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RESONANCE PHENOMENON AS A TRANSMUTATION OF THE QUASIENERGY SPECTRUM

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Явление резонанса как перестройка спектра квазиэнергии

Рассмотрен следующий известный парадокс. С одной стороны, для всякой квантовой системы с периодически зависящим от времени гамильтонианом существуют решения уравнения Шредингера, являющиеся периодическими функциями времени, модулированными множителем $e^{-itE/\hbar}$. С другой стороны, амплитуда колебаний гармонического осциллятора, на который воздействует периодическая во времени сила с резонансной частотой, неограниченно возрастает со временем при любых начальных условиях. Парадокс разрешен путем точного решения соответствующего уравнения Шредингера. Показано, что парадокс возникает, если не принять во внимание перестройку спектра квазиэнергии и базиса квазиэнергетических состояний в точке резонанса.

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Resonance Phenomenon as a Transmutation of the Quasienergy Spectrum

The following well-known paradox is considered. On one hand, for each quantum system with a time-periodic Hamiltonian, solutions of the Schrodinger equation exist which are time-periodic functions modulated by the factor $e^{-itE/\hbar}$. On the other hand, the amplitude of a harmonic oscillator, acted upon by a time-periodic force with a resonant frequency, increases with time for any initial condition. The paradox is resolved with the help of the exact solution of the corresponding Schrodinger equation. It is shown that the paradox arises, if one does not take into account the transmutation of the quasienergy spectrum and of the steady states (quasienergy states) basis in a resonant point.

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I. Introduction

The following fact is generally known in both classical and quantum mechanics. If a harmonic oscillator with a self-frequency ω_0 is acted upon by an external time-periodic force with a frequency Ω, and $\Omega \neq \omega_0/n$, where n is an integer, then a steady oscillation regime exists for the oscillator. This steady regime amplitude increases when $\Omega \rightarrow \omega_0/n$. When $\Omega = \omega_0/n$, an exact resonance sets in: the oscillation amplitude increases with time at any initial condition.

At the same time the considered oscillator is a system with a time-periodic Hamiltonian. It is known that each solution of the Schrödinger equation with a time-periodic Hamiltonian can be presented as a linear combination of some special solutions:

$$|\psi(t)>=\sum_\alpha c_\alpha e^{-itE_\alpha/\hbar}|\phi_\alpha(t)> \; ; \; |\phi_\alpha(t)>=|\phi_\alpha(t+T)> , \tag{1}$$

where t is time, $T = 2\pi/\Omega$ is the Hamiltonian period and $\hbar$ is Planck's constant. The time-periodic ket-vectors $|\phi_\alpha(t)>$, which are the analogues of the Bloch functions for the time-periodic case, were introduced in [1, 2] as quasienergy states and in [3] as steady states. The E_α value, the analogue of a quasimomentum, is called quasienergy.

The following orthogonality and completeness relations are proved for the steady states by the most general case:

$$<\phi_\alpha(t)|\phi_\beta(t)>=\delta_{\alpha\beta} \; ; \; \sum_\alpha|\phi_\alpha(t)><\phi_\alpha(t)|=I , \tag{2}$$

where I is the unit operator in the corresponding Hilbert space, δ is the Kronecker symbol if α and β are discrete parameters, and δ is the Dirac delta-function if α and β are continuous parameters [4]. These relations mean [5] that we can describe the evolution of each vector $|\psi(t)>$ using its representation on the $|\phi_\alpha(t)>$ basis:

$$|\psi(t)>=\sum_\alpha e^{-itE_\alpha/\hbar}|\phi_\alpha(t)><\phi_\alpha(0)|\psi(0)> . \tag{3}$$

If we choose one of the steady states as an initial condition for the corresponding Schrödinger equation: $|\psi(t=0)>=|\phi_{\alpha_0}(t=0)>$, the solution has the form:

$$|\psi(t)>=e^{-itE_{\alpha_0}/\hbar}|\phi_{\alpha_0}(t)> . \tag{4}$$

The question arises: how can we combine, in the resonant case, the time-periodic character of solution (4) and the oscillation amplitude increasing with time at any initial condition?

This paradox is the simplest representative of a circle of similar difficulties. The very interesting example connected with a parametrical resonance is discussed in

[6]. In that model the oscillator energy increases with time at any initial condition, if the external action frequency belongs to the interval surrounding the resonant point.

Ya.B. Zeldovich, one of the founders of the time-periodic systems theory, concluded after considering these paradoxes, that the steady states (the quasienergy states, as he wrote) cannot be introduced for every quantum system. He recommended the use of the quasienergy method in the discrete spectrum case only, when similar paradoxes do not arise [7]. H. Samble in his paper [3], where he introduced the steady states, also preferred to consider only vectors with finite norms, in order to avoid any difficulties.

But it is generally known that even in the simplest case of the time-independent Hamiltonian, for free particles, the Hamiltonian eigenvectors are un-normalizable. So we have to use un-normalizable steady states if we want to get a general theory for time-periodic systems. The orthogonality and completeness relations (2) are the mathematical base of such general theory. In order to get not only an abstract theory, but a real calculation method, we have to learn how to construct the steady states and the quasienergy spectrum for the difficult cases like the one described above. We also have to understand the physical cause of the difficulties which arise to know when it is necessary to be careful.

In the present paper a resolution of the formulated paradox is given. In Section II the detailed qualitative analysis is carried out, and the physical essence of the phenomenon is exposed. In Section III a simple mathematical model, reproducing the main features of the phenomenon, is solved exactly. In Section IV the use of the steady states conception in the case of the continuous quasienergy spectrum is discussed.

II. The Qualitative Analysis

The problem of constructing quasienergy and steady states for the oscillator acted upon by a time-periodic force was solved exactly by the auther not long ago [8]. The following facts were obtained:

(I) If $\Omega \neq \omega_0/n$, where n is any integer, the quasienergy spectrum is discrete and equidistant. The steady states have a finite norm.

(II) If $\Omega = \omega_0/n$, the quasienergy spectrum is continuous. The steady states satisfy the orthogonality and completeness relations (2) with the continuous parameters α, β. Therefore, the steady states are un-normalizable, and therefore, they do not belong to the corresponding Hilbert space.

Some important qualitative conclusions follow this mathematical result. We consider them below.

It is well-known that in the case of the time-independent Hamiltonian the discrete energy spectrum and normalizable Hamiltonian eigenvectors correspond to the finite motion of a particle; the continuous energy spectrum and un-normalizable Hamiltonian eigenvectors correspond to the infinite motion of a particle [9]. In the case of the time-periodic Hamiltonian quasienergy takes the place of energy and the steady states take place of the Hamiltonian eigenvectors [1, 2, 3]. It is easy to see that the behaviour of the steady states and the quasienergy spectrum, described above, corresponds to a situation analogous to the one in the time-independent case.

Indeed, if the external force frequency is not a resonant one, the oscillator motion is finite; its quasienergy spectrum is discrete and its steady states are normalizable. If the external force frequency is resonant, the oscillator motion is infinite (the amplitude increases with time); its quasienergy spectrum is continuous and its steady states are un-normalizable.

The peculiarity of the situation consists in the fact that the character of the oscillator motion, as a function of the external force frequency, changes abruptly — from finite to infinite and back — when the frequency crosses the resonant point.

The paradox under consideration connects with this quasienergy spectrum transmutation. Indeed, when one points out that the oscillation amplitude increases with time at any initial condition, the additional suggestion, that the initial function belongs to the corresponding Hilbert space (i.e., that it has the finite norm), is used. It is natural for the oscillator that is not acted upon by an external force, because if the wave function of such oscillator does not decrease fast enough with the distance, its potential energy is infinite. But the Hamiltonian of the oscillator acted upon by a resonant force is an absolutely different operator. It looks like the free particle Hamiltonian. Its steady states, periodically depending on time (except the factor $e^{-itE/\hbar}$), do not vanish anywhere in the whole space. Thus, the considered paradox connects with the fact that the initial conditions for the oscillator acted upon by a resonant force, are taken in the form that is natural for an absolutely different system — for the free oscillator.

In the end of our qualitative analysis let us note the following fact. The functions describing the initial conditions with finite norms cannot really be the steady states of the resonant perturbated oscillator. They are the linear combinations of the steady states only. So, their evolution is described by relation (3). The fact that the increasing with time function can be presented as a linear combination of the periodic functions and oscillating factors like (3), is illustrated by the well-known representation of the linear function by the Fourier integral:

$$t = -i \int_{-\infty}^{\infty} d\omega \frac{d\delta(\omega)}{d\omega} e^{-it\omega} . \qquad (5)$$

III. Mathematical Model

Let us demonstrate the oscillator properties described above, using a simple model with the following Hamiltonian:

$$H(t) = H_{osc} + \alpha e^{it\Omega}a + \alpha e^{-it\Omega}a^{+} \ , \ H_{osc} = \hbar\omega_0\left(a^{+}a + \frac{1}{2}\right) , \qquad (6)$$

where a, a^{+} are annihilation and creation operators, respectively, and α is a real number describing an external action. In the more general model the Hamiltonian has the following form: $H_{osc} + f(t)a + \bar{f}(t)a^{+}$, where $f(t) = f(t + 2\pi/\Omega)$ (the overline means the complex conjugation). The Hamiltonian (6) is usually named as the resonant approximation to the exact one. Model (6) has only one resonant frequency $\Omega = \omega_0$. In other respects this model reproduces all the qualitative features of the oscillator Hamiltonian with any time-periodic external force. All solutions for model (6) have a very simple form and are convenient for qualitative analysis. At the same time, the model (6) is virtually exact when $\Omega \to \omega_0$, i.e., near the resonance, which is the most interesting case of our consideration [8].

The steady states $|\phi(t)>$ and quasienergy E are the solutions to the following eigenvalue problem [1, 3]:

$$\left\{H(t) - i\hbar\frac{\partial}{\partial t}\right\}|\phi(t)>= E|\phi(t)> ; \ |\phi(t)>= |\phi(t+T)> . \qquad (7)$$

In order to solve equation (7), it is convenient in our case to use a complex representation [10]. If z is a complex number and $|z>$ is the corresponding coherent state, then for each ket-vector $|\phi>$ the function

$$\phi(z) = \exp\left\{|z|^2/2\right\} < \bar{z}|\phi> \qquad (8)$$

is the analitical function of z in the whole complex plane. In the complex representation, the eigenvalue problem (7) for the Hamiltonian (6) has the form:

$$\left\{\hbar\omega_0 z\frac{\partial}{\partial z} + \alpha e^{it\Omega}\frac{\partial}{\partial z} + \alpha e^{-it\Omega}z - i\hbar\frac{\partial}{\partial t}\right\}\phi(z,t) = E\phi(z,t)$$
$$\phi(z,t) = \phi(z,t+T) . \qquad (9)$$

If we use the complex variable $u = ze^{-it\Omega}$ instead of z, the problem (9) has the form:

$$\left\{\hbar\left(\omega_0 - \Omega\right)u\frac{\partial}{\partial u} + \alpha\frac{\partial}{\partial u} + \alpha u - i\hbar\frac{\partial}{\partial t}\right\}\phi(u,t) = E\phi(u,t)$$
$$\phi(u,t+T) = \phi(u,t) . \qquad (10)$$

As the variables u and t in equation (10) are separated, the solutions can be simply found. So we write the results using the variable z again.

In the nonresonant case, when $\Omega \neq \omega_0$, the quasienergies E_n and the steady states $\phi_n(z,t)$ are given by:

$$E_n = \hbar\left(\omega_0 - \Omega\right)n + \hbar\omega_0/2 + \hbar\alpha^2/\left(\Omega - \omega_0\right)$$

$$\phi_n(z,t) = (n!)^{-1/2}\left[ze^{-it\Omega} - \frac{\alpha}{(\Omega-\omega_0)}\right]^n \exp\left\{\frac{ze^{-it\Omega}\alpha}{(\Omega-\omega_0)} - \frac{\alpha^2}{2(\Omega-\omega_0)^2}\right\} , \qquad (11)$$

where $n = 0, 1, 2 ...$ One can see from (11) that when $\Omega \to \omega_0$, the quasienergy spectrum almost degenerates.

As the quasienergy defined modulo $q\hbar\Omega$ only, where q is any integer, one can use an equivalent representation for the quasienergy and the steady states [1, 2, 3]:

$$\tilde{E}_n = \hbar\omega_0(n + 1/2) + \hbar\alpha^2/(\Omega - \omega_0) \quad ; \quad \tilde{\phi}_n(z,t) = e^{itn\Omega}\phi_n(z,t) . \qquad (12)$$

One can see that the quasienergy spectrum (12) is equivalent to the unperturbated oscillator energy spectrum (except the ground state energy). So we can say that the nonperturbated oscillator and the nonresonant perturbated oscillator are similar systems.

One can represent the steady states (11) in a simple form in the following way. Let us introduce the complex function of time $O(t)$ by the rule:

$$O(t) = \alpha e^{-it\Omega}/(\Omega - \omega_0) . \qquad (13)$$

Let $|O(t)>$ be its corresponding coherent state. Let us also introduce operators $a(t)$, $a^{+}(t)$ by the rule:

$$a(t) = ae^{-it\Omega} - \alpha/(\Omega - \omega_0) \ ; \ a^{+}(t) = a^{+}e^{it\Omega} - \alpha/(\Omega - \omega_0) . \qquad (14)$$

Then the operators $a(t)$, $a^{+}(t)$ satisfy the canonical commutation relations, and the vector $|O(t)>$ is the vacuum vector for them, as $a(t)|O(t)>= 0$. Using the operators introduced, one can represent the ket-vectors $|\phi_n(t)>$, corresponding to the functions $\phi_n(z,t)$, in the following form:

$$|\phi_n(t)>= (n!)^{-1/2}\left\{a^{+}(t)\right\}^n|O(t)> . \qquad (15)$$

So, for each n, the vector $|\phi_n(t)>$ is the n^{th}-quantum excitation of the vacuum vector $|O(t)>$.

Let us return to the eigenvalue equation (10) and consider the resonant case, when $\Omega = \omega_0$. The quasienergy spectrum is continuous in this case: $-\infty < E < \infty$. The steady states are given by the formula:

$$\phi_E(z,t) = \left(2\pi\hbar^2\alpha^2\right)^{-1/4}\exp\left\{-\left(\frac{E - \hbar\Omega/2}{2\hbar\alpha}\right)^2 + \frac{ze^{-it\Omega}(E - \hbar\Omega)}{\hbar\alpha} - \frac{z^2e^{-2it\Omega}}{2}\right\} . \qquad (16)$$

It is easy to show by the elementary integrals calculation that the steady states (16) satisfy the following relations at any t:

$$\frac{1}{\pi}\int dz d\bar{z} e^{-|z|^2}\bar{\phi}_B(z,t)\phi_{B'}(z,t) = \delta(E-E')$$
$$\int_{-\infty}^{\infty} dE\bar{\phi}_B(\bar{z},t)\phi_B(z',t) = e^{\bar{z}z'} . \qquad (17)$$

These relations mean [11], that the corresponding ket- and bra-vectors satisfy orthogonality and completeness relations like (2):

$$<\phi_B(t)|\phi_{B'}(t)> = \delta(E-E') \;\; ; \;\; \int_{-\infty}^{\infty} dE|\phi_B(t)><\phi_B(t)| = I . \qquad (18)$$

So the complete orthogonal steady states basis exists in the resonant case as well as in the nonresonant one. But these steady states do not belong to the Hilbert space because of they are un-normalizable. It is easy to show by the calculation of the corresponding elementary integrals that the energy of each steady state is infinite. However, as the completeness relation (18) is fulfilled, each vector belonging to the Hilbert space can be presented as a linear combination of the steady states (16). So, we have proven all the statements used in our qualitative analysis in the section II.

IV. Discussion

The resolution of the concrete paradox, made in the present paper, is interesting in itself. But I suppose that a much more important subject needs to be broached.

The founders of the quasienergy method used it to solve the problems with a discrete spectrum, when the steady states norms are finite [3, 7]. The opinion was even expressed that the quasienergy method is useless in the continuous spectrum case [7].

In essence, the ability to apply the quasienergy method in the rather difficult concrete case of the continuous quasienergy spectrum is demonstrated in the present paper.

As the general orthogonality and completeness relations (2) are fulfilled, I suppose that expansion on the steady states basis is as universal a method in the time-periodic case as the expansion on the Hamiltonian eigenvectors basis is in the time-independent case. One can find more difficult examples confirming this opinion in [8, 12].

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