $\mathbf{J}$ being given by the standard expression (2.2b). The number density, however, is given by

$$J_0 = \rho - \rho_0. \tag{3.13}$$

It is straightforward to check that the model regularised in this way does possess the condensate solution: $\psi = \sqrt{\rho_0}$, $A_0 = -(1/e)U'(\rho_0)$, $\mathbf{A} = 0$.

B. Background charge and external magnetic field

In case the Chern-Simons constant κ is nonzero, the background charge density can be reinterpreted as the external static uniform magnetic field. Indeed, suppose that in addition to the gauge field A_μ we have an external, nondynamical gauge field $A_\mu^{\rm ext}$ (but no background charge). This situation is described by Jackiw-Pi's Lagrangian (2.3) where we should only change the expression for the covariant derivative:

$$D_\mu = \partial_\mu + ieA_\mu + ieA_\mu^{\rm ext}. \tag{3.14}$$

In terms of the resultant gauge field

$$\widetilde{A}_\mu = A_\mu + A_\mu^{\rm ext} \tag{3.15}$$

the Lagrangian (2.3) becomes

$$\mathcal{L} = \frac{i}{2}(\overline{\psi}D_0\psi - \psi\overline{D_0\psi}) - \frac{1}{2m}|D_k\psi|^2 - \frac{\mu}{4}\widetilde{F}_{\alpha\beta}\widetilde{F}^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}\widetilde{A}_\gamma\widetilde{F}_{\alpha\beta} - U(\rho) + \mathcal{L}_1, \tag{3.16a}$$

where $D_\mu = \partial_\mu + ie\widetilde{A}_\mu$, $\widetilde{F}_{\mu\nu} = \partial_\mu\widetilde{A}_\nu - \partial_\nu\widetilde{A}_\mu$,

$$\mathcal{L}_1 = \frac{\mu}{2}\widetilde{F}^{\alpha\beta}F_{\alpha\beta}^{\rm ext} - \frac{\kappa}{2}\epsilon^{\gamma\alpha\beta}\widetilde{A}_\gamma F_{\alpha\beta}^{\rm ext}, \tag{3.16b}$$

and we have dropped the term

$$-\frac{\mu}{4}F_{\alpha\beta}^{\rm ext}F_{\rm ext}^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\gamma\alpha\beta}A_\gamma^{\rm ext}F_{\alpha\beta}^{\rm ext} - \frac{\kappa}{2}\epsilon^{\gamma\alpha\beta}\partial_\alpha(A_\gamma^{\rm ext}A_\beta)$$

the variation of which is zero. We can rewrite $\mathcal{L}_1$ as

$$\mathcal{L}_1 = (\kappa\widetilde{A}_0 + \mu\widetilde{B})B^{\rm ext} - \kappa\widetilde{\mathbf{A}}\times\mathbf{E}^{\rm ext} - \mu\widetilde{\mathbf{E}}\cdot\mathbf{E}^{\rm ext}. \tag{3.17}$$

Here the quantities with tildes correspond to the field $\widetilde{A}_\mu$. If $B^{\rm ext} = $ const, the term $\mu\widetilde{B}B^{\rm ext}$ may be dropped from $\mathcal{L}_1$. Consequently, in the case of the purely magnetic ($\mathbf{E}^{\rm ext} = 0$) constant external field, $\mathcal{L}_1$ reduces simply to $\kappa\widetilde{A}_0B^{\rm ext}$. If we rename the constant $\kappa B^{\rm ext}$ as $e\rho_0$, eq. (3.16a) becomes nothing but our model with the background charge density, eq. (3.12), for the fields ψ and $\widetilde{A}_\mu$.

Note that the Chern-Simons term is essential for the equivalence to take place. If $\kappa = 0$, i.e. if we have a pure Maxwell gauge field, we cannot identify $\kappa B^{\rm ext}$ and $e\rho_0$ and the transformation breaks down. The Maxwell theory witb the external constant magnetic field and the Maxwell theory with the background electric charge turn out to be unrelated models. [Instead, as one can readily see from (3.17), the $\kappa = 0$ model with the background charge density can be reformulated as the Maxwell theory in the linearly growing *electric* field, $\mathbf{E}^{\rm ext} = -(e\rho_0/2\mu)\mathbf{r}$.]

IV. THE CONSTRAINED HAMILTONIAN FORMULATION

The Hamiltonian formulation is crucial for the construction of self-dual solutions and a variety of other purposes. In this section we give the Hamiltonian formulation of the model with the background charge (as well as of the "standard" model of Jackiw and Pi.) One immediately notices

that the Lagrangian (3.12) is singular (degenerate in velocities) which implies that we deal with constrained dynamics. In its treatment we follow Dirac-Bergmann's approach [13].

The momenta conjugate to the matter field ψ and its complex conjugate $\overline{\psi}$ are given by

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \frac{i}{2}\overline{\psi}(1 - \frac{\rho_0}{\rho}),$$
$$\overline{\pi} = \frac{\partial\mathcal{L}}{\partial(\partial_0\overline{\psi})} = -\frac{i}{2}\psi(1 - \frac{\rho_0}{\rho}),$$

and give rise to two primary constraints:

$$\xi_1 = \pi - \frac{i}{2}\overline{\psi}(1 - \frac{\rho_0}{\rho}) = 0, \tag{4.1a}$$
$$\xi_2 = \overline{\pi} + \frac{i}{2}\psi(1 - \frac{\rho_0}{\rho}) = 0. \tag{4.1b}$$

The momenta conjugate to the fields A^α are

$$\Pi_\alpha = \frac{\kappa}{2}\epsilon_{\alpha\beta 0}A^\beta - \mu F^{\alpha 0}. \tag{4.2}$$

Π_0 vanishes; therefore $\Pi_0 = 0$ is a third primary constraint. The canonical Hamiltonian density is the energy density T^{00} [see eq. (5.2a) below] expressed in terms of canonical variables $\psi, \overline{\psi}, \pi, \overline{\pi}, A^\mu$ and Π_μ. Integrating by parts (see sec.V) we write the canonical Hamiltonian as

$$H_c = \int\left[\frac{1}{2m}|\mathrm{D}\psi|^2 + U(\rho) + \frac{\mu}{2}(\mathbf{E}^2 + B^2)\right]d^2x - \int A_0\left[(\mu\,\mathrm{div}\,\mathbf{E} - \kappa B - e(\rho - \rho_0)\right]d^2x, \tag{4.3}$$

where $E^i = (\kappa\epsilon^{ij}A^j/2 - \Pi_i)/\mu$, and $B = \partial_1A^2 - \partial_2A^1$. The Poisson bracket between two functionals is

$$\{S,T\} = \int\left[\frac{\delta S}{\delta\psi}\frac{\delta T}{\delta\pi} - \frac{\delta S}{\delta\pi}\frac{\delta T}{\delta\psi} + \frac{\delta S}{\delta\overline{\psi}}\frac{\delta T}{\delta\overline{\pi}} - \frac{\delta S}{\delta\overline{\pi}}\frac{\delta T}{\delta\overline{\psi}} + \frac{\delta S}{\delta A^\alpha}\frac{\delta T}{\delta\Pi_\alpha} - \frac{\delta S}{\delta\Pi_\alpha}\frac{\delta T}{\delta A^\alpha}\right]d^2x.$$

The constraints (4.1) are second class, $\{\xi_1(\mathbf{x},t), \xi_2(\mathbf{y},t)\} = -i\delta^{(2)}(\mathbf{x}-\mathbf{y})$, and can be accommodated by the introduction of the Dirac bracket,

$$\{S,T\}_{\mathcal{D}} = \{S,T\} - \frac{1}{i}\int\Big[\{S,\xi_1(\mathbf{z})\}\{\xi_2(\mathbf{z}),T\} - \{S,\xi_2(\mathbf{z})\}\{\xi_1(\mathbf{z}),T\}\Big]d^2z. \tag{4.4}$$

The expression (4.4) can be simplified considerably if one solves the constraints (4.1) for π and $\overline{\pi}$ and treats S and T as functionals of ψ and $\overline{\psi}$ only. Denote

$$\widetilde{S}[\psi,\overline{\psi}] = S\left[\psi, \pi(\psi,\overline{\psi}), \overline{\psi}, \overline{\pi}(\psi,\overline{\psi})\right],$$
$$\widetilde{T}[\psi,\overline{\psi}] = T[\psi, \pi(\psi,\overline{\psi}), \overline{\psi}, \overline{\pi}(\psi,\overline{\psi})].$$

($\widetilde{S}$ and $\widetilde{T}$ also depend on A_α, Π_α, of course.) A straightforward calculation shows that

$$\{S,T\}_{\mathcal{D}} = \int\left[\frac{1}{i}\left(\frac{\delta\widetilde{S}}{\delta\psi}\frac{\delta\widetilde{T}}{\delta\overline{\psi}} - \frac{\delta\widetilde{S}}{\delta\overline{\psi}}\frac{\delta\widetilde{T}}{\delta\psi}\right) + \frac{\delta\widetilde{S}}{\delta A^\alpha}\frac{\delta\widetilde{T}}{\delta\Pi_\alpha} - \frac{\delta\widetilde{S}}{\delta\Pi_\alpha}\frac{\delta\widetilde{T}}{\delta A^\alpha}\right]d^2x. \tag{4.5}$$

Hereafter we omit the tilde and always understand S and T as functionals of only ψ, $\overline{\psi}$, A_α, and Π_α.

The constraint $\Pi_0 = 0$ is first class and the requirement of its conservation leads to a secondary constraint

$$\eta = \{\Pi_0, H_c\}_{\mathcal{D}} = \mu \,\mathrm{div}\, \mathbf{E} - \kappa B - e(\rho - \rho_0) = 0. \tag{4.6}$$

The secondary constraint $\eta = 0$ does not involve any further constraints since, as one can check, $\{\eta, H_c\}_{\mathcal{D}} = 0$.

The total Hamiltonian is now $H_T = H_c + \int v\Pi_0 d^2x$, where v is an arbitrary multiplier. A useful observation is that the variables A^0 and Π_0 have no physical significance, $\Pi_0 = 0$ all the time while A^0 can take arbitrary values. Accordingly we may drop them out from the set of dynamical variables of the model. This can be accomplished by discarding the term $v\Pi_0$ in H_T, the only rôle of which is to let A^0 vary arbitrarily, and by treating A^0 as an arbitrary multiplier. As a result the Hamiltonian formulation of our model can be given in terms of three pairs of canonical fields: $\psi, \overline{\psi}; A^i, \Pi_i$. The dynamics is described by the Hamiltonian H_c, with the bracket (4.5) (now α=1,2). The initial state should satisfy the constraint (4.6). The Hamilton's equations will contain an arbitrary function A^0 which can be determined only for a static problem. In the static case the problem reduces to finding stationary points of energy

$$H = \int \left[\frac{1}{2m} |\mathbf{D}\psi|^2 + U(\rho) + \frac{\mu}{2}(\mathbf{E}^2 + B^2) \right] d^2x \tag{4.7}$$

on the constraint manifold (4.6). In this case A^0 plays the rôle of a Lagrange multiplier and H_c is a Lagrange function.

So far we have assumed $\mu \neq 0$. For $\mu = 0$ (pure Chern-Simons model) the situation is somewhat different. The theory has two more second class primary constraints,

$$\xi_3 = \Pi_1 - \frac{\kappa}{2} A^2 = 0, \tag{4.8a}$$

$$\xi_4 = \Pi_2 + \frac{\kappa}{2} A^1 = 0, \tag{4.8b}$$

resulting from the definition of momenta Π_i. As $\{\xi_3(\mathbf{x}, t), \xi_4(\mathbf{y}, t)\} = -\kappa\delta^{(2)}(\mathbf{x} - \mathbf{y})$, these can be accommodated by modifying the bracket:

$$\{S, T\}_{\mathcal{D}} = \{S, T\} + \int \Big[-\frac{1}{i}\{S, \xi_1(\mathbf{z})\}\{\xi_2(\mathbf{z}), T\} + \frac{1}{i}\{S, \xi_2(\mathbf{z})\}\{\xi_1(\mathbf{z}), T\}$$
$$- \frac{1}{\kappa}\{S, \xi_3(\mathbf{z})\}\{\xi_4(\mathbf{z}), T\} + \frac{1}{\kappa}\{S, \xi_4(\mathbf{z})\}\{\xi_3(\mathbf{z}), T\} \Big] d^2z.$$

Solving (4.8) for Π_1 and Π_2, this can be brought to the form

$$\{S, T\}_{\mathcal{D}} = \int \left[\frac{1}{i} \left(\frac{\delta S}{\delta \psi} \frac{\delta T}{\delta \overline{\psi}} - \frac{\delta S}{\delta \overline{\psi}} \frac{\delta T}{\delta \psi} \right) + \frac{1}{\kappa} \left(\frac{\delta S}{\delta A^1} \frac{\delta T}{\delta A^2} - \frac{\delta S}{\delta A^2} \frac{\delta T}{\delta A^1} \right) \right] d^2x, \tag{4.9}$$

where S and T are considered as functionals of only ψ, $\overline{\psi}$, A^1 and A^2.

In fact in the pure Chern-Simons case a further reduction is possible. In this case one can explicitly resolve the constraint (4.6) and end up with ψ and $\overline{\psi}$ as the only canonical variables. This is the approach utilized by Jackiw and Pi [5,6]. In the general case, however, the fact that the constraint involves both Π_i and A^i makes such reduction impossible.

Our final remark in this section is that the fact that the model can be formulated as a *constrained* Hamiltonian system is a consequence of the local gauge invariance. The generator of the local gauge transformations is nothing but the first class constraint (4.6). The linear part of the transformation is

$$\psi(\mathbf{x}) \to \psi(\mathbf{x}) + a(\mathbf{x}_0)\{\psi(\mathbf{x}), \eta(\mathbf{x}_0)\} = \psi(\mathbf{x}) + iea(\mathbf{x}_0)\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)\psi(\mathbf{x})$$

$$\overline{\psi}(\mathbf{x}) \to \overline{\psi}(\mathbf{x}) + a(\mathbf{x}_0)\{\overline{\psi}(\mathbf{x}), \eta(\mathbf{x}_0)\} = \psi(\mathbf{x}) - iea(\mathbf{x}_0)\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)\overline{\psi}(\mathbf{x})$$

$$A^i(\mathbf{x}) \to A^i(\mathbf{x}) + a(\mathbf{x}_0)\{A^i(\mathbf{x}), \eta(\mathbf{x}_0)\} = A^i(\mathbf{x}) + a(\mathbf{x}_0)\frac{\partial}{\partial x^i}\delta^{(2)}(\mathbf{x} - \mathbf{x}_0)$$

$$\Pi_i(\mathbf{x}) \to \Pi_i(\mathbf{x}) + a(\mathbf{x}_0)\{\Pi_i(\mathbf{x}), \eta(\mathbf{x}_0)\} = \Pi_i(\mathbf{x}) + \frac{\kappa}{2}a(\mathbf{x}_0)\epsilon_{ij}\frac{\partial}{\partial x^j}\delta^{(2)}(\mathbf{x} - \mathbf{x}_0).$$

The formulas (4.10) pertain to the case where both Maxwell and Chern-Simons terms are present. The bracket here is the Dirac bracket (4.5). (Here and below we omit the subscript $\mathcal{D}$.) When A_μ is the pure Chern-Simons gauge field ($\mu = 0$), eqs. (4.10a–c) are still in place, while (4.10d) should be discarded. The bracket then is defined by eq. (4.9).

V. MOMENTUM

Trying to reformulate the 2D nongauged NLS equation as a Hamiltonian system, one faces several problems in $n \neq 0$ topological sectors. The energy of topologically nontrivial solutions, i.e. Gross-Pitayevski vortices, is divergent; the differentiable momentum is not available, and so on. The Hamiltonian formulation of the dynamics of vortices becomes sensible, however, if the gauge field is added. The vortex energy is now finite, and the aim of this section is to correctly define the momentum. The requisite momentum, $\mathbf{P}$, should satisfy the following fundamental criteria:

(i) $\mathbf{P}$ is a differentiable functional, conserved in view of the equations of motion: $\{\mathbf{P}, H_c\} = 0$;

(ii) $\mathbf{P}$ generates translations on fields: $\{\mathbf{P}, \chi(\mathbf{x})\} = \nabla\chi$;

(iii) $\mathbf{P}$ is gauge invariant on the constraint manifold.

In order to ensure that our conclusions are not mere artifacts of any particular gauge, we do not fix the gauge in this section. We start with the derivation of the energy-momentum tensor for the model with the background charge density, eq.(3.12). The corresponding results for the standard Jackiw-Pi's theory are recovered simply by letting $\rho_0 = 0$.

By Noether's theorem, the canonical energy-momentum tensor is

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\partial^\beta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\overline{\psi})}\partial^\beta\overline{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\gamma)}\partial^\beta A^\gamma - \mathcal{L}g^{\alpha\beta}, \tag{5.1}$$

where $g^{\alpha\beta} = \text{diag}\{1,-1,-1\}$. For the Lagrangian (3.12) eq. (5.1) yields

$$\begin{aligned} T^{00} = & \frac{1}{2m}|D_k\psi|^2 + U(\rho) + \frac{\mu}{4}F_{\alpha\beta}^2 \\ & - A_0\left(\mu\partial_k F^{k0} - \kappa F^{21} - e(\rho-\rho_0)\right) \\ & + \partial_k\left[A_0\left(\mu F_{0k} - \frac{\kappa}{2}\epsilon_{km}A^m\right)\right], \end{aligned} \tag{5.2a}$$

$$\begin{aligned} T^{0i} = & \frac{i}{2}(\psi\overline{D_i\psi} - \overline{\psi}D_i\psi) + e\rho_0\epsilon_{ij}x^jB + \mu\epsilon_{ij}F_{0j}F^{21} \\ & - A^i\left(\mu\partial_k F^{k0} - \kappa F^{21} - e(\rho-\rho_0)\right) \\ & + \partial_k\left[e\rho_0\epsilon_{mk}\epsilon_{ij}x^jA^m + A^i\left(\mu F_{0k} - \frac{\kappa}{2}\epsilon_{km}A^m\right)\right], \end{aligned} \tag{5.2b}$$

$$\begin{aligned} T^{i0} = & -\frac{1}{2m}\left(\overline{D_i\psi}D_0\psi + D_i\psi\overline{D_0\psi}\right) + \mu\epsilon_{im}F_{0m}F^{21} \\ & - \partial_0\left[A_0\left(\mu F_{0i} - \frac{\kappa}{2}\epsilon_{im}A^m\right)\right] \\ & + \epsilon_{ik}\partial_k\left\{\mu A_0F^{21} + \frac{\kappa}{2}(A_0)^2\right\}, \end{aligned} \tag{5.2c}$$

$$\begin{aligned} T^{ij} = & \frac{1}{2m}\left(\overline{D_i\psi}D_j\psi + D_i\psi\overline{D_j\psi}\right) - \mu F_{0i}F_{0j} \\ & + \left(\frac{\mu}{4}F_{\alpha\beta}^2 - \frac{\Delta\rho}{4m} + U'(\rho)\rho - U + e\rho_0A_0\right)\delta_{ij} \\ & - \partial_0\left[A^j\left(\mu F_{0j} - \frac{\kappa}{2}\epsilon_{ik}A^k\right)\right] \\ & + \epsilon_{ik}\partial_k\left\{\mu A^jF^{21} + \frac{\kappa}{2}A^0A^j\right\}. \end{aligned} \tag{5.2d}$$

The expressions for the fluxes, eqs (5.2c) and (5.2d) were simplified using the equations of motion. One may also be tempted to simplify the conserved densities, eqs (5.2a) and (5.2b), by dropping the multiples of $\mu\,\text{div}\,\mathbf{E} - \kappa B - e(\rho-\rho_0)$. However,

$$\mu\,\text{div}\,\mathbf{E} - \kappa B - e(\rho-\rho_0) \approx 0$$

is a *weak* equality in Dirac's sense and can be implemented only after all brackets have been calculated.

The last terms in the fluxes T^{i0} are two-dimensional curls. We can discard them as they cancel in the local conservation laws

$$\partial_0T^{0\alpha} + \partial_kT^{k\alpha} = 0; \quad \alpha = 0,1,2. \tag{5.3}$$

Next, terms in the square brackets in the conserved densities (5.2a) and (5.2b) cause the divergence or nondifferentiability of the corresponding integrals of motion. Fortunately the terms with the square brackets in (5.2a-d) are components of (2+1)-dimensional curls and so can be dropped without violating the local conservation laws (5.3).

Thus we arrive at the following expression for the linear momentum:

$$\begin{aligned} P^i = \int\Big\{ & \frac{i}{2}(\psi\overline{D_i\psi} - \overline{\psi}D_i\psi) + e\rho_0\epsilon_{ij}x^jB + \mu\epsilon_{ij}F^{j0}B \\ & + (A^i - \lambda^i)\left[-\mu\partial_kF_{0k} + \kappa B + e(\rho-\rho_0)\right]\Big\}\,d^2x. \end{aligned} \tag{5.4}$$

Here we have added arbitrary multipliers λ^i $(i = 1,2)$ whose rôle is to reflect the gauge freedom in the constrained Hamiltonian formulation. One can verify that P^i is indeed a differentiable functional, and by construction it commutes with the canonical Hamiltonian, eq.(4.3). What is also important, $\mathbf{P}$ is gauge invariant on the constrained manifold. It generates translations supplemented by local gauge transformations on gauge-noninvariant fields:

$$\begin{aligned} \psi(\mathbf{x}) \to \psi_{a_s}(\mathbf{x}) &= \psi(\mathbf{x}) + a_s\{\psi(\mathbf{x}), P^s\} \\ &= \psi(\mathbf{x}) - a_s\partial_s\psi(\mathbf{x}) + iea_s\lambda(\mathbf{x},a)\psi(\mathbf{x}), \end{aligned} \tag{5.5a}$$

$$\begin{aligned} \overline{\psi}(\mathbf{x}) \to \overline{\psi}_{a_s}(\mathbf{x}) &= \overline{\psi}(\mathbf{x}) + a_s\{\overline{\psi}(\mathbf{x}), P^s\} \\ &= \overline{\psi}(\mathbf{x}) - a_s\partial_s\overline{\psi}(\mathbf{x}) + iea_s\lambda(\mathbf{x},a)\overline{\psi}(\mathbf{x}), \end{aligned} \tag{5.5b}$$

$$\begin{aligned} \mathbf{A}(\mathbf{x}) \to \mathbf{A}_{a_s}(\mathbf{x}) &= \mathbf{A}(\mathbf{x}) + a_s\{\mathbf{A}(\mathbf{x}), P^s\} \\ &= \mathbf{A}(\mathbf{x}) - a_s\partial_s\mathbf{A}(\mathbf{x}) + a_s\nabla\lambda, \end{aligned} \tag{5.5c}$$

$$\begin{aligned} \mathbf{\Pi}(\mathbf{x}) \to \mathbf{\Pi}_{a_s}(\mathbf{x}) &= \mathbf{\Pi}(\mathbf{x}) + a_s\{\mathbf{\Pi}(\mathbf{x}), P^s\} \\ &= \mathbf{\Pi}(\mathbf{x}) - a_s\partial_s\mathbf{\Pi}(\mathbf{x}) + \frac{\kappa}{2}a_s\nabla\times\lambda, \end{aligned} \tag{5.5d}$$

where $s = 1,2$. (No summation over s). The fact that (5.5) is not a pure translation may seem to contradict the axiom (ii) above. However if we restrict ourselves to gauge-invariant fields, e.g. $\overline{\psi}(\mathbf{x})\psi(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$, and $B(\mathbf{x})$, the momentum (5.4) generates nothing but translations:

$$\begin{aligned} \{\mathbf{P}, \rho(\mathbf{x})\} &= \nabla\rho(\mathbf{x}), \\ \{\mathbf{P}, F_{\mu\nu}(\mathbf{x})\} &= \nabla F_{\mu\nu}(\mathbf{x}). \end{aligned}$$

The explicit dependence of the momentum density, eq. (5.4), on x and y results in a nonstandard commutation relation between P^1 and P^2:

$$\{P^1, P^2\} = \frac{d}{da_2}P^1[\psi_{a_2}, \overline{\psi}_{a_2}, \mathbf{A}_{a_2}, \mathbf{\Pi}_{a_2}]\Big|_{a_2=0} = 2\pi\rho_0 n, \tag{5.6}$$

where n is the topological charge of the configuration ψ, *viz.*

$$\begin{aligned} n &= \frac{e}{2\pi}\int Bd^2x = \frac{1}{2\pi}\oint d\,\text{Arg}\,\psi \\ &= \frac{1}{2\pi i\rho_0}\int(\partial_1\overline{\psi}\partial_2\psi - \partial_1\psi\partial_2\overline{\psi})d^2x. \end{aligned} \tag{5.7}$$

In fact the relation (5.6) can be obtained without invoking the explicit expressions for the momenta, eq. (5.4). All one needs to know are the brackets in (5.5), which are more fundamental than the explicit expressions (5.4). (The momentum densities are defined up to an addition of arbitrary curls whereas the brackets in (5.5) result simply from the requirement that the momentum generate translations.) Consequently the explicit coordinate dependence of P^i cannot be eliminated by re-defining the momenta. The linear momentum in our theory should *necessarily* be translationally noninvariant.

We close this section by mentioning the work of Lee [14] who has derived the momentum for the (relativistic) Higgs-Maxwell theory with the background charge density. His approach is entirely different, however, and the expression he arrives at has a different structure from eq. (5.4) above.

VI. VORTICES IN THE CONDENSATE

A. Background and asymptotic behaviour

The condensate solution has the form: $\psi = \sqrt{\rho_0}$, $A_0 = -(1/e)U'(\rho_0)$, $\mathbf{A} = 0$. This solution exists even when the potential $U(\rho)$ does not possess a symmetry-breaking minimum at ρ_0. We can, however, confine ourselves to potentials with $U'(\rho_0) = 0$ as this condition can always be accomplished by the transformation $A_0 \to A_0 - U'(\rho_0)/e$, $U(\rho) \to U(\rho) + U'(\rho_0)\rho$. The condensate solution is then $\psi = \sqrt{\rho_0}$, $A_0 = 0$, $\mathbf{A} = 0$, and it corresponds to an extremum of $U(\rho)$. By a singular gauge transformation the condensate generates a set of singular solutions,

$$\psi = \sqrt{\rho_0}e^{in\theta}; \quad A_0 = 0, \quad \mathbf{A} = \frac{n}{e}\frac{\mathbf{e}_\theta}{r} \tag{6.1}$$

which serve as $r \to \infty$ asymptotes for vortices. The magnetic flux of the vortex is quantized: $\Phi = \int B\, d^2x = \oint A_\theta\, r d\theta = 2\pi n/e$. Integrating eq. (4.6) over the entire plane we observe that the flux and the charge of the vortex are related:

$$-\kappa\Phi = Q. \tag{6.2}$$

Here $Q = eN$ and N is the (regularised) number of particles in the vortex, $N = \int(\rho - \rho_0)d^2x$. Hence the charge is quantized as well:

$$Q = e\int(\rho - \rho_0)\, d^2x = -\frac{2\pi\kappa}{e}n. \tag{6.3}$$

The exception is the pure Maxwell case, $\kappa = 0$, where vortices are electrically neutral: $Q = 0$ for all n.

Confining consideration to radially symmetric configurations we write

$$\psi(\mathbf{x}) = \psi(r)e^{in\theta}, \tag{6.4a}$$

$$A^j(\mathbf{x}) = \epsilon^{jk}x_k\frac{n/e - \phi(r)}{r^2}, \quad j,k = 1,2, \tag{6.4b}$$

$$A_0(\mathbf{x}) = A_0(r). \tag{6.4c}$$

Regularity at the origin requires $\psi(0) = 0$ when $n \neq 0$. For finite $\mathbf{A}(0)$ we should also have $\phi(0) = n/e$. Hence

$$\phi(r) = \frac{n}{e} - \int_0^r B(r)rdr.$$

Physically, $2\pi\phi(r)$ represents the magnetic flux through the exterior of the disc of radius r. As $r \to \infty$, $\phi(r) \to 0$. The system (2.1) reduces to

$$-\frac{1}{2m}\Delta\psi + U'(\rho)\psi + eA_0\psi + \frac{e^2}{2m}\frac{\phi^2}{r^2}\psi = 0, \tag{6.5a}$$

$$-\mu r\frac{d}{dr}\frac{\phi_r}{r} + \kappa r\frac{dA_0}{dr} + \frac{e^2}{m}\rho\phi = 0, \tag{6.5b}$$

$$-\mu\Delta A_0 + \kappa\frac{\phi_r}{r} - e(\rho - \rho_0) = 0. \tag{6.5c}$$

Eqs. (6.5) may exhibit both topological vortices, for which $n \neq 0$, and vorticity-free solutions, *bubbles*. Eq. (6.3) indicates that the bubbles have to be nodal in our model, i.e. $\rho(r) - \rho_0$ should necessarily change sign. In the pure Maxwell case ($\kappa = 0$) both bubbles and vortices should be nodal.

In the neighborhood of the origin solutions to (6.5) can be sought as power series in r. One readily verifies that there are two possibilities corresponding to bubble and vortex solutions respectively, *viz*.,

$$\psi(r) = \begin{cases} \psi_0 + O(r^2) & \text{(bubble)} \\ \psi_0 r^{|n|} + O(r^{|n|+1}) & \text{(vortex, } n \neq 0\text{);} \end{cases}$$

$$\phi(r) = n/e + O(r^2);$$

$$A_0(r) = A_0(0) + O(r^2).$$

(Here $\psi_0 \neq 0$.)

Having in mind future numerical solution of the system (6.5), it is also instructive to analyse the asymptotic behaviour at infinity. When $\kappa \neq 0$ eq. (6.5b) allows to express A_0 in terms of ϕ:

$$\kappa r\frac{dA_0}{dr} = \mu r\frac{d}{dr}\frac{\phi_r}{r} - \frac{e^2\rho_0}{m}\phi. \tag{6.6}$$

Using (6.6) and assuming that A_0 and ϕ decay at infinity one can readily conclude that asymptotically $\phi^2/r^2 \ll A_0$. Hence we can neglect the last term in (6.5a). Eq. (6.5a) then becomes

$$-\frac{1}{2m}\Delta\delta\psi + e^2\delta\psi + eA_0 = 0. \tag{6.7}$$

where $c^2 = 2\rho_0 U''(\rho_0)$ and $\psi(r) = \rho_0^{1/2}(1+\delta\psi)$. We assume that $U(\rho)$ has a *minimum* at ρ_0: $c^2 > 0$. Eq. (6.5c) also linearizes asymptotically:

$$-\mu\Delta A_0 + \kappa\frac{\phi_r}{r} - 2e\rho_0\delta\psi = 0. \tag{6.8}$$

Solving the homogeneous linear system (6.6)–(6.8) gives the requisite asymptotic behaviour:

$$\frac{\psi(r)}{\sqrt{\rho_0}} = 1 + \frac{e^{-\gamma r}}{\sqrt{r}}(g_0 + \frac{g_1}{r} + \ldots), \tag{6.9a}$$

$$\phi(r) = \sqrt{r}(\phi_0 + \frac{\phi_1}{r} + \ldots)e^{-\gamma r}, \tag{6.9b}$$

$$A_0(r) = \frac{e^{-\gamma r}}{\sqrt{r}}(\alpha_0 + \frac{\alpha_1}{r} + \ldots), \tag{6.9c}$$

where γ is a root of the following bi-cubic equation:

$$\gamma^2(2mc^2 - \gamma^2)\left(\frac{1}{r_0^2} - \gamma^2\right) = \frac{2e^2\rho_0}{\mu}\left(\frac{2e^2\rho_0}{\mu} - 2m\gamma^2\right), \tag{6.10}$$

with

$$\frac{1}{r_0^2} = \frac{e^2\rho_0}{m\mu} + \frac{\kappa^2}{\mu^2}. \tag{6.11}$$

(More precisely, we should pick up the root with the minimal value of $|\operatorname{Re}\gamma|$.)

Eq. (6.10) can obviously have no negative or zero roots γ^2. This means that $\operatorname{Re}\gamma$ is never zero and, provided $\kappa \neq 0$ and $c^2 > 0$, solutions to the system (6.5) are always exponentially localised.

At the Chern-Simons limit ($\mu = 0$), eq. (6.10) reduces to a bi-quadratic equation:

$$\gamma^2(2mc^2 - \gamma^2) = \left(\frac{2e^2\rho_0}{\kappa}\right)^2. \tag{6.12}$$

(The reason is that eqs. (6.5b) and (6.5c) are of the *first* order now.) The exponent can be easily found,

$$\gamma^2 = 2m\rho_0\left(U''(\rho_0) - \sqrt{U''(\rho_0)^2 - \frac{e^4}{m^2\kappa^2}}\right). \tag{6.13}$$

Clearly for $U''(\rho_0) \geq e^2/(m\kappa)$ the solution approaches the condensate monotonically, while for $0 < U''(\rho_0) < e^2/(m\kappa)$ it undergoes an oscillatory decay to the background.

An interesting phenomenon occurs in the pure Maxwell case ($\kappa = 0$). In this case we cannot solve eq. (6.5b) for A_0 and therefore cannot assert that $\phi^2/r^2 \ll A_0$ in eq. (6.5a). Eq. (6.5b) asymptotically decouples from (6.5a), (6.5c) and can be readily solved:

$$\phi(r) = \sqrt{r}\left(\phi_0 + \frac{\phi_1}{r} + \ldots\right)e^{-r/r_0}, \tag{6.14}$$

with r_0 as in (6.11). The other two equations become a *nonhomogeneous* linear system:

$$-\frac{1}{2m}\Delta\delta\psi + c^2\delta\psi + eA_0 = -\frac{e^2}{2m}\frac{\phi^2}{r^2}, \tag{6.15a}$$

$$-\mu\Delta A_0 - 2e\rho_0\delta\psi = 0. \tag{6.15b}$$

Its general solution is a sum of the general homogeneous solution, decaying as $e^{-\gamma r}$, and a non-homogeneous solution, decaying as ϕ^2/r^2:

$$\frac{\psi}{\sqrt{\rho_0}} = 1 + \frac{e^{-\gamma r}}{\sqrt{r}}\left(g_0 + \frac{g_1}{r} + \ldots\right) + \frac{e^{-2r/r_0}}{r}\left(\tilde{g_0} + \frac{\tilde{g_1}}{r} + \ldots\right)$$
$$A_0 = \frac{e^{-\gamma r}}{\sqrt{r}}\left(\alpha_0 + \frac{\alpha_1}{r} + \ldots\right) + \frac{e^{-2r/r_0}}{r}\left(\tilde{\alpha}_0 + \frac{\tilde{\alpha_1}}{r} + \ldots\right). \tag{6.16}$$

Here γ is a root of

$$\gamma^2(2mc^2 - \gamma^2) = \frac{4me^2\rho_0}{\mu} \tag{6.17}$$

with the minimal (positive) real part. Depending on which exponent decays slower in (6.16), the variation of the matter density $\rho - \rho_0$ and the electric field will be localised at distances $\sim r_0/2$ or $\sim 1/\operatorname{Re}\gamma$. On the other hand, the magnetic field is localised at distances $\sim r_0$ (see eq. (6.14)). Thus in the pure Maxwell case one may have two different localisation scales.

B. Self-dual limits

1. Static solutions correspond to stationary points of the Hamiltonian (4.7) on the constraint manifold (4.6). In the mixed CS-Maxwell case, using the flux-vorticity relation, $\Phi = 2\pi n/e$, and the Bogomol'nyi decomposition,

$$|\mathbf{D}\psi|^2 = |(D_1 \pm iD_2)\psi|^2 \pm \frac{1}{2}\nabla\times\mathbf{J} \pm eB\rho, \tag{6.18}$$

the energy (4.7) takes the form

$$H = \int\left\{\frac{1}{2m}|(D_1 \pm iD_2)\psi|^2 + \frac{\mu}{2}\left(B \pm \frac{e}{\mu}\frac{\rho-\rho_0}{2m}\right)^2 - \frac{e^2}{2\mu}\left(\frac{\rho-\rho_0}{2m}\right)^2 + U(\rho) + \frac{\mu}{2}(\nabla A_0)^2 \pm \frac{1}{4m}\nabla\times\mathbf{J}\right\}d^2x \pm \frac{\pi\rho_0}{m}n. \tag{6.19}$$

For $U(\rho) = e^2(\rho-\rho_0)^2/(4m\mu)$ (which corresponds to the Bose gas with δ-function pairwise repulsion) and fields approaching the condensate background (6.1) the energy can be rewritten as

$$H = \int\left\{\frac{1}{2m}|(D_1 \pm iD_2)\psi|^2 + \frac{\mu}{2}\left(B \pm \frac{e}{\mu}\frac{\rho-\rho_0}{2m}\right)^2 + \frac{\mu}{2}(\nabla A_0)^2\right\}d^2x \pm \frac{\pi\rho_0}{m}n. \tag{6.20}$$

The lower bound of energy, $H = \pm\pi\rho_0 n/m$ is saturated when the following self-duality equations are satisfied:

$$(D_1 \pm iD_2)\psi = 0, \tag{6.21a}$$

$$B \pm \frac{e}{\mu}\frac{\rho-\rho_0}{2m} = 0, \tag{6.21b}$$

$$A_0 = 0. \tag{6.21c}$$

The upper (lower) sign should be associated with the positive (negative) vorticity n. This can be concluded, for example, simply from the fact that the energy is positive for positive $U(\rho)$ (see eq. (4.7)). Comparing (6.21b) with (4.6) we see that solutions of eqs. (6.21) lie on the constraint manifold only for $\kappa = 2m\mu$ and only in the case of the upper sign. Thus, for $\kappa > 0$, only vortices with positive vorticities may exist. Eq. (6.21a) yields

$$A^i = \pm\frac{1}{2e}\epsilon_{ij}\partial_j \ln\rho + \frac{1}{e}\partial_i \mathrm{Arg}\psi \tag{6.22}$$

and we should retain only the upper sign here. Substituting this into (4.6) we arrive at

$$\nabla^2 \ln\rho = 2\frac{e^2}{\kappa}(\rho - \rho_0). \tag{6.23}$$

The corresponding magnetic field is then

$$B = -\frac{e}{\kappa}(\rho - \rho_0). \tag{6.24}$$

Eq. (6.23) appeared previously in the self-dual limit of the relativistic Higgs model with the Maxwell term [15] and is known to possess solutions with the "topological vortex" asymptotic behaviour: $\rho(\infty) = \rho_0$, $\rho(0) \sim r^{2n}$, $n \geq 1$. However, no explicit solutions have been found. For $\rho_0 \neq 0$, it does not pass the Painlevé test [16] and therefore is nonintegrable. Nevertheless eq. (6.23) was proved to possess multivortex solutions [17] for which no closed form representations exist but which can be obtained, for instance, numerically. In Fig.1, we plot the $n = 1$ solution of eq. (6.23) together with the corresponding $B(r)$, eq. (6.24).

2. The self-duality reduction is also possible in the pure Chern-Simons case ($\mu = 0$). Making use of the identity (6.18) and the constraint (4.6) the energy (4.7) can be represented as

$$H = \int \left[\frac{1}{2m}|(D_1 \pm iD_2)\psi|^2 \mp \frac{e^2}{2m\kappa}(\rho - \rho_0)\rho + U(\rho)\right] d^2x. \tag{6.25}$$

Invoking again the integrated constraint, eq. (6.3), we rewrite this as

$$H = \int \left[\frac{1}{2m}|(D_1 \pm iD_2)\psi|^2 \mp \frac{e^2}{2m\kappa}(\rho - \rho_0)^2 + U(\rho)\right] d^2x \pm \frac{\pi\rho_0}{m}n.$$

With the choice of $U(\rho) = \pm e^2(\rho - \rho_0)^2/(2m\kappa)$ one observes that the energy is minimal provided $(D_1 \pm iD_2)\psi = 0$ whence we have, as before, eq. (6.22). Substituting this into the constraint equation (4.6) we arrive at

$$\nabla^2 \ln\rho = \pm 2\frac{e^2}{\kappa}(\rho - \rho_0) \tag{6.26}$$

and

$$B = -\frac{e}{\kappa}(\rho - \rho_0). \tag{6.27}$$

Note that no equation for A_0 arises here. This is not surprising as A_0 is not a dynamical variable. (In our Hamiltonian formulation it is just a Lagrange multiplier.) For any combination of the Maxwell and Chern-Simons terms it can be determined from eq. (2.4b),

$$\mu\nabla \times B + \kappa\nabla \times A_0 = e\mathbf{J}. \tag{6.28}$$

The matter current is calculated from eq. (6.22): $\mathbf{J} = \mp(1/2m)\nabla \times \rho$. Substituting this into eq. (6.28) for $\mu = 0$ we can readily solve for A_0:

$$A_0 = \mp\frac{e}{2m\kappa}(\rho - \rho_0). \tag{6.29}$$

3. In contrast to the mixed CS-Maxwell case, *both* signs are allowed in (6.26). As a result the pure Chern-Simons model exhibits a wider variety of self-dual solutions. Besides conventionally shaped topological vortices of positive vorticity, arising in the case of the repulsive potential $U(\rho) = e^2(\rho - \rho_0)^2/(2m\kappa)$ (Fig. 1), it admits a new type of solutions corresponding to a self-gravitating gas, $U(\rho) = -e^2(\rho - \rho_0)^2/(2m\kappa)$ [negative sign in (6.26)]. These solutions approach the condensate in an oscillatory fashion:

$$\rho(r) \to \rho_0 \exp\left\{\frac{C_1}{\sqrt{r}} \cos\left(\sqrt{\frac{2e^2\rho_0}{\kappa}}r - C_2\right)\right\}, \tag{6.30}$$

as $r \to \infty$. Here $C_1, C_2 =$ const. At the origin $\rho(0)$ can be both zero and nonzero; more precisely, as $r \sim 0$,

$$\psi(r) = \sqrt{h_0}\left[1 + \frac{e^2(\rho_0 - h_0)}{4\kappa}r^2 + \ldots\right] \tag{6.31a}$$

or

$$\psi(r) = \sqrt{h_0}r^n\left(1 + \frac{e^2\rho_0}{4\kappa}r^2 + \ldots\right)e^{-in\theta}, \tag{6.31b}$$

with $h_0 =$ const and n positive integer. These two types of oscillating solutions are presented in Figs. 2a and 2b, respectively. Because of the slow approach of these solutions to ρ_0 (see eq. (6.30)) the corresponding flux, number of particles, and consequently energy, are infinite. Naturally, this fact reduces the interest in these solutions.

When $\rho_0 \to 0$, the period of oscillations in (6.30) becomes infinite and the oscillatory solutions pass to the lumps of Jackiw and Pi [5,6]. The vorticity-free solutions, eq. (6.31a) (Fig. 2a), pass to the lumps with $n = 0$, whereas solutions of the second type, eq. (6.31b) (Fig. 2b) pass to the lumps with negative vorticity.

4. Finally it is appropriate to mention that eq. (6.23) arising in both self-dual limits and pertaining to the repulsive potential $U = e^2(\rho - \rho_0)^2/(2m\kappa)$, admits also vorticity-free lump-like solutions [$\rho(r) \to 0$ as $r \to \infty$]. A straightforward phase space analysis shows that $\rho(0)$ can be any number between 0 and ρ_0. As $\rho_0 \to 0$, these solutions disappear.

VII. CONCLUDING REMARKS

Jackiw and Pi's model of self-attractive boson gas interacting with a Chern-Simons-Maxwell field is known to support asymptotically vanishing soliton solutions. This model, however, does not have the condensate, the ground state solution of uniform nonzero density. No condensate is formed even if the sign of the scalar self-interaction is reversed, i.e. if *repulsive* bosons are considered. In order to include asymptotically nonvanishing fields more fundamental modifications have to be made to the theory. In our previous paper we proposed to modify the "standard model" by adding the background electric charge of a uniform density. In the pure Chern-Simons limit ($\mu = 0$) the arising theory provides a phenomenological description of the fractional quantum Hall effect [12]. In the pure Maxwell case ($\kappa = 0$) it is relevant for the superconductivity. In fact, in real superconductors the condensate of Cooper pairs possesses a nonzero electric charge which is neutralised by the background positive ions. Therefore, our model can be more relevant to real superconductors than the standard Maxwell theory without the background charge density [18].

We reformulated the new model as a constrained Hamiltonian system. Apart from its importance for the quantization of the theory, the Hamiltonian formulation has two virtues on the classical level. Firstly, it allows to find self-dual solutions. Secondly, it provides a natural framework for the analysis of symmetries and conservation laws. In this paper we have mainly been concerned with the linear momentum. We have demonstrated that the momentum of topologically nontrivial field configurations can only be defined as a translationally noninvariant quantity. An important consequence of the translational noninvariance is that, as opposed to the relativistic system, there are no *travelling* vortices here. All vortices are "frozen" at their positions.

The anomalous commutation relation (5.6) deserves a special comment. A similar noncommutativity of translation operators — "spontaneous algebra breaking" — was observed by Chen *et al* in the second-quantised description of the anyon gas in a constant external magnetic field [19]. The algebra is restored after the gauge field has been made dynamical. (For a generalisation to CS-Maxwell systems and for the analysis on a torus see [20] and [21], respectively.) Banerjee, however, calls these conclusions "naive and ill founded" on the grounds that even "a proper candidate for the translation generator is unavailable" and "it is difficult to give a proper definition for the momentum operator" [22].

In sec.V we have demonstrated how can the "proper" momentum be defined in the model with the background charge density. The momentum we define is functionally differentiable and gauge-invariant; it commutes with the Hamiltonian and generates translations on gauge-invariant fields. Although this definition is not unique, the relation

$$\{P^1, P^2\} = 2\pi\rho_0 n \tag{7.1}$$

will hold true for *any* "proper" choice of P^i.

The source of the noncommutativity is clearly the term $e\rho_0\mathbf{x} \times B(\mathbf{x})$ in the momentum density (5.4). Physically, this term is related to the momentum acquired by the charged background in the magnetic field of the vortex, $B(\mathbf{x})$.

If the Lagrangian of our model comprises the Chern-Simons term, it can be reformulated as a theory in the external magnetic field $B^{\text{ext}} = e\rho_0/\kappa$. The electric charge of the configuration with the topological charge n is given by eq. (6.3): $Q = -2\pi\kappa n/e$. Eliminating ρ_0 and n from (7.1), we obtain

$$\{P^1, P^2\} = -QB^{\text{ext}},$$

which is exactly the quantum mechanical result of Chen *et al* [19].

ACKNOWLEDGMENTS

It is a pleasure to thank J. Burzlaff, J. Dziarmaga, L. S. O'Raifeartaigh and T. Tchrakian for instructive discussions. We have also benefited from comments by A. Jevicki, M. Plushchaj, P. Wiegmann and P. Winternitz. One of the authors (I. Barashenkov) wishes to cordially thank F. Lambert and his colleagues at Vrije Universiteit Brussel for their warm hospitality. A. Harin gratefully acknowledges support from the Visiting Lecturer's Fund of the University of Cape Town. This research was supported by FRD of South Africa and a research grant from UCT.

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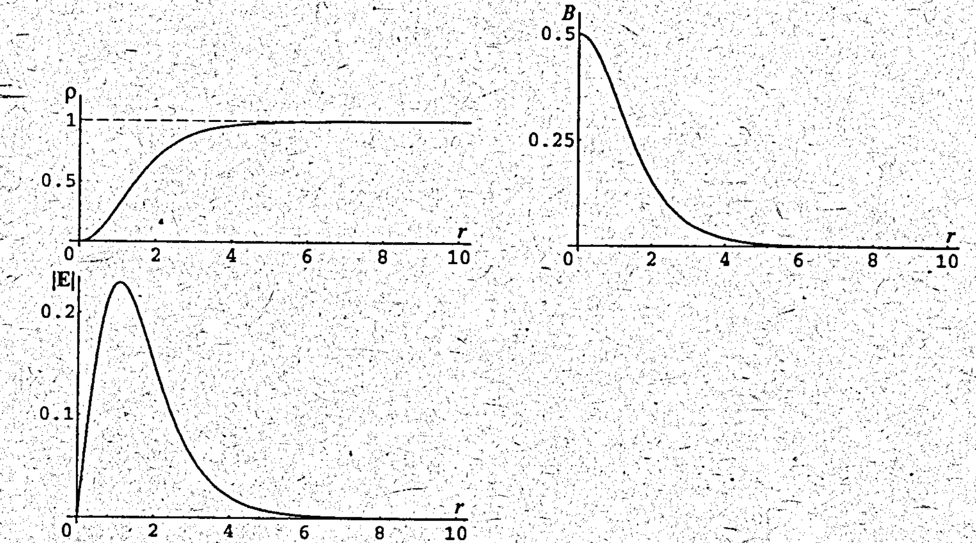

FIG. 1. The $n = 1$ self-dual vortex. 1a and 1b correspond to both the Maxwell-Chern-Simons ($\kappa = 2m\mu$) and the pure Chern-Simons ($\mu = 0$) case. Fig. 1a shows the solution of the Liouville-like equation (6.22a) while on 1b the corresponding magnetic field is plotted, eq. (6.22b). Fig. 1c shows the electric field carried by the vortex in the pure CS case, $\mathbf{E} = -\nabla A_0$ with $A_0 = -(e/2m\kappa)(\rho - \rho_0)$. (In the Maxwell-Chern-Simons case the vortex carries *no* electric field.)

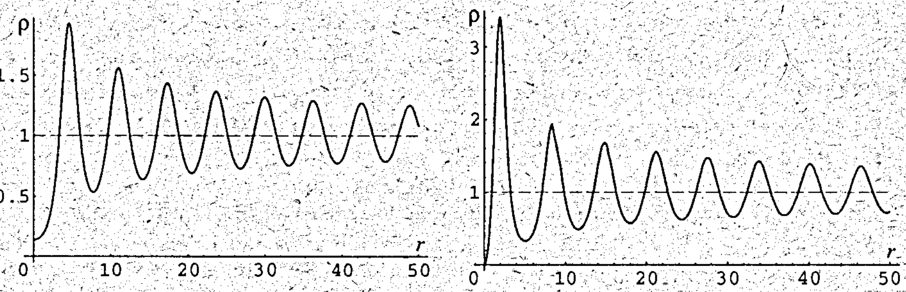

FIG. 2. Oscillating solutions in the case of attraction, $U = -(e^2/2m\kappa)(\rho - \rho_0)^2$. 1a. Vorticity-free solution. 1b. Solution with a negative vorticity.

Received by Publishing Department
on July 12, 1995.