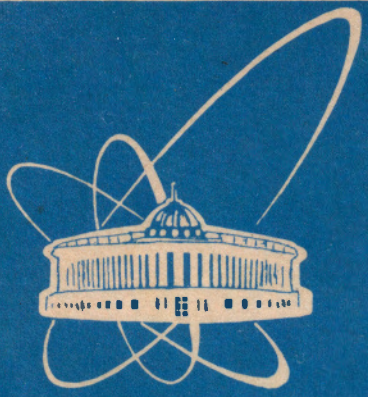

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MEASUREMENT OF SPIN STATE USING STERN-GERLACH DEVICES

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1. Introduction

Quantum measurement of an individual physical system gives a particular value of the observable. The value is usually of little interest. It is the probabilities of particular values which are of real importance. But the probabilities are the property of the state of the system under measurement and not of the observable. This paper deals with the determination of the unitial spin wave function, i.e., modules and phases of a finite number of the function components (two for spin 1/2). This determination is possible using the known Stern-Gerlach device (see [1] and sect. 3.5. below) which is the popular example of the canonical scheme of quantum measurement, e.g., see [2—4]. The first purpose of this paper is to show that this determination can be done in a way, which is beyond the canonical scheme, by using a version of the Stern-Gerlach experiment with a weak purely inhomogeneous magnetic field $\vec{b}(\vec{x})$ (for definition see sect.2.2 below). The second purpose is to discuss under what condition the initial spin state does not change during its measurement and remains the same after the measurement as before it.

Section 2 presents a model of the Stern-Gerlach experiment with a weak inhomogeneous field $\vec{B}(\vec{x}) = \vec{B}_c + \vec{b}(\vec{x})$. New peculiarities that arise in this case (as compared to the conventional Stern-Gerlach case) are stated in sect.5. They are basically represented (and are most prominent) when $\vec{B}(\vec{x})$ is the purely inhomogeneous field $\vec{b}(\vec{x})$ (the case $\vec{B}_c = 0$). This is the reason for calling section 2 «The Stern-Gerlach experiment with a weak purely inhomogeneous magnetic field». This version is compared in sect.3 with the usual Stern-Gerlach measurement.

In both the sections 2 and 3 we use from the outset the inhomogeneous magnetic field $\vec{B}(\vec{x})$ which satisfies the equations div $\vec{B} = 0$ and rot $\vec{B} = 0$ (we consider the most general case of this type of a field provided it is linear in $\vec{x}$). The so-called «impulsive measurement» approximation [2], [1] is not employed in this paper.

In sect.4, we discuss a reductionless way of the spin state determination suggested by Aharonov, Anandan and Vaidman [5]. The peculiarities of the spin state determination proposed in sect.2 are commented in sect.5.

2. Version of the Stern-Gerlach Experiment with a Weak Purely Inhomogeneous Magnetic Field

Global movement of atoms or other neutral particles in the Stern-Gerlach magnetic field depends upon particles' spin state. So the measurement of the particles' position or momentum allows one to obtain information on a spin state. This measurement is supposed to obey the usual postulates of the quantum measurement.

2.1. In the real Stern-Gerlach experiment the beam of atoms passes through a region where there is a constant inhomogeneous magnetic field. The following model of the experiment is used here.

Unitially at $t = 0$ the atom state is described by a wave packet, e.g., being at rest. At $t = 0$ a magnetic field $\vec{B}$ is turned on in the region of the packet localization. The field is turned out at $t = T$ and then the atom momentum is measured. The model Hamiltonian is

$$H = p^2/2M + g(t)\,\mu\vec{\sigma}\cdot\vec{B}. \tag{1}$$

Here $\vec{p} = -i\vec{\nabla}_x$ is the atom momentum operator conjugated to the atom center-of-mass coordinate $\vec{x}$; M is the atom mass; 2μ is its magnetic moment; $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices. The function $g(t)$ is zero outside the interval $(0, T)$ and realizes the smooth turning on and out of the magnetic field. In the most part of $(0, T)$ the function $g(t)$ is unity so that $\int_0^T g(t)dt \cong T$.

There are more realistic models of the Stern-Gerlach device, see, e.g., [6], but the described one is simple and is of use [1].

2.2. Let us stress that the equations div $\vec{B} = 0$ and rot $\vec{B} = 0$ must be satisfied when describing the inhomogeneous part of $\vec{B} = \vec{B}(\vec{x})$ (rot $\vec{B} = 0$ is the Maxwell equation in the region where the current density generating the magnetic field is zero). The homogeneous ($\vec{x}$ independent) part of $\vec{B}(\vec{x})$, let us call it $\vec{B}_c$, of course satisfies these equations. The simplest inhomogeneous magnetic field (i.e., linear in $\vec{x}$ and vanishing at $\vec{x} = 0$) is of the form $b_i(\vec{x}) = \sum_j L_{ij}x_j$; $i,j = 1,2,3$. The equations div $\vec{b} = 0$ and rot $\vec{b} = 0$ lead to $\sum_i L_{ii} = 0$ and $L_{ij} = L_{ji}$, see [7]. The real symmetric matrix L is hermitian and can be diagonalized, i.e., be represented as

$$L_{ij} = \sum_\alpha \langle i|\nu_\alpha\rangle \beta_\alpha \langle \nu_\alpha|j\rangle, \quad \alpha = 1,2,3,$$
$$\beta_1 + \beta_2 + \beta_3 = 0, \tag{2}$$

where its eigenvectors are orthonormal and real: $\langle i|\nu_\alpha\rangle = \langle \nu_\alpha|i\rangle = (\vec{\nu}_\alpha)_i$. So

$$b_i(\vec{x}) = \sum_\alpha \sum_j \langle i|\nu_\alpha\rangle \beta_\alpha \langle \nu_\alpha|j\rangle x_j$$

$$\vec{B}(\vec{x}) = \vec{B}_c + \vec{b}(\vec{x}) = \vec{B}_c + \sum_\alpha \beta_\alpha(\vec{x}\cdot\vec{\nu}_\alpha)\vec{\nu}_\alpha. \tag{3}$$

The field is called purely inhomogeneous. For its realization see [8].

2.3. Let the initial atom state be described by the vector $\psi_0 = \varphi_0(\vec{x})\chi_0(m)$, φ_0 and χ_0 having a unit norm. The function $\varphi_0(\vec{x})$ is a wave packet located in a region between current wires creating $\vec{B}(\vec{x})$. The spin wave function $\chi_0(m)$, m being spin projection, is unknown and must be determined. Let us represent the solution of the Schroedinger equation

$$i\partial_t\psi(t) = [p^2/2M + g(t)\,\mu\,\vec{\sigma}\cdot\vec{B}(\vec{x})]\psi(t), \quad \psi(t=0) = \psi_0 \tag{4}$$

as $\psi(t) = \exp(-iH_0t)\psi_I(t)$, $H_0 = p^2/2M$. Then, one gets for $\psi_I(t)$ the interaction picture equation

$$i\partial_t\psi_I(t) = H_I(t)\psi_I(t) = g(t)\,\mu\,\vec{\sigma}\cdot\vec{B}(\vec{x}(t))\psi_I(t), \tag{5}$$

$$H_I(t) \equiv \exp(iH_0t)H_I\exp(-iH_0t),$$
$$\vec{x}(t) = \exp(iH_0t)\vec{x}\exp(-iH_0t) = \vec{x} + t\vec{p}/M. \tag{6}$$

Note that $\vec{\sigma}$ commutes with H_0 and, therefore,

$$\vec{\sigma}(t) \equiv \exp(iH_0t)\,\vec{\sigma}\exp(iH_0t) = \vec{\sigma}. \tag{7}$$

The solution of eq. (5) is represented with the help of T-exponent

$$\psi_I(t) = T\exp[-i\mu\int_0^t dt' g(t')\,\vec{\sigma}\cdot\vec{B}(\vec{x})(t'))]\psi_I(0)$$
$$\psi_I(0) = \psi_0 \tag{8}$$

because $[H_I(t_1), H_I(t_2)] \neq 0$. For the T exp definition see, e.g., eq. (21) below.

Let us assume that the interaction H_I is weak and, therefore, one may calculate $\psi_I(t)$ using perturbation theory. This means that the magnetic field or/and particle magnetic moment are supposed to be small. The approximation is contrary to the usual impulsive one, when one supposes that H_0 may be

neglected as compared to H_I in the interval $(0, T)$, see, e.g., [1,2]. Then, one has

$$\psi_I(t) \cong \left\{1 - i\mu \int_0^t dt' g(t') \vec{\sigma}\cdot\vec{B}(\vec{x}(t'))\right\} \psi_0. \tag{9}$$

2.4. Let us show that the measurement of the atoms' momentum distribution at $t > T$ allows one to determine χ_0. For this purpose, calculate the average value of the momentum operator $\vec{p}$ at $t > 0$ in the state $\psi(t) = \exp(-iH_0 t)\psi_I(t)$

$$\langle p_k \rangle \equiv \langle e^{-iH_0 t} \psi_I(t) | p_k | e^{-iH_0 t} \psi_I(t) \rangle = \langle \psi_I(t) | p_k | \psi_I(t) \rangle \cong$$

$$\cong \langle \psi_0 | \left\{1 + i\mu \int_0^t dt' g(t) \vec{\sigma}\cdot\vec{B}(\vec{x}(t'))\right\} p_k \left\{1 - i\mu \int_0^t dt' g(t') \vec{\sigma}\cdot\vec{B}(\vec{x}(t'))\right\} | \psi_0 \rangle \cong$$

$$\cong \langle \varphi_0 | p_k | \varphi_0 \rangle + i\mu \int_0^t dt' \langle \psi_0 | [\vec{\sigma}\cdot\vec{b}(\vec{x}(t')), p_k] | \psi_0 \rangle =$$

$$= \langle p_k \rangle_0 - \mu \int_0^t dt' g(t') \sum_\alpha \beta_\alpha \vec{\zeta}_0 \cdot \vec{v}_\alpha (\vec{v}_\alpha)_k, \tag{10}$$

$$\vec{\zeta}_0 \equiv \langle \chi_0 | \vec{\sigma} | \chi_0 \rangle.$$

We have used here eqs. (9) and (3). Let us direct the coordinate axes along $\vec{v}_\alpha$ so that $(\vec{v}_\alpha)_k = \delta_{\alpha k}$. Then, at $t > T$ one can rewrite eq. (10) as

$$\langle p_k \rangle = \langle p_k \rangle_0 - \mu T \beta_k \zeta_{0k}, \tag{11}$$

where $\langle p_k \rangle_0$ is the k-th projection of the initial average atom's momentum $\vec{p}_0$. Note that $\langle p_k \rangle$ does not depend upon $\vec{B}_c$ but only on $\vec{b}(\vec{x})$. If one knows μ, T, $\beta_k \neq 0$ and measures $\langle \vec{p} \rangle - \vec{p}_0$, then one can determine ζ_{0k}, i.e., the initial polarization vector. This is equivalent to the determination of the initial spinor χ_0 which is the $\vec{\sigma}\cdot\vec{\zeta}_0$ eigenfunction describing the spin directed along $\vec{\zeta}_0$, see Appendix. Note that if $\beta_k = 0$ for some value of k, then the related projection ζ_{0k} cannot be found.

Note that the result (11) is true also in the case when the atom's initial spin state is described by a density matrix ρ_0 or by the polarization vector $\vec{\zeta}_0 = \mathrm{Tr}\,\rho_0 \vec{\sigma}$.

Using eq. (9) one may calculate

$$\tilde{\psi}_I(\vec{p}, m, t) = \int d^3x\, e^{-i\vec{p}\vec{x}} \psi_I(\vec{x}, m, t) \tag{12}$$

and show that the maximum of the momentum distribution $\sum_m |\tilde{\psi}_I(\vec{p}, m, t)|^2$ is shifted as compared to the maximum of $|\tilde{\varphi}_0(\vec{p})|$, and the shift value is the same as $\langle \vec{p} \rangle - \vec{p}_0$, i.e., is equal to $\mu T \beta_k \zeta_{0k}$.

2.5. Let us investigate under what conditions the atom spin state after measurement is the same as the initial one. First of all, the state must not change when atom interacts with the magnetic field (during «premeasurement« [9]). After that it must not change when measuring the atoms' momentum distribution after this interaction (this is needed to calculate $\langle p_k \rangle$).

For this investigation one must use the description of the spin state at $t > 0$ by the density matrix

$$\rho(m_1, m_2) = \int d^3x\, \psi(\vec{x}, m_1, t)\, \psi^*(\vec{x}, m_2, t), \tag{13}$$

because the atom wave function $\psi(\vec{x}, m, t)$ has not the form of a product $\varphi(\vec{x}, t)\chi(m, t)$ see, e.g., eq. (9). Instead of ρ one can equivalently use the polarization vector [10]

$$\vec{\zeta} = \mathrm{Tr}\,\rho\vec{\sigma} = \int d^3x \sum_{m_1 m_2} \psi^*(\vec{x}, m_2, t)\, \vec{\sigma}_{m_2, m_1} \psi(\vec{x}, m_1, t) \tag{14}$$

because $\rho = (1 + \vec{\zeta}\vec{\sigma})/2$.

Using eq. (9) one can calculate $\vec{\zeta}$ in complete analogy with the $\langle p_k \rangle$ calculation in sect. 2.4

$$\zeta_k = \langle \psi_I(t) | \sigma_k | \psi_I(t) =$$

$$= \zeta_{0k} - \mu \int_0^t dt' g(t') \sum_{m,n} \varepsilon_{mkn} \langle \varphi_0 | B_{cm} + b_m(\vec{x}(t')) | \varphi_0 \rangle \zeta_{0n}. \tag{15}$$

The term $\sim |B|^2$ is neglected, ε_{mkn} is the unit antisymmetric tensor. The second term in the r.h.s. of eq. (15) vanishes if $\vec{B}_c = 0$ or $\vec{B}_c \parallel \vec{\zeta}_0$ and if $\langle \varphi_0 | b_m(\vec{x}(t')) | \varphi_0 \rangle = 0$. The calculation of $\langle \varphi_0 | b_m | \varphi_0 \rangle$ due to eqs. (3) and (6) is reduced to the integrals

$$\int d^3x\, \varphi_0^*(x) \vec{x} \varphi_0(\vec{x}), \quad \int d^3x\, \varphi_0^*(\vec{x})(-i\vec{\nabla}_x)\, \varphi_0(\vec{x}). \tag{16}$$

They vanish if $\varphi_0(\vec{x}) = \varphi_0(-\vec{x})$ and average momentum of the state φ_0 is zero, e.g., $\varphi_0 = f(x^2)$, $f(x^2) \sim \exp(-x^2/l^2)$. The first integral vanishes also in

the case of the moving packet $\varphi_0(\vec{x}) = \exp(i\vec{p}_0\vec{x})\, f(x^2)$ but the second is then equal to $\vec{p}_0$. The second term in the r.h.s. of eq. (15) will also vanish in this case if one takes instead of (3) the «running» field $\vec{b}_t$ which moves together with the packet

$$\vec{b}(\vec{x}) \to \vec{b}_t(\vec{x}) = \vec{b}(\vec{x} - \vec{\vartheta}t), \quad \vec{\vartheta} = \vec{p}_0/M. \tag{17}$$

So we have approximately $\vec{\zeta} = \vec{\zeta}_0$ in the case of purely inhomogeneous magnetic field and under the conditions described above. In particular, the spin state does not change provided the atom packet is at rest and is located symmetrically in the field, i.e., $\varphi_0(\vec{x}) = \varphi_0(-\vec{x})$.

This result can also be obtained by using $\tilde{\psi}_I(\vec{p}, m, t)$, see eq. (12), instead of $\psi_I(\vec{x}, m, t)$. Then, the integrals

$$\int d^3p\, \tilde{\varphi}_0^*(p)\, i\vec{\nabla}_p\, \tilde{\varphi}_0(\vec{p}), \quad \int d^3p\, \tilde{\varphi}_0^*(\vec{p})\, \vec{p}\, \tilde{\varphi}_0(\vec{p}) \tag{18}$$

appear instead of (16), the integration being over all possible $\vec{p}$ values belonging to supp $\tilde{\varphi}_0$.

Let us now examine the change of the spin state when the atom momentum is measured. To determine the average momentum $\langle \vec{p} \rangle$, see eq. (11), one must find the atoms' momentum distribution, i.e., the probabilities of finding the momentum in the regions $S(\vec{p}, \Delta)$, $\vec{p}$ being the region center and Δ its dimension. Each of these probabilities is given by the expectation value of the projection observable

$$\prod_p (\Delta) = \int\limits_{S(\vec{p},\Delta)} d^3p' |\vec{p}'\rangle\langle\vec{p}'|$$

in the state $\psi(T)$ (momentum distribution does not change after T). For the subensemble of atoms which suffer the reduction in $S(\vec{p}, \Delta)$ the equality $\vec{\zeta} = \vec{\zeta}_0$ does not hold in general, because the integrals (18) do not vanish in general if they are taken over $S(\vec{p},\Delta)$. The exception is the case when the center $\vec{p}$ of $S(\vec{p},\Delta)$ coincides with $\langle \vec{p} \rangle$ and its dimension Δ is of an order of momentum uncertainty in the state $\psi(T)$, i.e., when $S(\vec{p}, \Delta)$ is the effective supp $\tilde{\psi}(T)$. The expectation value of the corresponding projector is almost unity: almost all atoms reduce in the exceptional momentum region. Therefore, $\vec{\zeta} = \vec{\zeta}_0$ can be valid for the case. But the exceptional region is unknown until $\langle \vec{p} \rangle$ is determined.

3. Comparison with the Conventional Stern-Gerlach Measurement

The treatment of the conventional Stern-Gerlach measurement needs an explicit expression for $\psi(t) = \exp(-iH_0t)\psi_I(t)$ while only average values have been calculated in sect. 2.

3.1. Let us argue that under some condition one can simplify significantly eq. (8) for $\psi_I(t)$ replacing $\vec{\sigma}\cdot\vec{B}(\vec{x}(t'))$ by $\vec{\sigma}\cdot\vec{B}_c + \vec{\sigma}\cdot\vec{b}(\vec{x})$, i.e., neglecting the term $t\vec{\sigma}\cdot\vec{b}(\vec{p})/M$, see eq. (6) for $\vec{x}(t)$.

In the first approximation of perturbation theory one has

$$\psi_I(t) = 1 - i\mu \int_0^t dt' g(t') \vec{\sigma}\cdot\vec{B}(\vec{x}(t'))\psi_c = \psi_0 + \psi_B + \psi_x + \psi_p, \tag{19}$$

$$\psi_x = -i\mu \int_0^t dt' g(t') \vec{\sigma}\vec{b}(\vec{x})\varphi_0\chi_0,$$
$$\psi_p = -i\mu \int_0^t dt' g(t') \frac{t'}{M} \vec{\sigma}\vec{b}(\vec{p})\varphi_0\chi_0. \tag{20}$$

Let us compare the ψ_x and ψ_p norms in the case of the immoveable packet of the dimension l, e.g., $\varphi_0(\vec{x}) = f(x^2) = N \exp(-x^2/l^2)$.

Suppose l to be a macroscopic length, e.g., $l = 0{,}1$ cm. In the first integral of eqs. (20) we have $x\varphi_0 \approx l\varphi_0$ whereas in the second one

$$\frac{t}{M} p_k \varphi_0 \sim \frac{t}{M}\frac{h}{l}\varphi_0 = ct\frac{\lambda_c}{c}\varphi_0,$$

where $\lambda_c = h/Mc$ (the possible momentum values in the initial state are confined in the sphere of the radius h/l). So the ratio $\|\psi_p\|/\|\psi_x\|$ is of the order $\lambda_c/l \cdot cT/l$ where $\lambda_c/l \sim 10^{-13}$ (the hydrogen mass is subsituted for M).

It is reasonable to impose the requirement $q \lesssim h/l$ for the momentum change $\vec{q}$ of the atom beam after its interaction with the magnetic field during the time interval $(0, T)$ (see eq. (26) below for $\vec{q}$). As $q_k \sim \mu T \beta_k$ we derive an estimation on T and cT/l:

$$cT/l \equiv cT\mu\beta_k/l\mu\beta_k \gtrsim c\frac{h}{l}(l\mu\beta_k)^{-1}.$$

The inhomogeneous field $b(\vec{x})$ changes from zero to $l\beta_k$ in the region of the packet localization, see eq. (3). Assuming $l\beta_k = 1$ gauss, $l = 0{,}1$ cm and using $\mu \cong 10^{-20}$ erg/gauss, $c = 3\cdot 10^{10}$ cm/sec, $h \cong 10^{-27}$ ergsec we get the estimate $cT/l \sim 10^4$. So the ratio $\|\psi_p\|/\|\psi_x\|$ turns out to be of the order $10^{-13}\cdot 10^4 \ll 1$ (note that the ratio $\sim b^{-1}$).

We conclude that ψ_p in eq.(19) can be neglected as compared to ψ_x in the case of immoveable packet.

Note that the packet diffusion is quite negligible during the estimated time $T \sim 10^4 l/c \sim 10^{-7}$ sec.

The inequality $||\psi_p|| << ||\psi_x||$ may fail in the case of a moveable packet $\varphi_0(x) = \exp(i\vec{p}_0\vec{x}) f(x^2)$ if $p_0 >> h/l$. In this case, one must replace $\vec{b}$ by the «running» field b_t, see eq.(17). Then, $\vec{b}(\vec{p} - \vec{p}_0)$ substitutes $\vec{b}(\vec{p})$ in ψ_p and $||\psi_p|| << ||\psi_x||$ as before.

One can apply the above considerations to the expression

$$\psi_I(t) = T \exp\left[-i \int_0^t dt' H_I(t')\right]\psi_0 =$$

$$= \lim_{n \to \infty} [1 - i \int_{t_n}^t dt' H_I(t')] \ldots [1 - i \int_{t_1}^{t_2} dt' H_I(t')][1 - i \int_0^{t_1} dt' H_I(t')]\psi_0 \quad (21)$$

to justify the omission of the term $t\vec{\sigma}\cdot\vec{b}(\vec{p})/M$ beyond perturbation theory.

3.2. The omission of the term $t\vec{\sigma}\cdot\vec{b}(\vec{p})/M$ in eq.(8) turns T exp into the simple exp because $[H_I'(t_1), H_I'(t_2)] = 0$ if

$$H_I'(t) = g(t)\mu\vec{\sigma}\cdot\vec{B}(\vec{x}), \quad \vec{B}(\vec{x}) = \vec{B}_c + \vec{b}(\vec{x}). \quad (22)$$

Now one gets

$$\psi_I(t) \cong \exp\left[-ig_1(t)\mu\vec{\sigma}\cdot\vec{B}(\vec{x})\right]\varphi_0\chi_0 = \varphi_0(\vec{x}) \exp\left[-ih\vec{\sigma}\cdot\vec{n}(\vec{x})\right]\chi_0, \quad (23)$$

$$g_1(t) \equiv \int_0^t dt' g(t'), \quad h \equiv g_1(t)\mu|\vec{B}(\vec{x})|, \quad \vec{n}(\vec{x}) = \vec{B}(\vec{x})|\vec{B}(\vec{x})|^{-1}. \quad (24)$$

The operator $\exp(-ih\vec{\sigma}\cdot\vec{n})$ in eq.(23) is the known operator of the spin function rotation around the $\vec{n}$ direction, see, e.g., [11]. So eq.(23) describes precession of the atom spin (directed initially along the unit vector $\vec{\xi}_0$) around $\vec{n}$. If $B_c >> b(\vec{x})$ at $\vec{x} \in \operatorname{supp} \varphi_0(\vec{x})$, then $\vec{n}$ depends on $\vec{x}$ weakly, $\vec{n} \cong \vec{B}_c/B_c$, and the rotation axis is common for the whole packet.

3.3. Several exact expressions for the r.h.s. of eq.(23) can be written. Using $(\vec{\sigma}\cdot\vec{n})^2 = 1$ one obtains [11]

$$\psi_I(t) = \varphi_0(\vec{x})\{\cos h - i\vec{\sigma}\cdot\vec{n}(\vec{x}) \sin h\}\chi_0. \quad (25)$$

Eq. (25) allows one to obtain the results of sect.2. As before, $\vec{\xi} = \vec{\xi}_0$ follows if terms of the order h_2 are neglected.

The expression for $\psi_I(t)$ which is employed when discussing the conventional Stern-Gerlach experiment can be obtained by a transformation of eq.(25) but a simpler way is to expand χ_0 in eigenstates $\chi_\pm$ of the operator $\vec{\sigma}\cdot\vec{n}(\vec{x})$

$$\chi_0 = A_+\chi_+ + A_-\chi_-, \quad \vec{\sigma}\cdot\vec{n}(\vec{x})\chi_\pm = \pm\chi_\pm, \quad (26)$$

$$A_\pm = \langle\chi_\pm|\chi_0\rangle, \quad \chi_0 = \begin{pmatrix}\cos\theta_0/2 \\ \sin\theta_0/2 \exp i\phi_0\end{pmatrix},$$

$$\chi_+ = \begin{pmatrix}\cos\theta/2 \\ \sin\theta/2 \exp i\phi\end{pmatrix}, \quad \chi_- = \begin{pmatrix}\sin\theta/2 \\ -\cos\theta/2 \exp i\phi\end{pmatrix}, \quad (27)$$

$$\psi_I(t) = \varphi_0(\vec{x})[A_+e^{-ih}\chi_+ + A_-e^{+ih}\chi_-]. \quad (28)$$

Here, θ_0 and ϕ_0 are spherical angles of $\vec{\xi}_0$, θ and ϕ are spherical angles of $\vec{n}(\vec{x})$ which practically do not depend upon $\vec{x}$ if $B_c >> b$. One can show that $\exp(\pm ih)$ in eq.(28) may be approximated by the plane waves if $B_c >> b(\vec{x})$ and $B_c \cdot \vec{b}(\vec{x}) >> b^2(\vec{x})$. For this purpose use eq.(24) for h and the equation

$$B(\vec{x}) = \sqrt{[\vec{B}_c + \vec{b}(\vec{x})]^2} \cong B_c + \vec{n}\cdot\vec{b}(\vec{x}) = B_c + \Sigma_k n_k\beta_k x_k = B_c + \vec{\pi}\cdot\vec{x}$$

$$\pi_k = n_k\beta_k, \quad \vec{n} = \vec{B}_c/B_c. \quad (29)$$

If $\varphi_0 = f(x^2) \exp(i\vec{p}_0\vec{x})$ is the packet having the (almost exact) momentum $\vec{p}_0$, then $\varphi_0 \exp ih$ describes the packet $f(x^2) \exp(i(\vec{p}_0 + \vec{q})\vec{x})$ having the (almost exact) momentum $\vec{p}_0 + \vec{q}$,

$$q_k = q_1(t)\mu\pi_k = g_1(t)\,\mu n_k\beta_k. \quad (30)$$

Here $g_1(t) = T$ at $t > T$, see sect 2.1.

3.4. The vector $\vec{q}$ is not parallel to $\vec{B}_c$ in general. But if one of the $\vec{b}(\vec{x})$ eigenvectors $\vec{v}_\alpha$ (see sect 2.2.), say $\vec{v}_3$, is directed along $\vec{B}_c$, then $n_1 = n_2 = 0$ along with $q_1 = q_2 = 0$, i.e., $\vec{q} \,||\, \vec{B}_c$. In this case $\vec{q}$ and $\psi_I(t)$ depend, in fact, only on β_3, i.e., on the $\vec{b}(\vec{x})$ component parallel to $\vec{B}_c$. This fact gives a simple justification for the approach which from the beginning uses the unrealizable supposition $\vec{b}(x) \,||\, \vec{B}_c$, see, e.g., [2,3]. For another discussion of this topic see, e.g., [1,3].

3.5. At $t > T$ one can rewrite eq. (28) as

$$\psi_I(\vec{x}, m, t) = \varphi_0(\vec{x})[A_+ e^{-i\mu TB_c} e^{-i\vec{q}\vec{x}}\chi_+(m) + A_- e^{+i\mu TB_c} e^{i\vec{q}\vec{x}}\chi_-(m)]. \quad (31)$$

The function $\psi(t) = \exp(-iH_0 t)\psi_I(t)$ describes the propagation and diffusion of the packets $\psi_0(x) \exp(\pm i\vec{q}\vec{x})$.

Eq. (31) reveals the specific correlation (entanglement): the atom has the momentum $\vec{p}_0 - \vec{q}$ if its spin state is χ_+, similarly $\vec{p}_0 + \vec{q}$ is correlated with χ_-, cf. [2,3]. The expansion in eigenstates $\chi_\pm$ of the operator $\vec{\sigma}\cdot\vec{n}$ means that it is this operator which is measured [4]. This can be realized by means of the atom momentum measurement. According to quantum postulates, the probability to find the momentum value in a sphere S_- centered at the point $\vec{p}_0 - \vec{q}$ turns out to be equal to

$$\int_{S_-} d^3p\Sigma_m \left| \int d^3x e^{-i\vec{p}\vec{x}}\psi_I(\vec{x}, m, t) \right|^2 = |A_+|^2. \quad (32)$$

The sphere S_- radius must be greater than the momentum uncertainty h/l in the initial packet φ_0 but smaller than $|\vec{q}|$ (one must suppose that $q > h/l$). Reduction of the atom momentum to S_- corresponds to the reduction of the atom spin state to χ_+. One can show that

$$|A_\pm|^2 = [1 \pm \vec{\zeta}_0\cdot\vec{n}]/2. \quad (33)$$

So the momentum measurement gives information on the $A_\pm$ modules.

Eq. (33) is valid also in the case when the initial spin state is described by a density matrix (the case $\zeta_0^2 < 1$) [1]. In the real Stern-Gerlach experiment, the change of atom's coordinate is measured which is determined by the momentum change $\vec{q}$.

It has been noted in [1] that one can determine also $A_\pm$ phases, i.e., one can find $\chi_0 = A_+\chi_+ + A_-\chi_-$. For this purpose one must repeat the measurement of the same atom's ensemble using two other magnetic fields $\vec{B}'(\vec{x})$ and $\vec{B}''(\vec{x})$. $\vec{B}'(\vec{x})$ is obtained by a rotation of the Stern-Gerlach magnet about the beam axis (it is convenient to have $\vec{n}' \perp \vec{n}$). $\vec{B}''$ must have $\vec{n}''$ which is linearly independent of $\vec{n}$ and $\vec{n}'$. Determination of $|A_\pm|^2$ in three experiments gives three angles between the unknown $\vec{\zeta}_0$ and $\vec{n}, \vec{n}', \vec{n}''$. This allows one to find $\vec{\zeta}_0$, which is equivalent to the χ_0 determination, see Appendix. In the real experiment, to obtain $\vec{n}''$ one should rotate about an axis perpendicular to the

beam. This would result in the beam impinging on the magnetic poles. To circumvent the difficulty, one can turn the atom spin in a known and needed way before atom's entering the Stern-Gerlach magnet (using, e.g., an additional constant magnetic field) [1].

3.6. Precession of the spin about the $\vec{B}_c$ direction (see sect.3.2.) means that the atom's spin state alters when the atom interacts with the magnetic field. The exception is the case $\vec{\zeta}_0 \parallel \vec{B}_c$. This allows one to measure the spin state without changing it in the following way. Determine $\vec{\zeta}_0$ in the way described above. Take the Stern-Gerlach device with $\vec{B}_c \parallel \vec{\zeta}_0$. In this case, $\chi_0 = \chi_+$ and, therefore, $A_- = \langle \chi_- | \chi_0 \rangle = 0$. This means that all atoms acquire momenta belonging to S_- being in the initial spin state χ_0. So having measured χ_0, one may indicate a measurement which determines χ_0 without altering it. Essentially, this is the well-known case when the measured wave function is the eigenstate of the observable. The distinction is that one now fits the observable using some preliminary information.

3.7. The peculiarities of the version considered in sect.2, as compared to the conventional Stern-Gerlach case, are rooted in that the Stern-Gerlach magnetic field $\vec{B}(\vec{x})$ is strong enough for splitting the initial packet φ_0 into two ones $\varphi_0 \exp(\pm i\vec{q}\vec{x})$. One cannot describe the splitting using the first nonvanishing order of the perturbation theory. In this section, we have used instead the exact solution (after simplifying the interaction Hamiltonian, see eq. (22)) together with the inequality $\vec{B}_c >> \vec{b}(\vec{x})$.

There is no such a splitting when atoms interact with the weak field $\vec{B}(\vec{x})$, which was considered in sect.2 (to justify the use of perturbation theory). Instead, φ_0 suffers a shift $\langle p_k \rangle - p_{0k}$, see eq. (11) and a spreading. Measuring the shift allows one to determine χ_0 without the rotations of the Stern-Gerlach device described in sect.3.5.

4. On the «Protective» Spin State Measurement by Aharonov, Anandan and Vaidman

«Protective» measurement suggested by Aharonov, Anandan and Vaidman (AAV) [5] is based upon eq. (30) for the momentum change $\vec{q}$ acquired by atoms when interacting with the magnetic field. Note that to find $|A_+|^2$ using eq. (32), one does not need a precise measurement of $\vec{q}$. The radius of the sphere S_- can be rather large: one must only require that the spheres S_- and S_+

centered at $\vec{p}_0 - \vec{q}$ and $\vec{p}_0 + \vec{q}$ do not intersect. What additional information is provided by a precise $\vec{q}$ measurement? Eq.(30) allows, e.g., determining n_k if β_k and q_k are known. So if the purely inhomogeneous part $\vec{b}(\vec{x})$ of the field $\vec{B}(\vec{x}) = \vec{B}_c + \vec{b}(\vec{x})$ is known, then the atom momentum measurement can play the role of the magnetometer which determines the direction of $\vec{B}_c$ (under the condition $B_c >> b$).

Now the AAV suggestion may be represented as follows. Somebody (he may be called preparater) prepares atoms in a spin state χ_0 (polarization vector $\vec{\zeta}_0$) and simultaneously imposes on the atoms a homogeneous magnetic field $\vec{B}_0 \parallel \vec{\zeta}_0$. The field $\vec{B}_0$ is much stronger than the Stern-Gerlach field $\vec{B}(\vec{x})$ used by another person (experimenter) to determine χ_0 that is unknown to him. $\vec{B}_0$ acts on the atoms at all times. The result is the absence of a sensible spin procession in the field $\vec{B}_0 + \vec{B}(\vec{x})$ during the experiment (see sect. 3.2). Spin state does not change when measuring the atom momentum: one has $\chi_0 \cong \chi_+$ and, therefore, $A_- = \langle \chi_- | \chi_0 \rangle = 0$. This means that all atoms acquire the same (almost precise) momentum $\vec{p}_0 - \vec{q}$ and, therefore, one needs only one atom to measure $\vec{q}$ and then to find $\vec{n}$ which is now practically the $\vec{B}_0$ direction. As $\vec{B}_0 \parallel \vec{\zeta}_0$, this is equivalent to the $\vec{\zeta}_0$ or χ_0 determination. AAV say that $\vec{B}_0$ «protects» χ_0 against changing during the measurement.

I note that the AAV «protection» allows one to measure $\vec{\zeta}_0$ without measuring even a single atom. An experimenter may simply determine the $\vec{B}_0$ direction using a macroscopic magnetometer. It is unreasonable to suppose that the experimenter knows the field $\vec{b}(\vec{x})$ but he is unable to employ a usual magnetometer to determine the stronger field $\vec{B}_0$ which is present along with $\vec{b}(\vec{x})$. So if «protection» is present, then one need not do just the quantum measurement. It is sufficient to measure only «protection».

Note a similarity between the conditions for the reductionless spin state measurement discussed in sections 2.5, 3.6 and the AAV protective condition: somebody (the preparater) must know a spin state in order its reductionless determination can be realized.

5. Comments

5.1. The way of the spin state determination proposed in sect.2 differs in many respects from the canonical scheme of quantum measurement, e.g., see

[2,3,4,9]. The cause of the peculiarities was discussed in sect.3.7, the peculiarities themselves can be stated as follows.

One cannot point out any definite spin observable which is measured, e.g., such as the operator $\vec{\sigma}\cdot\vec{n}$ in the conventional Stern-Gerlach case, see sect.3.5. In this respect, our proposition differs also from the «general quantum theory of measurement» by A.Fine [12].

Our interaction Hamiltonian $H_I = g(t)\,\mu\vec{\sigma}\cdot\vec{B}(\vec{x})$ is not a product of a single spin observable and a single apparatus observable [2]. Three independent spin operators $\sigma_1, \sigma_2, \sigma_3$ enter into H_I. Note that to determine χ_0 using the conventional Stern-Gerlach device, one must also measure successively three independent observables $\vec{\sigma}\cdot\vec{n}$, $\vec{\sigma}\cdot\vec{n}'$, $\vec{\sigma}\cdot\vec{n}''$, see sect.3.5.

The atom wave function $\psi(t)$ during and after the magnetic field action is not an expansion of the kind $\varphi_+(x, t)\chi_+ + \varphi_-(x, t)\chi_-$, χ_+ and χ_- being independent of $\vec{x}$. This expansion is characteristic of the conventional Stern-Gerlach case, see eq.(31). In our case, the expansion is replaced by eq.(28) provided the inequality $\vec{B}_c >> \vec{b}(\vec{x})$ does not hold. Then, $\chi_\pm$ entering into eq.(28) depend appreciably upon $\vec{x}$. This means that there is no strong correlation (entanglement) between the atom spin state and the atom momentum.

The so-called «impulsive measurement» approximation [2] has not been used in this paper. The interval , during which the interaction H_I takes place, is not considered to be small and H_I is not supposed to be stronger than $H_0 = p^2/2M$.

5.2. It has been shown in sections 2.5 and 3.6 that there are exceptional measurements which determine the spin state χ_0 without altering it (so that χ_0 suffers no reduction under the measurement). They must satisfy some conditions. To realize these measurements, one must preliminarily determine χ_0 employing usual measurements which alter χ_0.

The «protection» used in the AAV way of the reductionless spin state determination seems to be an example of too restrictive condition. Under this condition the problem of quantum measurement of the wave function is reduced to a classical measurement of the «protection», see sect.4.

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Appendix. Spin State Description

One can show that an arbitrary normalized spinor $\chi_0 = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ may be represented in the form given in eq.(27) (the common phase of the spinor being irrelevant).

One can verify also that any χ_0 is the eigenstate of a spin operator $\vec{\sigma}\cdot\vec{n}_0$ belonging to the eigenvalue + 1. The spherical angles θ_0, ϕ_0 of the unit vector $\vec{n}_0$ are determined by the spinor components α and β which can be parametrized as $\alpha = \cos\theta_0/2$, $\beta = \sin\theta_0/2 \exp i\phi_0$.

The words «spin is directed along $\vec{n}_0$» may have the sense that the related spin state χ_0 is the eigenstate of the operator $(\vec{\sigma}\cdot\vec{n}_0)$. Another possible sense is that the related polarization vector $\vec{\xi}_0 = \langle \chi_0 | \vec{\sigma} | \chi_0 \rangle$ coincides with $\vec{n}_0$.

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Note added in proof.

Recently two comments on AAV paper [5] have appeared. W.Unruh (Phys.Rev.A, 1994, vol.50, 882) discusses the way of «protection» (measuring interaction switches on and out adiabatically slow) which is different from that used in reality by AAV in their sect.VB. The paper by C.Rovelli (Phys.Rev.A, 1994, vol.50, 2788) deals with the AAV protection by means of the strong magnetic field $\vec{B}_0$. Rovelli argues that AAV suggestion can be reduced to the well-known case of measuring the observable whose eigenstate coincides with the measured state χ_0. I share Rovelli's criticism though my comment on AAV is somewhat different (see the end of my sect.4): the AAV «protection» is so restrictive a supposition that experimenter needs only to measure $\vec{B}_0$ (using a classical magnetometer) in order to obtain the information on χ_0 which already has the preparater. The problem of getting the information is really moved aside to the preparater.

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