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QUASICLASSICAL DESCRIPTION OF THE NUCLEAR PARTICLE SCATTERING AT LARGE ANGLES

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Introduction

To calculate elastic scattering of alpha-particles, light, and heavy ions in the nuclear potential field at energies $E \gg V$, a large number of partial waves $l \simeq kR \gg 1$, where R is the radius of a nucleus (or the interaction range) is to be taken into account. At large scattering angles the cross section is as a rule exponentially decreasing and the solution of the problem becomes much more difficult. Indeed, in this case every term of the corresponding sign-alternating set of partial waves for the scattering amplitude compensates its neighbor and, as a result of summation, we obtain a small value of the calculated cross section. So, with the aim to get a reliable result it is necessary to achieve a very large accuracy in numerical calculations, required in rather large computational time. One way to avoid these difficulties is to use the semiclassical approach for the one- dimensional radial wave equation. Methods of that kind for elastic scattering were reviewed in [1]. As to the semiclassical wave functions, the corresponding approach was suggested in [2], where also the sum over partial waves was replaced by the integral over collision parameters, so that the wave function was obtained in a three- dimensional form.

However, there exist other possible ways to resolve the mentioned problems. Indeed, the initial conditions $E \gg V$, $kR \gg 1$ may be used themselves for developing the approaches where it is not necessary to use the partial wave expansion for the elastic scattering amplitude [3–5]. In particular, one can develop a method based on three- dimensional semiclassics, that operates not with the one- dimensional partial waves but directly with the three-dimensional action function [5]. We use the condition $E \gg V$, which simplifies the problem and permits us to use the straight- line classical trajectories in calculating the action integrals.

Below we will apply this method to the process of elastic scattering in a field of the real nuclear potential $V = V_N$, an example, that demonstrates the main features of that kind of calculations. In the framework of the High Energy Approximation (HEA) [5] we start with the elastic scattering amplitude of the following form:

$$f(\vartheta) = \frac{1}{4\pi}\int \Psi_f^{(-)*}(\vec{r})\Psi_i^{(+)}(\vec{r})U(\vec{r})d\vec{r}, \qquad (1)$$

where $U = \frac{2m}{\hbar^2}V_N$, V_N is the nuclear potential; and $\Psi^{(\pm)}$, the incoming and outgoing wave functions, respectively. The amplitude (1) is applied in the region of large angles $\vartheta > \frac{1}{kR}$, i.e. practically in the whole interesting interval of scattering angles.

Wave Functions

When the condition $V/E \ll 1$ is fulfilled, the semiclassical wave functions as solutions of the wave equation have the following form:

$$\Psi^{(\pm)} = e^{iS^{(\pm)}}, \qquad (2)$$

where

$$S^{(\pm)} = \vec{k}\vec{r} \pm \Phi_N^{(\pm)}(\vec{r}), \qquad (3)$$

$$\Phi_N^{(\pm)} = -\frac{1}{\hbar v}\int_0^\infty V_N(\vec{r} \mp \hat{k}s)ds. \qquad (4)$$

It is natural to conclude that at high energies the main region of formation of the phase (4) is the space $r < R_N$, where R_N is the radius of a potential. Due to the integration in (4) the detailed features of the nuclear potential near its periphery at $r \approx R_N$ will not be important. Therefore in calculating the phase (4) this allows us to use an approximation of the step potential

$$V_N \approx V_0\theta(R_0 - r) = \begin{cases} V_0 & \text{if } R_0 - r > 1 \\ 0 & \text{if } R_0 - r < 1, \end{cases} \qquad (5)$$

where $V_0 = -|V_0|$ is the parameter of the potential well. Now choosing the straight line trajectories of the classical motion we replace $\vec{r} = \vec{\rho} + \vec{z}$

($\vec{\rho} \perp \vec{z}$), where ρ means a collision parameter. Bearing in mind that the trajectory is directed along the momentum of motion, i.e. $\vec{z} \uparrow\uparrow \vec{k}_i$, we obtain, with the help of (4),(5), the phase for the incoming channel:

$$\Phi_{i,N}^{(+)} = -\frac{1}{\hbar v_0}\int_0^\infty V_N(\vec{r} - \hat{k}s)ds =$$

$$= -\frac{V_0}{\hbar v}\int_{-z_i}^\infty \theta(R_0 - \sqrt{\rho_i^2 + \lambda^2})d\lambda = -\frac{V_0}{\hbar v}(\sqrt{R_0^2 - \rho_i^2} - z_i), \qquad (6)$$

Then, using smallness $\rho/R_0 < 1$ we expand (6) in the intrinsic region:

$$\Phi_{i,N}^{(+)} \simeq -\frac{V_0}{\hbar v}(R_0 - \frac{\rho_i^2}{2R_0} + z_i) \qquad (7)$$

In the same way we find the phase in the outgoing channel:

$$\Phi_{f,N}^{(-)} \simeq -\frac{V_0}{\hbar v}(R_0 - \frac{\rho_f^2}{2R_0} - z_f) \qquad (8)$$

One can note that these phases have a meaning only in the region of action of the corresponding potential. Then, one can establish a connection between R_0 and the radius of a realistic potential as, for instance, the Saxon-Woods potential

$$V_N = \frac{V_0}{1 + \exp\frac{r-R}{b}}. \qquad (9)$$

For this aim one should compare the mean squared radii of two potentials (5) and (8) to obtain

$$R_0 = R\sqrt{1 + \frac{7}{3}(\frac{\pi b}{R})^2} \qquad (10)$$

In addition, we note that using the approximate expressions for phases (7),(8), obtained by an expansion in ρ/R_0, we have actually increased the radius of the step potential, so the function (7) is more spreaded over ρ as compared with the exact phase (6). This shift of the radius can be taken into account in (7) if one changes R_0 by the effective radius R_e, that must be smaller in its value. This effective radius can be derived from the condition of an identity of the positions of a semislope of functions (6) and (7), which gives us the relation $R_e = \sqrt{\frac{3}{4}}R_0$. Inserting here the expression (10), we obtain the effective radius to be used instead of R_0 in the approximate phases (7) и (8):

$$R_0 \to R_e = R\sqrt{\frac{3}{4}[1+\frac{7}{3}(\frac{\pi b}{R})^2]} \qquad (11)$$

The scattering amplitude (1) contains the product of semiclassical wave functions, that can be presented with the help of (2),(7),(8) in the following way:

$$\Psi_f^{(-)*}\Psi_i^{(+)} = e^{i\vec{q}\vec{r}+i\Phi_N} = e^{i\tilde{\Phi}}, \qquad (12)$$

where

$$\Phi_N = \Phi_{i,N}^{(+)} + \Phi_{f,N}^{(-)} = 2a_{0,N} + \frac{a_{1,N}}{k}\vec{q}\vec{r} + \frac{a_{2,N}}{k^2}[(\vec{k}_i\vec{r})^2 + (\vec{k}_f\vec{r})^2], \qquad (13)$$

and $\vec{q} = \vec{k}_i - \vec{k}_f$, $q = 2k\sin\frac{\vartheta}{2}$ is the transfer momentum dependent on the scattering angle ϑ and $k = k_i = k_f$. Here we also have:

$$a_{0,N} = -\frac{V_0}{\hbar v}(R_e - \frac{r^2}{2R_e}), \quad a_{1,N} = -\frac{V_0}{\hbar v}, \quad a_{2,N} = -\frac{V_0}{\hbar v}\frac{1}{2R_e}. \qquad (14)$$

To calculate the scalar products in (13), we select the coordinate system with axes $oz \uparrow\uparrow \vec{q}$ and $ox \uparrow\uparrow \vec{K} = \vec{k}_i + \vec{k}_f$. Then,

$$\vec{k_i}\vec{r} = kr(\alpha\mu + \sqrt{1-\alpha^2}\sqrt{1-\mu^2}\cos\bar{\varphi}), \qquad (15)$$

$$\vec{k_f}\vec{r} = kr(-\alpha\mu + \sqrt{1-\alpha^2}\sqrt{1-\mu^2}\cos\bar{\varphi}), \qquad (16)$$

where $\alpha = \sin\frac{\vartheta}{2}$, and $\bar{\theta}$ and $\bar{\varphi}$ are the angles that determine the position of vector $\vec{r}$ in a spheric coordinate system. Inserting (15),(16) into (13) and (12), we obtain the whole phase

$$\tilde{\Phi} = \vec{q}\vec{r} + \Phi_N = 2a_{0,N} + \beta_n\mu + n_1\mu^2 + n_2(1-\mu^2)\cos^2\bar{\theta}, \qquad (17)$$

where

$$\beta_n = 2(k + a_{1,N})\alpha r; \quad n_1 = 2a_{2,N}\alpha^2 r^2; \quad n_2 = 2a_{2,N}(1-\alpha^2)r^2. \qquad (18)$$

Thus, the integrand (1) contains a product of semiclassical wave functions of the form $\exp i\tilde{\Phi}$ with the phase (17) having the typical linear dependence on the variables of integration r and $\mu = \cos\bar{\theta}$.

The Scattering Amplitude

We will consider the potential $U(\vec{r})$ in the amplitude (1) as a spherically symmetrical function. Then, keeping in mind that $d\vec{r} = -r^2 dr d\mu d\bar{\varphi}$, we first integrate in (1) over $d\mu$ by parts. Here we should take into account the semiclassical conditions, which mean in particular $\beta_n \gg 1$. Then, $\partial\tilde{\Phi}/\partial\mu \approx \beta_n \approx kR \gg 1$ and therefore we can write

$$I = \int_{-1}^{1} d\mu e^{i\tilde{\Phi}(r,\mu,\bar{\varphi})} =$$

$$= \frac{e^{i\tilde{\Phi}}}{i\frac{\partial\tilde{\Phi}}{\partial\mu}}\Big|_{-1}^{1} + i\int \frac{e^{i\tilde{\Phi}}\frac{\partial^2\tilde{\Phi}}{d\mu^2}}{(\frac{\partial\tilde{\Phi}}{\partial\mu})^2}d\mu = -i\frac{e^{i\tilde{\Phi}}}{\partial\tilde{\Phi}/\partial\mu}\Big|_{-1}^{1} + O(\frac{1}{(kR)^2}), \qquad (19)$$

where the second term has a higher order of smallness in $(kR)^{-1}$ and can be neglected. As a result, we have:

$$I = -ie^{2ia_{0,N}}[I_{(+)} - I_{(-)}], \qquad (20)$$

where

$$I_{(\pm)} = \frac{e^{\pm i(\beta_n \pm n_1)}}{\beta_n \pm 2n_1 \mp 2n_2\cos^2\bar{\varphi}} \qquad (21)$$

The integration in (20) over $d\bar{\varphi}$ can be performed with the help of the table integral:

$$\int_0^{2\pi} \frac{d\bar{\varphi}}{A^2 + (B^2 - A^2)\cos^2\bar{\varphi}} = \frac{4\pi}{AB}, \qquad (22)$$

This lets us to obtain the scattering amplitude (1) in a form of the one-dimensional integral:

$$f(\theta) = -\frac{im}{\hbar^2}\int r^2 dr e^{2ia_{0,N}+in_1}V_N[F_{(+)}^N - F_{(-)}^N], \qquad (23)$$

where

$$F_{(\pm)}^N = \frac{e^{\pm i\beta_n}}{2\alpha r\sqrt{(k \pm 2a_{2,N}\alpha r)(k \mp 2(a_{2,N}/\alpha)r)}} \qquad (24)$$

The root singuliarities in the denominator are situated far from the radius of interaction R, namely, at $r_s = \frac{ME}{20} R_0$, where M is the mass of a scattered particle in the proton mass units, and E, its energy in MeV. At these distances the potential V_N in the integrand is decreasing very fast, and the previously suggested approximations for calculations of phases do not work. This forces us to introduce a prescription when the integration in (23) should be cut off at distances less than the point r_s, that is to exclude the increase of an integrand due to the nonphysical singuliarities. In this way one can suggest the method of calculating the integral (23) by using the properties of the potential (9) on the complex r-plane. One can easily see that it has simple poles at $r_{\varepsilon,k} = R + i\varepsilon\pi b(2l+1)$, where $\varepsilon = \pm 1$, and $l = 0, 1, 2, \ldots$ It is possible to show that the integration contour for $\varepsilon = +1$ should be drawn in the first quadrant of the complex plane and goes along the imaginary axis and then over the circles of an infinite radius. The same contour, but in the fourth quadrant, must be used in the case of ($\varepsilon = -1$). Thus, the result is expressed through a sum of the corresponding residues at the above-mentioned poles. However, in practice it is enough to take into account only the couple of poles r_{11} and r_{-11} nearest to the real axis, one in the first and the other in the fourth quadrant, because each next pair contributes approximately an order smaller than the previous one.

Conclusion

The final expression for the scattering amplitude (23) in terms of poles of the Saxon-Woods potential has the following typical form:

$$f(\vartheta) \sim e^{-2\pi bk \sin\frac{\vartheta}{2}} \cos(2kR_e \sin\frac{\vartheta}{2} + X(R_e, b, k, \vartheta)), \quad (25)$$

where X is a known function of the mentioned parameters. One can see that the amplitude and cross section fall down exponentially at large angles. Herewith the "diffuseness" parameter of a potential b determines a slope of the exponent, and a radius R regulates "oscillations" of the amplitude, like in diffraction models of scattering. At the same time the X-function shifts and smoothes these oscillations. To understand the influence of the scattering phase $\Phi_N^{(+)}$ on the behaviour of the scattering

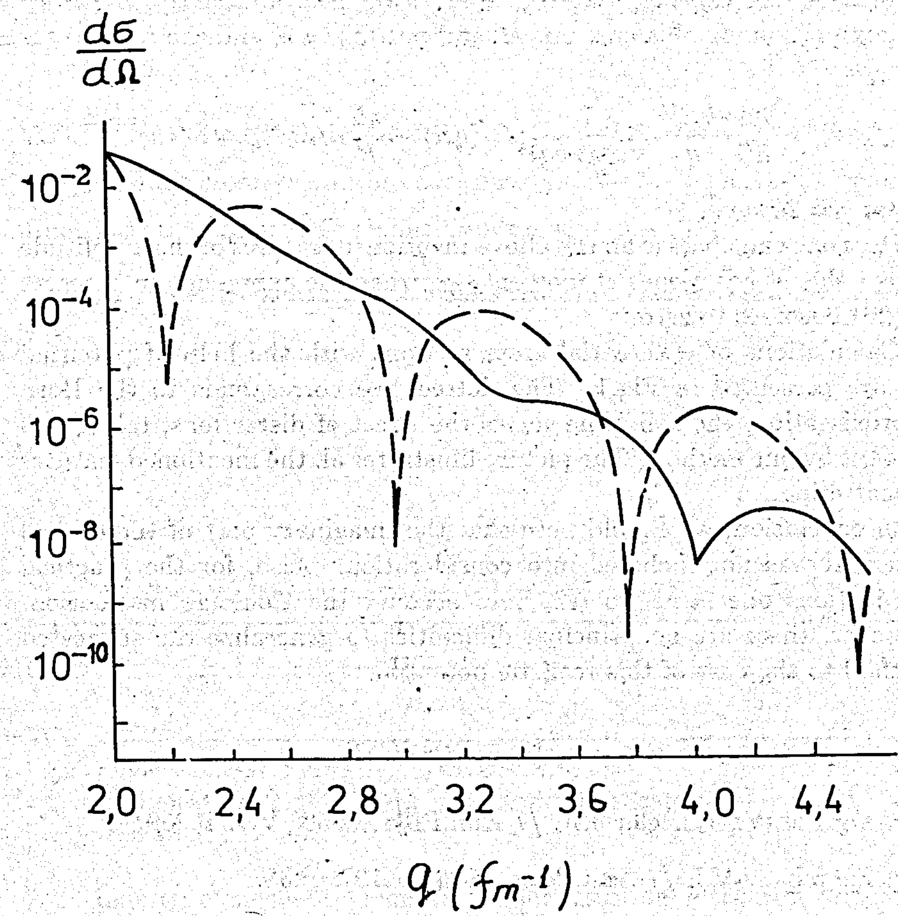

Fig.1. Differential cross section (in arbitrary units) as a function of the transfer momentum $q = 2k\sin\frac{\vartheta}{2}$. Dotted line is the Born approximation, solid line is a calculation with semiclassical functions. Parameters: $V_0 = -50$MэB, $k = 3fm^{-1}$, $R = 4fm$, $b = 0.7fm$.

amplitude, one can compare the result with that obtained in the Born approximation. In this case one should put $\Phi_N = 0$, and the result is as follows:

$$f_B(\vartheta) = \frac{2m}{\hbar^2}\frac{\pi bRV_0}{q}\frac{1}{\sinh(\pi bq)}[\cos(qR) - \frac{\pi b}{R}\sin(qR)\coth(\pi bq)] \qquad (26)$$

where $q = 2k\sin\frac{\vartheta}{2}$.

Here one can observe all the above-mentioned features of the amplitude except that it has zeroes at momenta q_0, when the expression in brackets in (26) becomes to zero.

Calculations of differential cross sections with the help of potential (9) are presented in Fig.1. The dotted line corresponds to the Born approximation, the solid line shows the effect of distortions, taken into account by our method. The picture illustrates all the mentioned features of scattering.

In conclusion, we should note that the imaginary part of the optical potential was not included into consideration. Also, for the practical applications one needs to take into account the Coulomb interaction. However, there are no principal difficulties to generalize the suggested method to the case of this realistic potential.

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