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FIVE-LEPTON DECAY MODES OF μ AND τ MESONS

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Introduction

A number of c/τ fabrics are planned to be built in the nearest future [1]. Arrangements of such type with sufficiently high numbers of produced π, K, τ, μ ($> 10^8\ sec^{-1}$) will provide precision investigation of the problems of bound state physics, the theoretical description of which is ambiguous now. The measurement of rare decays may put essential restrictions on possible deviations from the Standard Model (SM).

Of great importance are reference processes which have a small background and have to be calculated with high accuracy. For this purpose we calculate in this work the widths of radiative decays μ and τ into five leptons: two neutrinos, one recoil charged lepton and a lepton pair conversed from a "heavy" photon.

Some of these processes have been discussed in detail during the last 30 years [2-11]. Unfortunately their analytical description was rather poor.

We consider the muon decay into five leptons: $\mu^+ \longrightarrow e^+\nu_e\bar{\nu_\mu}e^+e^-$. On the base of a detailed analysis of the kinematics of secondary leptons, we calculate the spectral distribution on the energy fractions of the electron and the positrons as well as the inclusive distribution on the energy of the positron $y = 2\varepsilon^-/M_\mu$.

In the calculation we take into account the identity of the positrons in the final state. We extend our results also for different five-lepton τ decay modes. The terms quadratic and linear in parametrically large logarithms of the masses ratio are calculated.

The decay

$$\mu^+ \longrightarrow e^+\nu_e\bar{\nu_\mu}e^+e^- \tag{1}$$

was observed [2,6] and now may be studied in detail experimentally. The decay presents a background in the study of muonium-antimuonium conversion [7-9] and in the search for neutrinoless decays $\mu^+ \to e^+e^+e^-$ [10] and $\tau^+ \to e^+e^+e^-$ [11]. It may be used also as metrical in the study of radiative and rare pion decays (π^+ β-decay, $\pi \to \mu\nu\gamma$, $\mu\nu e^+e^-$). The experimental branching [6]

$$R = \frac{\Gamma(\mu \to ee\bar{e}\nu\bar{\nu})}{\Gamma_{total}} = (3.4 \pm 0.4) \cdot 10^{-5} \tag{2}$$

is in agreement with our results as well as with the Monte-Carlo calculations [4] in the Standard Model.

Five-Lepton Decay Modes

The five-lepton μ decay

$$\mu^+(p) \to \nu_\mu(q_1) + \bar{\nu}_e(q_2) + e^+(p_2) + e^+(p_1) + e^-(p_3) \tag{3}$$

is described by four Feynman diagrams (see Fig. 1). The differential width in the SM frame after the standard integration over the (nonobservable) neutrino momenta [12]

$$\int q_{1\alpha} q_{2\beta} \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \delta^4(q_1 + q_2 - q) \sum |M|^2 = \frac{\pi}{6}(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta)|\bar{M}|^2 \tag{4}$$

may be expressed by the kinematical characteristics of three charged leptons in the rest frame of the muon

$$\frac{d\Gamma}{\Gamma_0} = \frac{\alpha^2}{2\pi^2} R x_1 x_2 y \, dx_1 dx_2 dy dc_1 \frac{d\Omega_2}{8\pi}, \tag{5}$$

where $d\Omega_2$ is the solid angle element of one of the electrons,

$$\Gamma_0 = \frac{G^2 m_\mu^5}{192\pi^3}, \qquad R = \frac{3}{2^9 \pi^3 \alpha^2} |\bar{M}|^2,$$

$$R = (1 + P(1,2))(R_{1212} + R_{1234}).$$

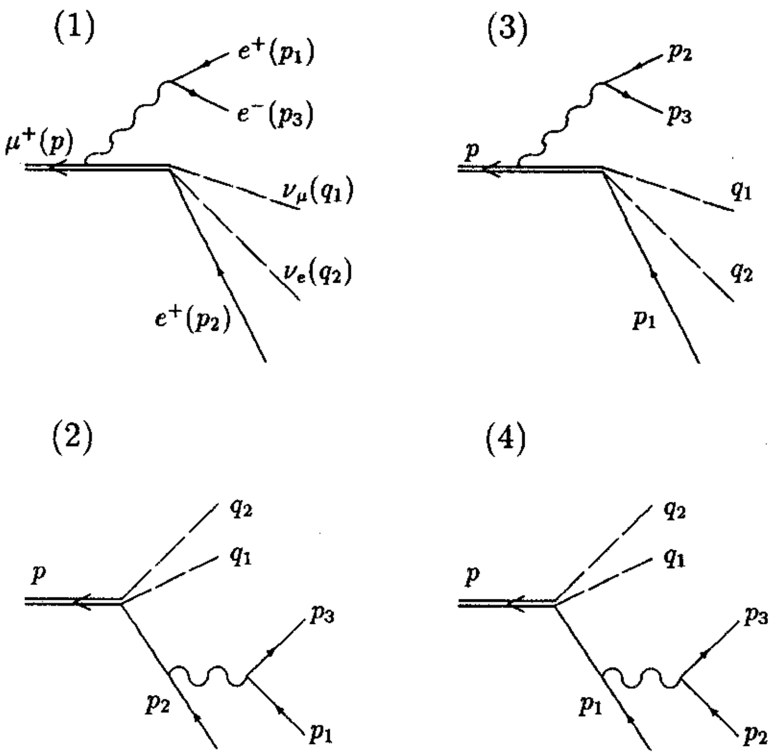

Fig. 1: Feynman diagrams for five-lepton decay

The operator $P(1,2)$ transposes the four-momenta of the identical fermions. R_{1212} represents the contribution of the first and second diagrams, R_{1234} – the interference of the first and the second diagrams with the third and the fourth ones. The energy fractions of the leptons $x_{1,2}$, y and the angular parameters are defined as

$$
\begin{aligned}
x_i &= \frac{2\varepsilon_i^+}{M_\mu},\ i=1,2; \quad y=\frac{2\varepsilon^-}{M_\mu}; \quad \frac{2m_e}{M_\mu} \le x_{1,2}; y \le 1; \\
c_i &= \cos(\widehat{\vec{p}_3\vec{p}_i}), \quad c_{12} = c_1c_2 - \sqrt{1-c_1^2}\sqrt{1-c_2^2}\,\cos\phi, \qquad (6)
\end{aligned}
$$

azimuthal angle ϕ is an angle between planes containing the momentum pairs $(\vec{p}_1, \vec{p}_3)$ and $(\vec{p}_2, \vec{p}_3)$. Note that we take into account the identity of two positrons in the final state by involving the third and the fourth diagrams. The following consideration is also applicable (using $\mu - e - \tau-$ universality) to the τ lepton decays. In τ decays $\tau \to \nu_\tau \mu e \bar{e} \bar{\nu}_\mu$, $\tau \to \nu_\tau \mu \bar{\mu} e \bar{\nu}_e$ in (4) it is necessary to put $P(1,2) = 1$ and omit R_{1234}. Besides, in the case of light pair creation ($\tau \to \nu_\tau \mu e \bar{e} \bar{\nu}_\mu$) the quantity R_{1212} receives an additional contribution δR. For the decays $\tau \to \nu_\tau \mu \mu \bar{\mu} \bar{\nu}_\mu$, $\tau \to \nu_\tau e e \bar{e} \bar{\nu}_e$ the formulae given below (with the evident substitutions) are valid. The quantity R_{1212} is proportional to the summed on spin states squared module of the matrix element, corresponding to Feynman diagrams (1), (2). It may be brought to the form

$$R_{1212} = \epsilon_{\mu,\nu} M^{\mu,\nu}/(k_1^2)^2 \tag{7}$$

$$\epsilon_{\mu,\nu} = Sp(p_1 - m)\gamma_\nu(p_3 + m)\gamma_\mu = 4(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - \frac{k_1^2}{2} g^{\mu\nu}),$$

$$k_1 = p_1 + p_3.$$

The integrated over neutrino momenta muon tensor may be put in the explicitly gauge-invariant form:

$$M^{\mu\nu} = a_1 \bar{g}^{\mu\nu} + 2a_2 \bar{p}_2^\mu \bar{p}_2^\nu + 2a_3 \bar{p}^\mu \bar{p}^\nu + 2a_4(\bar{p}_2^\mu \bar{p}^\nu + \bar{p}^\mu \bar{p}_2^\nu), \tag{8}$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \frac{k_1^\mu k_1^\nu}{k_1^2}, \qquad \bar{p} = p - \frac{pk_1}{k_1^2} k_1.$$

The quantities a_i may be expressed in terms of the invariants:

$$A = (p - k_1)^2 - 1, \qquad B = (p_1 + p_2 + p_3)^2 - m^2, \tag{9}$$

$$p_1^2 = p_3^2 = m_e^2, \qquad p_2^2 = m^2, \qquad p^2 = M_\mu^2 \equiv 1,$$

they are presented in Appendix 1.

From the condition of the positivity of the neutrinos invariant mass square

$$q^2 = (p - p_1 - p_2 - p_3)^2 > 0, \tag{10}$$

one may obtain the restriction on the phase space of the variables defined in (6):

$$1 - x_1 - x_2 - y + x_1 x_2 z_{12} + x_1 y z_1 + x_2 y z_2 > 0, \tag{11}$$

$$z_i = \frac{1 - c_i}{2}, \qquad z_{12} = \frac{1 - c_{12}}{2}.$$

In Fig. 2. one can see the phase space in the variables x_1, x_2, z_2 when $z_1 \to 0$ and $y = 1/2$.

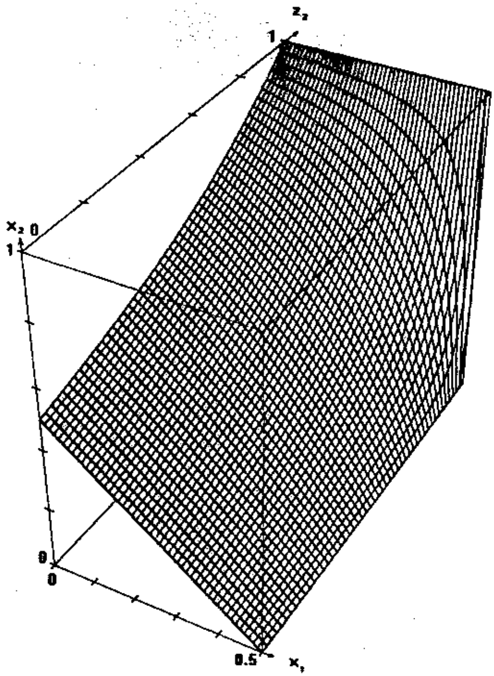

Fig. 2: The phase space in the variables x_1, x_2, z_2

We used a method developed by one of us (M.N.P.) in an analysis of the measurement process of two hard photons and of the production of hard e^+e^--pair in the deep inelastic ep-scattering [14].

The distribution on the energy fractions of charged particles, neglecting the terms of the order m_e^2/M_μ^2, may be put in the form:

$$\frac{d\Gamma}{dx_1 dx_2 dy} = (1 + P(1,2))r, \qquad r = f_1 L^2 + f_2 L + f_3, \tag{12}$$
$$L = \ln \frac{M_\mu^2}{m_e^2} \approx 10.6 .$$

The contributions, proportional to the "large logarithm" L, arise from the angular integration of the phase space. The main contribution of the order of L^2 goes from the peak kinematical region when all the invariants p_1p_2, p_1p_3, p_2p_3 are small compared with unity (in terms of μ mass square):

$$p_1p_2,\ p_1p_3,\ p_2p_3 \ll M_\mu^2 \equiv 1, \qquad m_e^2 \equiv m^2. \tag{13}$$

The angular phase space and invariants in this region are

$$\begin{aligned}
&d\Omega_1 d\Omega_2 d\Omega_3 = 32\pi^2 dz_1 dz_2 d\phi, \qquad \frac{m^2}{M_\mu^2} \ll z_1;\ z_2 < \sigma \ll 1,\\
&k_1^2 = m^2[\frac{(x_1+y)_2}{x_1 y} + x_1 y z_1],\\
&B = m^2[\frac{(x_1+y)^2}{x_1 y} + \frac{x_2^2+y^2}{x_2 y} + \frac{x_1^2+x_2^2}{x_1 x_2}]\\
&\quad + z_1 x_1 (x_2 + y) + z_2 x_2 (x_2 + y) + 2\sqrt{z_1 z_2} x_1 x_2 \cos\phi,\\
&a_{23} = 2(p_2p_3) = m^2[\frac{x_2^2+y^2}{x_2 y}] + x_2 y z_2, \qquad 2(p_1p_2) = B - k_1^2 - 2(p_2p_3),
\end{aligned} \tag{14}$$

In the square of the matrix element of the process in this region the terms of the order M_μ^4/m_e^4 are important. The details of the integration by z_1 and z_2 and all necessary integrals are contained in Appendix 2. Note that in this region the sum of the charged leptons energies does not exceed unity.

Contributions of the order L go from the double-collinear as well as from the semi-collinear region in which just one invariant, p_1p_3 or p_2p_3, is bounded:

$$p_1p_3 \ll 1, \quad p_2p_3 \sim p_1p_2 \sim 1, \quad or \quad p_2p_3 \ll 1, \quad p_1p_3 \sim p_1p_2 \sim 1. \tag{15}$$

Thus all the logarithms arise from the kinematical region when at least one of the values z_1, z_2 is small. In this case the permitted region for variables may be analyzed explicitly. Consider for definiteness the case where z_1 tends to zero. The restriction of positivity of neutrinos mass square will take the form:

$$1 - x_1 - x_2 - y + x_2(x_1 + y)z_2 > 0\,. \qquad (16)$$

The corresponding figure is drawn for $y = 0.5$ in fig.1.

The constant term goes from all parts of the whole phase space. We calculate the analytic form of f_1 and f_2 as well as the spectral distribution with respect to the energy fraction of the final electron taking into account only terms proportional to L^2 and L. The functions f_1 and f_2 are smooth ones of the order of unity and grow at the boundaries of the spectrum. The function f_3 has probably the same properties and so will not change spectrum distributions considerably.

Performing the angular integration with logarithmical accuracy (see Appendix 2) one obtains the following expressions for the functions f_1 and f_2:

$$\begin{aligned} f_1 &= \Theta\frac{1}{2}(2\Delta - 3)\frac{x_1x_2y}{t_1^2}\left[\frac{2\Delta}{t_1^2} + \frac{x_1y - x_2\Delta - \frac{1}{2}t_1^2}{x_1x_2y}\right], \qquad (17)\\ f_2 &= f_1 \ln\frac{x_1x_2y}{\Delta} + \left[(r^1_{1212} + r_{1234})\Theta + r^2_{1212}\Theta_1\right]x_1x_2y, \end{aligned}$$

$$\begin{aligned} r^1_{1212} = {} & (2\Delta - 3)\left[-\frac{1}{x_2t_1^2} + \frac{1}{2x_1x_2y} + \frac{1}{yt_1^2} - \frac{\Delta}{x_1x_2t_1^2} - \frac{6\Delta}{t_1^4} + \frac{x_1x_2}{yt_1^4}\right.\\ & \left. + \frac{y\Delta^2}{x_1x_2t_1^4}\right] - \frac{2}{t_1^4}\left[2\Delta(2\Delta - 3) - \frac{1}{3}x_2^2t_1^2 + \frac{1}{2}x_2t_1(4\Delta - 3)\right]\\ & + \frac{1}{x_1y}\left[A_2 + \frac{1}{2}A_3x_2t_1 + \frac{1}{3}x_2^2t_1^2A_4\right]; \end{aligned}$$

$$\begin{aligned} A_1 &= (2\Delta - 3)\left[-\frac{1}{2}t_1 + \frac{x_1y}{t_1} - \frac{x_2\Delta}{t_1}\right];\\ A_2 &= (1 - t_1)\left(-1 + \frac{2x_1y}{t_1}\right) + \frac{\Delta(2\Delta - 3)}{t_1^2} + \frac{x_2}{t_1}(4\Delta - 3);\\ A_3 &= (2\Delta - 3)\left[-\frac{1}{2t_1} + \frac{x_1y}{t_1^3}\right] - \frac{1}{t_1^2}(4\Delta - 3) - \frac{2}{t_1}x_2;\\ A_4 &= \frac{1}{t_1} - \frac{2x_1y}{t_1^3} + \frac{2}{t_1^2}; \end{aligned}$$

$$r^2_{1212} = \frac{1}{x_1 y}\Big[\frac{A_1}{x_2 t_1}\ln\frac{1}{\lambda} + (1-\lambda)A_2 + \frac{1}{2}(1-\lambda^2)x_2 t_1 A_3 + \frac{1}{3}x_2^2 t_1^2(1-\lambda^3)A_4\Big] - \frac{2}{t_1^4}\Big[\Delta(3-2\Delta)\ln\frac{1}{\lambda} + (1-\lambda)\big(2\Delta^2 - 3\Delta + x_2 t_1(4\Delta-3)\big) - \frac{1}{2}x_2 t_1(1-\lambda^2)\big(4\Delta - 3 + 2x_2 t_1\big) + \frac{2}{3}x_2^2 t_1^2(1-\lambda^3)\Big];$$

$$r_{1234} = \frac{8(2\Delta-3)}{x_1 x_2 y}\Big[\Big(\frac{x_1 x_2}{t_1 t_2} + \frac{x_2 - x_1}{t_1} + 1\Big)\ln\frac{y\Delta}{t_1 t_2} + \frac{y\Delta}{t_1^2}\Big(\frac{x_1 x_2}{y\Delta} - 1\Big) + 1\Big],$$

where

$$\Theta \equiv \Theta(1 - x_1 - x_2 - y), \quad \Theta_1 \equiv \Theta(1 - x_1 - y)\Theta(x_1 + x_2 + y - 1),$$
$$\lambda \equiv \frac{\Delta - 1}{x_2 t_1}, \qquad t_{1,2} \equiv x_{1,2} + y, \qquad \Delta \equiv x_1 + x_2 + y. \tag{18}$$

The results obtained may be extended to the case of five-lepton modes of τ-decay in the way discussed above. For the energy fractions spectrum of the decays $\tau \to ee\bar{e}\nu\bar{\nu}$ and $\tau \to \mu\mu\bar{\mu}\nu\bar{\nu}$ the formulae (14), (18) are valid with the substitutions $L \to \ln M_\tau^2/m_e^2$ and $L \to \ln M_\tau^2/M_\mu^2$, respectively. For the process $\tau \to e\mu\bar{\mu}\nu\bar{\nu}$ it is necessary to put in (14) and (18)

$$P(1,2) = 1, \qquad r_{1234} = 0, \qquad L \to \ln M_\tau^2/M_\mu^2. \tag{19}$$

For the process $\tau \to \mu e\bar{e}\nu\bar{\nu}$ — again

$$P(1,2) = 1, \qquad r_{1234} = 0, \qquad L \to \ln M_\tau^2/m_e^2 \tag{20}$$

and, besides,

$$r^{(1)}_{1212} \to r^{(1)}_{1212} + \delta r_{1212},$$

where

$$\delta r_{1212} = (2\Delta - 3)\Big\{2\Big[\frac{1}{x_1 x_2 y t_1^2}\Big(-\frac{1}{2}t_1^2 + x_1 y - x_2\Delta\Big) + \frac{2\Delta}{t_1^4}\Big]\ln\frac{t_1}{x_1 y} + \frac{1}{x_2 t_1^2} - \frac{1}{2x_1 x_2 y} + \frac{\Delta}{x_1 y t_1^2}\Big\}. \tag{21}$$

Using this distribution we obtain the distribution with respect to the total energy of charged leptons $\varepsilon_1^- + \varepsilon_2^- + \varepsilon^+ = M_\mu \Delta/2$. For the region $1 < \Delta < 2$ only one-logarithm contribution remains:

$$\frac{d\Gamma}{\Gamma_0 d\Delta} = \int_0^1 dx_2 \int_0^{t_1} dx_1 \frac{d^3\Gamma}{dx_1 dx_2 dy} = \left(\frac{\alpha}{\pi}\right)^2 L\Phi(\Delta), \qquad 1 < \Delta < 2, \quad (22)$$

where

$$\Phi(\Delta) = (3-2\Delta)\frac{2}{3}\Delta^2\Big[\frac{1}{2}\ln^2(\Delta-1) + 2\int_\Delta^2 \frac{dy}{y}\ln(y-1)\Big]$$
$$+\ln(\Delta-1)\Big[-\frac{4}{9}\Delta^3 + \frac{2}{3}\Delta^2 + \frac{2}{3}\Delta + \frac{5}{18}\Big] - \frac{1}{270}\Delta^6$$
$$+\frac{7}{180}\Delta^5 - \frac{13}{24}\Delta^4 + \frac{199}{54}\Delta^3 - \frac{323}{36}\Delta^2 - \frac{529}{90}\Delta + \frac{104}{45}. \quad (23)$$

When $\Delta \to 2$

$$\frac{d\Gamma}{\Gamma_0 d\Delta} \approx \left(\frac{\alpha}{\pi}\right)^2 L\frac{13}{36}(2-\Delta)^2. \quad (24)$$

This formula may be used for the estimation of the background in the search of the decay $\mu \to e^+e^+e^-$.

Using (14) we calculate also the distribution on the electron energy fraction for the positively charged muon decay. It has the form

$$\frac{d\Gamma^{\mu\to e e\bar{e}\nu\bar{\nu}}}{\Gamma_0 dy} = \left(\frac{\alpha}{\pi}\right)^2 (L^2\Phi_1(y) + L\Phi_2(y)) \equiv \left(\frac{\alpha}{\pi}\right)^2\Phi(y), \quad (25)$$

where

$$\Phi_1(y) = \frac{1}{6y} + \frac{17}{36} + \frac{3}{4}y - \frac{7}{6}y^2 - \frac{2}{9}y^3 + \left(\frac{5}{12} + y + y^2\right)\ln y,$$

$$\Phi_2(y) = \left(\int_y^1 \frac{dx}{x}\ln(1+x) + \ln y \ln(1+y)\right)\left[\frac{272}{3}y^3 + 136y^2 + 32y - \frac{40}{3}\right]$$
$$+\left(\int_y^1 \frac{dx}{x}\ln(1-x)\right)\left[-2y^2 - 2y - \frac{5}{6}\right]$$

$$
\begin{aligned}
&+(16y^3+24y^2)\int_y^1 \frac{dt_1}{t_1}\int_0^{1-t_1}\frac{dx_2}{x_2+y}\ \ln\left(\frac{(x_2+y)t_1}{(x_2+t_1)y}\right)\\
&+\left[-\frac{4}{3}y^3-\frac{7}{3}y^2+\frac{3}{2}y+\frac{17}{18}+\frac{1}{3y}\right]\ln(1-y) \qquad (26)\\
&+\left[-\frac{371}{6}y^2-\frac{277}{12}y+\frac{34}{3}+\frac{2}{3y}\right]\ln(y)\\
&+\left[-\frac{136}{3}y^3-50y^2-5y+\frac{55}{12}\right]\ln^2(y)\\
&+\frac{877}{54}-\frac{1949}{36}y-\frac{1}{6y}-\frac{631}{36}y^2+\frac{1501}{27}y^3,
\end{aligned}
$$

In the region $y \to 1$ one obtains from (25)

$$
\Phi(y)\Big|_{y\to1} \approx (1-y)^2\left[\frac{1}{8}L^2-(\frac{13}{12}-\frac{1}{4}\ln(1-y))L\right]. \qquad (27)
$$

The result may be used in an estimation of the background for the conversion muonium-antimuonium [7-9] as far as the process considered may play the role of a source of fast electrons:

$$
\begin{aligned}
\mu^+e^- \longrightarrow \quad & \mu^-e^+\\
& \hookrightarrow \ e^-\nu\nu.
\end{aligned}
$$

We note that our results differ in one-logarithm terms from the estimation obtained in [8]:

$$
\begin{aligned}
\frac{d\Gamma}{\Gamma_0 dy}\Big|_{y\to1} &= (\frac{\alpha}{\pi})^2[\frac{1}{8}L^2+L(-1+\frac{1}{2}\ln(1-y))+\frac{1}{2}\ln^2(1-y) \qquad (28)\\
&-2\ln(1-y)+3y-\frac{\pi^2}{6}].
\end{aligned}
$$

We put here also the distribution on the electron energy fraction for the decay $\tau^+ \to \mu^+e^+e^-\nu\nu$. It has the form (25) with the substitutions

$$
\begin{aligned}
&L \to \ln\frac{M_\tau}{m_e}, \qquad \Phi_2 \to \Phi_2+\delta\Phi_2,\\
&\delta\Phi_2 = (\frac{5}{3}+4y^2+4y)\left(\int_y^1\frac{dx}{x}\ln(1-x)-\ln^2 y\right)
\end{aligned}
$$

$$+(-\frac{17}{9}+\frac{8}{9}y^3+\frac{14}{9}y^2-3y-\frac{2}{3y})\ln(1-y)$$
$$+(-\frac{49}{18}+\frac{14}{3}y^2-3y-\frac{2}{3y})\ln y$$
$$+\frac{8}{9}-\frac{7}{18}y^3-\frac{35}{6}y^2+\frac{46}{9}y+\frac{2}{9y};$$
$$\delta\Phi_2(y)\Big|_{y\to 1}\approx(1-y)^2(1-\frac{1}{2}\ln(1-y)). \qquad (29)$$

To estimate the contribution to the total muon decay width from the pair creation channel, it is necessary to consider first the case when the created pair is soft:

$$\frac{2(\varepsilon_1+\varepsilon_3)}{M_\mu}\leq\eta, \qquad \eta\ll 1. \qquad (30)$$

One may use the approximation of classical currents [15] and omit effects of fermions identity:

$$\frac{\Gamma^S}{\Gamma_0}=\frac{d\Gamma^H}{\Gamma_0\,dt}\,dt=\frac{2\alpha^2}{3\pi^2}\left\{\frac{1}{3}((\frac{1}{2}L+\ln\eta)^3-\frac{4}{3}((\frac{1}{2}L+\ln\eta)^2\right. \qquad (31)$$
$$\left.((-\frac{\pi^2}{6}+\frac{61}{18})((\frac{1}{2}L+\ln\eta)+O(1)\right\}.$$

For the contribution of the kinematics of a hard pair creation one obtains from (14):

$$\frac{\Gamma^H}{\Gamma_0}=\int\limits_\eta^1\frac{d\Gamma^H}{\Gamma_0\,dt}\,dt=\frac{1}{2}(\frac{\alpha}{\pi})^2\,L\,(-\frac{2}{3}\ln^2\eta-\frac{1}{3}L\ln\eta$$
$$+\frac{16}{9}\ln\eta-\frac{7}{18}L-12\zeta_2+8\zeta_3+\frac{3565}{216}), \qquad (32)$$

$$\zeta_2=\sum_1^\infty n^{-2}=\frac{\pi^2}{6}, \qquad \zeta_3=\sum_1^\infty n^{-3}\approx 1.202\ .$$

The result has the form

$$\frac{\Gamma^H+\Gamma^S}{\Gamma_0}=(\frac{\alpha}{\pi})^2\{\frac{1}{36}L^3-\frac{5}{12}L^2+L(\frac{1351}{144}+4\zeta_3-\frac{19}{3}\zeta_2)+O(1)\}. \quad (33)$$

The parameter η cancelled in the total sum.

There are other contributions in the same order of the perturbation theory. We have also to consider a "virtual pair" creation – the diagram with the photon binding the initial muon and the final recoil electron. "Virtual pair" creation means the self-energy fermion-antifermion loop insertion in the photon propagator. We obtain for this contribution the following expression

$$\frac{d\Gamma^V}{d\Gamma_0} = (\frac{\alpha}{\pi})^2 \left\{ -\frac{1}{36} L^3 + \frac{3}{8} L^2 + \frac{1}{6} \left(\ln^2 \frac{M_W^2}{m_\mu^2} + L \ln \frac{M_W^2}{m_\mu^2} \right) \right\}, \quad (34)$$

$$L = \ln(\frac{m_\mu^2}{m_e^2}) \approx 10.7, \qquad \ln(\frac{M_W}{m_\mu}) \approx 6.6 \ .$$

We have to put M_W as the cut-off parameter in our calculations, as was done in [13]. The sum of these three contributions is

$$\begin{aligned} \frac{\Gamma^H + \Gamma^S + \Gamma^V}{\Gamma_0} &= (\frac{\alpha}{\pi})^2 \{ -\frac{1}{24} L^2 + \frac{1}{6} (\ln^2 \frac{M_W^2}{m_\mu^2} \\ &+ L \ln \frac{M_W^2}{m_\mu^2}) + O(L) \} \approx 0.02\% \ . \end{aligned} \quad (35)$$

Appendix 1

Here we present the coefficients a_i from Eq. (8).

$$\begin{aligned} a_1 = A + B - 2k_1^2 + m^2 + M_\mu^2 + \frac{1}{A} [B^2 + B(-3k_1^2 - 2pp_2 \\ + \frac{3}{2}(m^2 + M_\mu^2)) + k_1^2(2k_1^2 - \frac{5}{2}(m^2 + M_\mu^2))] + \frac{1}{B} [A^2 \\ + A(-3k_1^2 - 2pp_2 + \frac{3}{2}(m^2 + M_\mu^2)) + k_1^2(2k_1^2 - \frac{5}{2}(m^2 + M_\mu^2))] \\ + \frac{1}{A^2} [-k_1^2 B^2 + k_1^2 B(2k_1^2 + 4pp_2 - \frac{3}{2}(m^2 + M_\mu^2)) \\ + \frac{1}{2} k_1^2(-2(k_1^2 + 2pp_2)^2 + 3(k_1^2 + 2pp_2)(m^2 + M_\mu^2) - 4m^2 M_\mu^2)] \\ + \frac{1}{B^2} [-k_1^2 A^2 + k_1^2 A(2k_1^2 + 4pp_2 - \frac{3}{2}(m^2 + M_\mu^2)) \\ + \frac{1}{2} k_1^2(-2(k_1^2 + 2pp_2)^2 + 3(k_1^2 + 2pp_2)(m^2 + M_\mu^2) \end{aligned}$$

$$-4m^2M_\mu^2)] + \frac{k_1^2}{AB}\big[(k_1^2+2pp_2)(4pp_2+m^2+M_\mu^2)$$
$$-4pp_2(m^2+M_\mu^2)-4m^2M_\mu^2)\big],$$

$$a_2 = -2 + 2\frac{k_1^2}{A} + \frac{k_1^2}{AB}\big[-2k_1^2-4pp_2+3(m^2+M_\mu^2)\big]$$
$$+\frac{1}{B}\big[-4A+6k_1^2+8pp_2-3(m^2+M_\mu^2)\big]$$
$$+\frac{1}{B^2}\big[-2A^2+A(4k_1^2+8pp_2-3(m^2+M_\mu^2))-2(k_1^2+2pp_2)^2$$
$$+3(k_1^2+2pp_2)(m^2+M_\mu^2)-4m^2M_\mu^2)\big],$$

$$a_3 = -2 + 2\frac{k_1^2}{B} + \frac{k_1^2}{AB}\big[-2k_1^2-4pp_2+3(m^2+M_\mu^2)\big] \qquad (36)$$
$$+\frac{1}{A}\big[-4B+6k_1^2+8pp_2-3(m^2+M_\mu^2)\big]$$
$$+\frac{1}{A^2}\big[-2B^2+B(4k_1^2+8pp_2-3(m^2+M_\mu^2))-2(k_1^2+2pp_2)^2$$
$$+3(k_1^2+2pp_2)(m^2+M_\mu^2)-4m^2M_\mu^2)\big],$$

$$a_4 = -4 + \frac{1}{A}\big[-2B+4k_1^2+8pp_2-3(m^2+M_\mu^2)\big] \qquad (37)$$
$$+\frac{1}{B}\big[-2A+4k_1^2+8pp_2-3(m^2+M_\mu^2)\big]$$
$$+\frac{1}{AB}\big[-2(k_1^2+2pp_2)^2+2(k_1^2+2pp_2)(m^2+M_\mu^2)$$
$$+2pp_2(m^2+M_\mu^2)-4m^2M_\mu^2)\big].$$

We present also the expression for R_{1234}:

$$R_{1234} = \frac{x_1x_2y}{2k_1^2k_2^2}\Big\{\frac{Sp_{13}}{A\bar{A}} + \frac{Sp_{24}}{B^2} + \frac{(1+P(1,2))}{B}\Big(\frac{Sp_{14}}{A}\Big)\Big\}$$
$$k_2^2 = (p_2+p_3)^2, \qquad \bar{A} = (p-k_2)^2 - M_\mu^2, \qquad (38)$$

where

$$Sp_{24} = 16(pq)(p_1p_2)\big[(qp_3)\big((p_1p_2)+(p_1k_1)+(k_2p_2)+(k_1k_2)\big)$$

$$
\begin{aligned}
&-\big((qp_2)+(qk_1)\big)\big((p_1p_3)+(k_2p_3)\big)\\
&-\big((qp_1)+(qk_2)\big)\big((k_1p_3)+(p_2p_3)\big)\big]\\
&+8(qq)(p_1p_2)\big[(pp_3)\big((p_1p_2)+(p_1k_1)+(k_2p_2)+(k_1k_2)\big)\\
&-\big((pp_2)+(pk_1)\big)\big((p_3p_1)+(p_3k_2)\big)\big)\\
&-\big((pp_1)+(pk_2)\big)\big((p_3p_2)+(p_3k_1)\big)\big];
\end{aligned}
$$

$$
\begin{aligned}
Sp_{13} = {} & 16(pp_3)\big[(pq)\big((qp_1)(pp_2)+(pp_1)(qp_2)-(p_1p_2)((qp)\\
&-(qk_1)-(qk_2))\big)+(qq)\big((pp_2)(pp_1)+3/2\big((k_2p_2)(k_1p_1)\\
&-(k_1p_1)(pp_2)-(k_2p_2)(pp_1)\big)+1/2\big((p_1p_2)(-(k_1p)\\
&-(k_2p)+(k_1k_2))+(k_1p_2)(pp_1)+(k_2p_1)(pp_2)-(k_1p_2)(k_2p_1)\big)\\
&-(qk_1)(qk_2)(p_1p_2)-(k_1k_2)(qp_1)(qp_2)+(qk_1)(qp_2)(p_1k_2)\\
&+(qk_2)(qp_1)(k_1p_2)+(p_1q)((k_1p)(qp_2)-(k_1p_2)(qp))\\
&+(p_2q)((k_2p)(qp_1)-(k_2p_1)(qp))-(k_1q)(qp_2)(pp_1)\\
&-(qk_2)(qp_1)(pp_2)\big]+4M^2\big[2(qp_3)\big((p_1q)(k_1p_2)+(p_2q)(k_2p_1)\\
&+(pq)(p_1p_2)-(k_1q)(p_1p_2)-(qk_2)(p_1p_2)\big)+2(qp_1)\big(((qk_2)\\
&-(pq))(p_2p_3)-(p_2q)(k_2p_3)\big)+2(qp_2)\big((k_1q)(p_1p_3)\\
&-(pq)(p_1p_3)-(p_1q)(k_1p_3)\big)+(qq)\big(3(k_1p_1)(p_2p_3)+3(k_2p_2)(p_1p_3)\\
&+(p_1p_2)((k_2p_3)+(k_1p_3)-(pp_3))\\
&+(p_2p_3)(-(k_2p_1)-(pp_1))+(p_1p_3)(-(k_1p_2)-(pp_2))\big)\big];
\end{aligned}
$$

$$
\begin{aligned}
Sp_{14} = {} & 8M^2\big[-(qq)(p_1p_2)(p_1p_3)+(qp_1)\big((p_1p_2)(qk_1)+(qp_2)(k_1p_1)\\
&-(qp_1)(k_1p_2)\big)+(p_2p_1)\big((qp_2)(k_1p_1)+(qp_1)(k_1p_2)\\
&-(qk_1)(p_2p_1)\big)\big]+8(qq)\big[(pp_1)\big((p_2p_1)((pp_3)+(qk_1)-(qp_1))\\
&+(p_1p_3)((pp_2)+(qp_2))-(p_2p_3)((pp_1)+(qp_1))\big)\\
&-(p_2p_3)\big((p_2p_1)(pp_3)+(pp_2)(p_1p_3)-(p_2p_3)(pp_1)\big)\\
&-(pp_2)\big((p_2p_3)(pp_1)+(p_2p_1)(pp_3)-(pp_2)(p_1p_3)\big)\\
&-2(p_1p_3)\big((pp_2)(p_1p_3)+(p_2p_1)(pp_3)-(pp_1)(p_2p_3)\big)\big]\\
&+16(qp)(pp_1)\big((qp_3)(p_2p_1)-(qp_1)(p_2p_3)\big)\\
&+16(pq)(p_1p_3)\big((qp_1)(pp_2)-(qp)(p_2p_1)\big)\\
&+16(pq)(p_2q)\big((pp_2)(p_1p_3)-(p_2p_1)(pp_3)-(p_2p_3)(pp_1)\big)\\
&+16(pq)(p_1q)\big((qp_1)(k_1p_2)-(p_2q)(p_1k_1)-(qk_1)(p_2p_1)\big)
\end{aligned}
$$

$$+8(k_1k_1)(p_2p_1)\big((qp)(p_2p_1)+(qp_1)(pp_2)-(qp_2)(pp_1)\big)$$
$$+16(p_2p_1)(pp_2)\big((qk_1)(pp_1)-(qp_1)(k_1p)-(qp)(k_1p_1)\big)$$
$$+16(p_2q)(p_2p_3)\big((qp_3)(pp_1)-(qp_1)(pp_3)\big)$$
$$+16(p_2q)(p_1p_3)\big((qp_2)(pp_3)-(qp_3)(pp_2)\big).$$

Appendix 2

Here we present the angular integrals. The most complicated calculations take place in the double-collinear region. At first we give the results of the calculations and then an example. In the left side of equations we imply

$$\int_0^\sigma dz_1 \int_0^\sigma dz_2 \int_0^{2\pi} \frac{d\phi}{2\pi}$$

where σ is the auxiliary parameter chosen as $m_e/M_\mu \ll \sigma \ll 1$. We may omit the terms of the order σ compared with unity. Invariants k_1^2, B, a_{23} are defined above (14):

$$\frac{1}{k_1^2 B} = \frac{\Theta}{x_1x_2yt_1}\{\frac{1}{2}L^2 + L\ln\sigma + L\ln\frac{x_1x_2y}{\Delta}\};$$
$$\frac{k_2^2}{k_1^2 B} = \frac{\Theta}{x_1x_2t_1^2}\{\frac{1}{2}L^2 + L\ln\sigma + L\ln\frac{x_1x_2y}{\Delta} + L(-1+\frac{x_1x_2}{y\Delta})\};$$
$$\frac{1}{k_1^2k_2^2} = \frac{\Theta}{x_1x_2y^2}\{L^2 + 2L(\ln\sigma + 2L\ln\frac{x_1x_2y^2}{t_1t_2})\};$$
$$\frac{1}{k_1^2k_2^2} = \frac{\Theta}{x_1x_2y^2}\{L^2 + 2L(\ln\sigma + \ln\frac{x_1x_2y^2}{t_1t_2})\};$$
$$\{\frac{a}{B} - \frac{a_0}{B_0}\}\frac{1}{(k_1^2)^2} = \frac{\Theta}{yt_1^3}\{(1-\frac{y\Delta}{x_1x_2})(\frac{1}{2}L^2 + L\ln\frac{x_1x_2y}{\Delta})$$
$$-2L + (1-\frac{y\Delta}{x_1x_2})L\ln\sigma\};$$
$$\{\frac{a^2}{B^2} - \frac{a_0^2}{B_0^2}\}\frac{1}{(k_1^2)^2} = \frac{\Theta}{t_1^4}\{2(2-\frac{y\Delta}{x_1x_2})[\frac{1}{2}L^2 + L\ln\sigma + L\frac{x_1x_2y}{\Delta}]$$
$$+\frac{x_1x_2}{y\Delta} - 10 + \frac{y\Delta}{x_1x_2}\};$$
$$\frac{m_e^2}{(k_1^2)^2 B} = \frac{\Theta}{x_2t_1^3}L; \qquad \frac{1}{B^2} = \frac{\Theta}{x_1x_2y\Delta}L.$$

Here

$$\frac{a_0}{B_0} = \frac{a}{B}\Big|_{k_1^2=0} = \frac{y}{t_1}$$

is a regularization of divergence integrals. One can show that the total contribution from the additional terms is equal to 0.

When the created pair is heavy ($m_2 \ll m_l \ll m_\tau$) all of the previous results remain. In the case of a light pair production ($m_l \ll m_2 \ll m_\tau$), the previous integrals obtain additional terms δI_k:

$$\delta(\frac{1}{k_1^2 B}) = -\frac{\Theta l}{x_1 x_2 y t_1} 2 \ln \frac{x_1 y}{t_1}; \qquad \delta(\frac{k_2^2}{k_1^2 B}) = -\frac{\Theta l}{x_1 x_2 t_1^2} 2 \ln \frac{x_1 y}{t_1};$$
$$\delta(\frac{a}{B} - \frac{a_0}{B_0}) = \frac{\Theta l}{y t_1^3} 2(1 - \frac{y\Delta}{x_1 x_2}) \ln \frac{t_1}{x_1 y};$$
$$\delta(\frac{a^2}{B^2} - \frac{a_0^2}{B_0^2}) = \frac{\Theta l}{t_1^4} 4(2 - \frac{y\Delta}{x_1 x_2}) \ln \frac{t_1}{x_1 y};$$
$$\delta(\frac{m_e^2}{(k_1^2)^2 B}) = -\frac{\Theta l}{x_2 t_1^3}; \qquad \delta(\frac{1}{B^2}) = -\frac{\Theta l}{x_1 x_2 y \Delta};$$
$$l \equiv \ln \frac{m_2^2}{m_e^2}.$$

Besides, the contribution of the order $l = \ln \frac{m_2^2}{m_e^2}$ comes from the integral

$$\frac{m^2}{k_1^2 B^2}\Big|_{m^2 \gg m_e^2} = \frac{\Theta l}{x_1 y t_1^2}.$$

As an example let us consider the following integral

$$I = \left\{\frac{a}{B} - \frac{y}{t_1}\right\} \frac{1}{(k_1^2)^2},$$

in the region $z_1 < \sigma$, $0 < z_2 < 1$, when $m_2 = m_l = m$.

After the integration over ϕ we obtain (see (17))

$$I = \int_0^\sigma \frac{dz_1}{(m_e^2 \gamma + \delta z_1)^2} \left\{ \Theta \int_0^1 dz_2 (\frac{m^2 \alpha + \beta}{\sqrt{R}} - \frac{\beta}{b_3}) + \Theta_1 \int_\lambda^1 dz_2 (\frac{\beta}{\sqrt{R}} - \frac{\beta}{b_3}) \right\}, \tag{39}$$

where

$$R = \left(m^2b_1 + b_3z_2 + z_1(b_2 - 2b_4z_2)\right)^2 - 4b_4^2z_1z_2(1-z_2), \quad \lambda = \frac{\Delta - 1}{x_2t_1},$$
$$\Theta = \Theta(1-\Delta), \qquad \Theta_1 = \Theta(\Delta - 1)\Theta(1 - x_1 - y).$$

The integration over z_2 gives (see (19))

$$\int_0^\sigma \frac{dz_1}{(m^2\gamma + \delta z_1)^2}\Big\{\Big[\frac{m^2(\alpha b_3 - \beta b_1)b_3 + z_1\beta(b_2b_4 - 2b_4^2)}{b_3^3}$$
$$\times \ln\frac{b_3^2}{m^2b_1b_3 + z_1(b_2b_3 - b_4^2)} + \frac{\beta z_1}{b_3^3}(2b_3b_4 - 4b_4^2)\Big]\Theta$$
$$+\Theta_1\frac{\delta^2 z_1}{yt_1^3}\int_\lambda^1 \frac{dz_2}{z_2}\Big[\Big(\Big(1 - \frac{y\Delta}{x_1x_2}\Big) + 2\frac{y}{x_1}z_2\Big]\Big\},$$

where

$$\alpha = \frac{1}{x_2y}(x_2^2 + y^2), \qquad \beta = x_2y, \qquad \gamma = \frac{t_1^2}{x_1y}, \qquad \delta = x_1y$$
$$b_1 = \frac{(x_1+x_2)t_1t_2}{x_1x_2y}, \qquad b_2 = x_1t_2, \qquad b_3 = x_2t_1, \qquad b_4 = x_1x_2.$$

The final result for the integral is

$$I = \frac{1}{yt_1^3}\Big\{\Theta\Big(1 - \frac{y\Delta}{x_1x_2}\Big)\Big(\frac{1}{2}L^2 + L\ln\frac{x_1x_2y}{\Delta}\Big) - \frac{2(x_1 - y)}{x_1}L\Theta$$
$$+\Theta_1 L\Big[\ln\frac{1}{\lambda}\Big(1 - \frac{y\Delta}{x_1x_2}\Big) + 2\frac{y}{x_1}(1-\lambda)\Big]\Big\}.$$

In the double-collinear region $z_1, z_2 < \sigma \ll 1$ there are terms in R_{1212}, which give the contribution of the order M_μ^2/m_e^2. The sum of the terms of this kind is equal to zero. In order to see this, we present their sum in the form

$$R_{1212} \to \frac{1}{(k_1^2)^2A^2B^2}\left(x_12(p_2p_3) - y2(p_1p_2)\right)^2\left[-2(A+B)^2\right. \quad (40)$$
$$\left.+(4x_2 - 3)(A+B) - 2x_2 + 3x_2\right].$$

The contribution in the parentheses may be performed as (see (7))

$$\left(2(p_2p_3)x_1 - 2(p_1p_2)y\right)^2 = 4x_1^2x_2^2y^2z_1z_2(1-z_2)\cos^2\phi\,\left(1 + O(\sqrt{z_1})\right). \quad (41)$$

We see that all the mass-singular terms disappear.

One may also be convinced that the terms of the order L^2 do not appear from R_{1234}. The next surprising fact is the cancellation of L-terms from R_{1234} in the semi-collinear region $z_1 < \sigma, z_2 \sim 1$.

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