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ON A NOVEL EQUAL-TIME RELATIVISTIC QUASIPOTENTIAL EQUATION FOR TWO SCALAR PARTICLES

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1. – Introduction

The present work deals with the old problem of a relativistic description of two interacting particles. The quasipotential approach is presumably the most popular one among others. This approach to the two-particle relativistic problem has been developed first by Logunov and Tavkhelidze [1]. Quasipotential equations are differential ones with the structure of a one-body equation, with an energy-dependent quasipotential, valid for stationary states in the centre-of-mass (CM) system. In most applications, particle 2 has been treated non-relativistically. The first equation which avoided non-relativistic expansions in the quasipotential and which still allowed exact solution was that of Todorov [2]. So, the most popular solution of the two-body spinless problem is given by the Todorov-Komar-Droz-Vincent equation [2-4]. Similar equations, for the two-fermion and fermion-boson cases, have been proposed by Sazdjian [5]. In modern works [6-7] the authors consider the Todorov equation as the base one has to compare with.

In the second section of this paper we will construct an equation for two free particles that keeps full relativistic kinematics. Then we will introduce the electromagnetic interaction. It will be done in several steps: a pure Coulomb potential at first, then a transversal (magnetic) contribution, and the third – α^2 contributions. In the third section we will consider the properties of the solutions in various limit cases: the non-relativistic case, the case of equal masses, the case of one particle being at rest and the ultra-relativistic case. In the fourth section we will discuss the problem of retardation and make some conclusions.

2. – The equation

Let us start with the Schrödinger wave equation of the general form ($c = \hbar = 1$)

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi = \hat{H}\Psi\,, \tag{1}$$

where $\Psi = \Psi(t, \vec{x}_1, \vec{x}_2)$ is the wave function of two scalar particles. We begin our consideration in the equal-time formalism, later on the problem of the retardation will be discussed. At first we can take the Hamiltonian $\hat{H}$ as the sum of energies:

$$\frac{1}{i}\frac{\partial\Psi}{\partial t} = \left(\sqrt{\vec{p}_1^{\,2} + m_1^2} + \sqrt{\vec{p}_2^{\,2} + m_2^2}\right)\Psi\ . \qquad (2)$$

Here $\vec{p}_i$ has to be considered as an operator. We don't know how to work with square roots of operators. So, we take the time derivative once more and obtain

$$\frac{\partial^2\Psi}{\partial t^2} + \left(\sqrt{\vec{p}_1^{\,2} + m_1^2} + \sqrt{\vec{p}_2^{\,2} + m_2^2}\right)^2\Psi = 0\ . \qquad (3)$$

There are square roots again. But we can avoid the problem if we choose the frame of reference where

$$\vec{p}_2 = -\vec{p}_1\frac{m_2}{m_1}. \qquad (4)$$

Then all square roots disappear and we obtain the following equation

$$\frac{\partial^2\Psi}{\partial t^2} + (\vec{p}_1^{\,2} + m_1^2)(1 + \frac{m_2}{m_1})^2\Psi = 0\ . \qquad (5)$$

Now we substitute the variables: $(\vec{p}_1, \vec{p}_2, \vec{x}_1, \vec{x}_2) \longrightarrow (\vec{P}, \vec{R}, \vec{k}, \vec{r})$:

$$\vec{P} = \vec{p}_1 + \vec{p}_2,\quad \vec{R} = \frac{\vec{x}_1 + \vec{x}_2}{2},$$

$$\vec{k} = \frac{\vec{p}_2 - \vec{p}_1}{2},\quad \vec{r} = \vec{x}_2 - \vec{x}_1. \qquad (6)$$

It is convenient, because we will consider interactions depending on r.

We can now consider $\vec{p}_1$ as the differential operator

$$\vec{p}_1^{\,2} \longrightarrow -\frac{\partial^2}{\partial\vec{r}^{\,2}}\left(\frac{2m_1}{m_1 + m_2}\right)^2. \qquad (7)$$

So, we rewrite eq. (5) in the form

$$\frac{\partial^2\Psi}{\partial t^2} - 4\frac{\partial^2\Psi}{\partial\vec{r}^{\,2}} + (m_1 + m_2)^2\Psi = 0\ . \qquad (8)$$

This two-particle equation is analogous to the one-particle Klein-Gordon one, but it can be considered only in the chosen reference frame.

Now we will introduce an interaction. We will discuss a simple but interesting example: the electromagnetic interaction between two scalar particles. We use the minimal substitution:

$$p^\mu \rightarrow p^\mu - eA^\mu, \qquad (9)$$

where

$$\vec{A} = -\frac{e\vec{u}}{4\pi r},\qquad A^0 = -\frac{e}{4\pi r};\quad r \equiv |\vec{r}|. \qquad (10)$$

Note that in the reference frame (4) there is an important relation between the relativistic velocities of our particles:

$$\vec{u}_1 = \frac{\vec{p}_1}{\sqrt{\vec{p}_1^{\,2} + m_1^2}} = -\frac{\vec{p}_2}{\sqrt{\vec{p}_2^{\,2} + m_2^2}} = -\vec{u}_2 \equiv \vec{u}. \qquad (11)$$

It will be shown that this property is the main point of our approach.

At first we will take only the Coulomb potential (without $\hat{\alpha}^2$ terms). We can introduce it by the substitution

$$\frac{\partial}{\partial t} \longrightarrow \left(\frac{\partial}{\partial t} - i\frac{\hat{\alpha}}{r}\right);\qquad \hat{\alpha} \equiv Z_1 Z_2\frac{e^2}{4\pi}, \qquad (12)$$

where $Z_1 e$ and $Z_2 e$ are the charges of our particles. Neglecting the $\hat{\alpha}^2$ term, we obtain the equation

$$\left[-E^2 + 2\frac{E\hat{\alpha}}{r} - 4\frac{\partial^2}{\partial\vec{r}^{\,2}} + (m_1 + m_2)^2\right]\Psi = 0\ . \qquad (13)$$

Now taking E as an eigenvalue we derive the following equation

$$\left[\vec{k}^2 - \frac{E\hat{\alpha}}{2r} + \frac{\partial^2}{\partial\vec{r}^{\,2}}\right]\Psi = 0, \qquad (14)$$

E — the sum of the energies of the particles (see (2)). Note that in our approach for that interaction we have no reason to go off the mass shell.

The form of this equation is just the same as the one of the Schrödinger equation for the hydrogen atom. The difference is only in definitions of parameters and in the choice of a reference frame. The exact solution of an equation like that is well-known (see for example [9]). For a partial wave we have the radial equation:

$$\left[\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\rho + 1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right]R_l(\rho) = 0\ . \qquad (15)$$

$$\eta = \frac{\hat{\alpha}}{v}, \; v = \frac{4|\vec{k}|}{E}, \; \rho = (\vec{k}\vec{r}),$$

η — the Sommerfeld parameter for electromagnetic interactions. We can call v a generalized relative velocity, in the non-relativistic limit v is the common relative velocity. In a relativistic case, as we see, v depends on the way of the introduction of an interaction. So, in the solution we have two main parameters: v and $\hat{\alpha}$.

Now we will try to complete the minimal substitution. We will take into account the transversal (magnetic) interaction. Equation (5) can be rewritten in a symmetrical form:

$$\frac{\partial^2 \Psi}{\partial t^2} + [(\vec{p}_1 - \vec{p}_2)^2 + (m_1 + m_2)^2]\Psi = 0. \tag{16}$$

We can make this transformation due to the correlation between momenta (4). There is a serious problem: we have to avoid the "double counting". It would arise if we put the interaction twice: the action of the first particle on the second one and the other way round. Note that we do not meet with this problem for the Coulomb potential, while we have only one time variable for two particles. The solution is well-known in the conventional electrodynamics — we have to put in the Hamiltonian the half of the total interaction. So, let us try to put the minimal substitution only into $\vec{p}_1$:

$$\vec{p}_1 \longrightarrow \vec{p}_1 + \frac{\hat{\alpha}\vec{u}_2}{r}, \qquad \vec{p}_2 \longrightarrow \vec{p}_2 \tag{17}$$

We obtain the following modification of eq. (14)

$$\left[\vec{k}^{\,2} - \frac{E\hat{\alpha}}{2r} + \frac{\partial^2}{\partial \vec{r}^{\,2}} - \frac{\hat{\alpha}}{r}\frac{u^2}{\sqrt{1-u^2}}(m_1 + m_2)\right]\Psi = 0. \tag{18}$$

This equation differs from the Coulomb one (14) by the substitution in the quasipotential:

$$E \longrightarrow E + \frac{u^2}{\sqrt{1-u^2}}(m_1 + m_2) \equiv E + \Delta E,$$

where u is a function of the eigenvalue E. The form of the equation remains unchanged. We have only a shift in the value of the parameter v:

$$\begin{aligned} v &= \frac{4|\vec{k}|}{E + \Delta E} = \frac{v_C}{1 + v_C^2/4} \\ &= 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left(1 + \frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right)^{-1}, \end{aligned} \tag{19}$$

where v_C – the value of our generalized relative velocity in the case of a pure Coulomb potential.

There are some good features of the result obtained. At first we note that we would obtain just the same value of v if we make the minimal substitution only for $\vec{p}_2$. Secondly, we see that the result is symmetrical under the permutation of masses ($m_1 \to m_2$, $m_2 \to m_1$). This condition is very important for the description of all possible relations between the masses. And it is easy to show that if we try to put the magnetic interaction in both the momenta we would obtain a result without this symmetry.

There is an additional interesting feature of the result: the meaning of a relative velocity is restored in eq. (19). Now v is just the relativistic relative velocity of two particles. It is the relativistic sum of $\vec{u}_1$ and $\vec{u}_2$ (note that $\vec{u}_1$ and $\vec{u}_2$ are collinear):

$$\vec{v} = \frac{\vec{u}_1 + \vec{u}_2}{1 + (\vec{u}_1\vec{u}_2)}. \tag{20}$$

One can see that v_C goes to 2 with the growth of s, but v — only to 1 (in the units of the light velocity).

If we take now into account the $\hat{\alpha}^2$ terms from the Coulomb and the magnetic parts of the interaction, we obtain an additional term in eq. (15):

$$\frac{\hat{\alpha}^2}{4r^2}(-1 + u^2), \tag{21}$$

where "-1" comes from the Coulomb part and u^2 – from the magnetic one. The effect can be considered as a shift of the quantum number l:

$$\begin{aligned} &\frac{l(l+1)}{r^2} - \frac{\hat{\alpha}^2}{4r^2}(-1 + u^2) = \frac{l'(l'+1)}{r^2}, \\ &l' = \sqrt{(l + \frac{1}{2})^2 - \frac{\hat{\alpha}^2}{4}(1 - u^2)} - \frac{1}{2}. \end{aligned} \tag{22}$$

The breaking point at $\hat{\alpha}_{cr} = (1 - u^2)^{-1}$ — corresponds to the "critical charge". Our value for it differs from the Todorov one ($\hat{\alpha}_{cr} = 1/2$). The $\hat{\alpha}^2$ terms cause some other problems like a singularity of a wave function at the point $r = 0$. These problems are now beyond the scope of our consideration. Note only that in the relativistic case ($u \to 1$) all problems disappear.

3.– The limits

Now we will discuss the properties of the solution. There are several points where we have to compare the results with the previous ones.

At first we can consider the non-relativistic limit. In this case we have to compare the results of the equation with the spectra of hydrogen-like atoms. We can easily obtain conditions on bound states in an analogy with the Schrödinger equation. From our equation we have the bound state energy in reference frame (4). It is not difficult to make a relativistic transformation into the CM system. In the first order in α^2 the relativized Balmer formula is reproduced:

$$E_{CM} = m_1 + m_2 - \frac{\hat{\alpha}^2}{2n^2}\mu\,, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{23}$$

μ – the reduced mass. The next approximation reads

$$E_{CM} = m_1 + m_2 - \frac{\hat{\alpha}^2}{2n^2}\mu + \frac{\hat{\alpha}^4}{8n^4}\left(3\mu - \frac{m_1^2 m_2^2}{(m_1+m_2)^3} - \frac{8\mu n}{(2l+1)}\right). \tag{24}$$

In the last formula we take into account the Coulomb potential as well as the magnetic interaction and the $\hat{\alpha}^2$ terms. But we have omitted in eq. (14) and in eq. (18) terms arising from commutators of E and $1/r$. This remains to be our problem.

The equal-mass limit turns out to be important in our consideration. For the case $m_1 = m_2 \equiv m$ the Todorov equation gives

$$v_T = 2\sqrt{\frac{s-4m^2}{s}}\,\frac{s}{2(s-2m^2)}\,; \tag{25}$$

the Schrödinger equation gives

$$v_{Sch} = \frac{2p}{m} = 2\sqrt{\frac{s-4m^2}{s}}\,\frac{\sqrt{s}}{2m}, \qquad p = |(\vec{p}_2 - \vec{p}_1)/2|; \tag{26}$$

and, finally, our equation (the pure Coulomb potential) —

$$v_C = 2\sqrt{\frac{s-4m^2}{s}}. \tag{27}$$

The non-relativistic limits are equal for all these values.

Here we can involve an additional check. In the paper by Baier and Fadin [11] the correction due to the Coulomb interaction of two charged final (or initial) particles to the cross section of a process is considered. In the case of equal masses the infinite sum of ladder diagrams is calculated in [11], omitting the diagrams with crossed photon lines. It means taking into account only the Coulomb part of the interaction. The correction has the same form as that one obtained by Sakharov [12] from the non-relativistic Schrödinger equation:

$$T = \frac{2\pi\alpha/v}{1 - \exp(-2\pi\alpha/v)}. \tag{28}$$

Here we see our parameter v. Sakharov receives v as the non-relativistic velocity, Baier and Fadin — just (27). We suggest that the comparison with the result of the summation of the infinite ladder diagram series provides us with a good check in the equal-mass limit.

We have also to consider the limit when one of the particles is at rest. Todorov [2] suggested that any quasipotential equation in the case m_1 (or m_2) $\to \infty$ has to give the same results as the Klein-Gordon equation with the Coulomb interaction. This means that the parameter v in this case has to be equal to the relativistic velocity of the "light" particle:

$$v\big|_{m_1\to\infty} = \frac{|\vec{p}_2|}{\sqrt{\vec{p}_2^{\;2} + m_2^2}}. \tag{29}$$

We suggest a more general condition — an equation in the reference frame where one of the particles is at rest has to reproduce the value (29). For the Coulomb potential we have expression (27). It is written in the form which is suitable to make any choice of a reference frame. We take s in the case $\vec{p}_1 = 0$:

$$s = (p_1 + p_2) = (p_1^0 + p_2^0)^2 - (\vec{p}_1 + \vec{p}_2)^2 = m_1^2 + m_2^2 + 2m_1\sqrt{\vec{p}_2^{\;2} + m_2^2}, \tag{30}$$

and obtain

$$v_C\big|_{\vec{p}_1=0} = \frac{2|\vec{p}_2|}{\sqrt{\vec{p}_2^{\;2} + m_2^2} + m_2}. \tag{31}$$

This result does not coincide with (29) in a relativistic case. But we have to remember that there are also magnetic interactions and they are really important. Note that there is no magnetic term in the frame of reference where one particle is at rest, because it is proportional to the product of two velocities. When we choose another reference frame, we have to make a full relativistic transformation of interactions starting from full equation (18) in the reference frame (4). So, we use (19) and obtain

$$v|_{\vec{p}_1=0} = \frac{v_C}{1+v_C^2/4} = \frac{|\vec{p}_2|}{\sqrt{\vec{p}_2^{\ 2}+m_2^2}}. \tag{32}$$

Our approach allows us to describe also the ultra-relativistic case. In this case we can compare our results with the ones of the corresponding Feynman diagram. Let us take two scalar particles scattering on each other by exchanging with a photon. We can consider the case of arbitrary masses. In the ultra-relativistic case we have

$$|\vec{p}_i|, |\vec{k}_i| \gg m_i, |\vec{q}| \tag{33}$$

$$p_i^0 \approx |\vec{p}_i| \approx k_i^0 \approx |\vec{k}_i|, \quad i = 1, 2. \tag{34}$$

Here k_i^0, $\vec{k}_i$ and p_i^0, $\vec{p}_i$ are, respectively, the energies and momenta of the initial and final particles. $\vec{q}$ is the transfer momentum.

The differential cross section of the process can be written in the form

$$d\sigma = \frac{1}{|\vec{u}_1-\vec{u}_2|}\frac{1}{4k_1^0k_2^0}|\mathcal{M}|^2\frac{d^3p_1}{2p_1^0(2\pi)^3}\frac{d^3p_2}{2p_2^0(2\pi)^3}(2\pi)^4\delta^4(k_1+k_2-p_1-p_2). \tag{35}$$

$\mathcal{M}$ is the matrix element of the process,

$$\sqrt{|\mathcal{M}|^2} = \frac{e^2}{q^2}(k_2+p_2)^\mu(k_1+p_1)_\mu, \tag{36}$$

$$q^2 = (k_2-p_2)^2 \approx -4\vec{p}_2^{\ 2}\sin^2(\Theta_2/2), \tag{37}$$

where Θ_2 is the scattering angle of the second particle. For the differential cross sections depending on the parameters of the second particle we obtain:

$$\frac{d\sigma}{d\Omega_2} = \frac{\alpha^2}{4\sin^4(\Theta_2/2)}\frac{1}{\vec{p}_2^{\ 2}}. \tag{38}$$

It has the form of the Rutherford formula [8].

From our equation we have an exact solution for the wave function. It has the usual form and the asymptotics is well-known (see for example [10]). So, we can write at once from eq. (18)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{(E+\Delta E)v^2/2}\right)^2\frac{1}{\sin^4(\Theta/2)}, \tag{39}$$

but this result is written for an effective particle with the momentum $\vec{k}$ (see (6)). We have to return now to the parameters of real particles:

$$\vec{p}_2^{\ 2} = \vec{k}^{\ 2}\left(\frac{2m_2}{m_1+m_2}\right)^2, \qquad \Theta_2 = \Theta\frac{m_1+m_2}{2m_2}, \qquad \Omega_2 = \Omega\frac{m_1+m_2}{2m_2}. \tag{40}$$

(Note that in the ultra-relativistic case we have to consider small scattering angles.) Then we obtain the result coinciding with (38).

There is an interesting point here. We take now into account both the Coulomb interaction and the magnetic one. But if we took only the Coulomb one (use eq. (14)), we would obtain for the differential cross section the result being four times as small as (38). We can see this also from the diagram itself. The matrix element (36) has two parts in the product of the four-momenta: $(k_1^0+p_1^0)(k_2^0+p_2^0)$ and $(\vec{k}_1+\vec{p}_1)(\vec{k}_2+\vec{p}_2)$. The first part corresponds to the Coulomb interaction and the second — to the magnetic one. And in the ultra-relativistic case they give equal contributions.

So, we can conclude that in an ultra-relativistic case for arbitrary masses the equation gives a good description of charged particles scattering. Note that in this case spins of the particles are not important, the calculation of the Feynman diagram for two spinors gives also (38).

4.– The discussion and conclusions

Now we have to clarify the reasons for our use of the equal-time formalism. At first, we note that the correct ultra-relativistic limit means that the retardation was taken into account in a certain way. (A Feynman diagram "knows all" about the retardation). Let us look at the retarded potentials

$$A_1^0 = -\frac{Z_2e}{4\pi r_1'}, \qquad \vec{A}_1 = -\frac{Z_2e\vec{u}_2}{4\pi r_1'}, \qquad r_1' \equiv r(1+|\vec{u}_2|).$$

The second particle acts on the first one via this potential. We see that only in the frame of reference we have chosen (see (11)) the retardation factors are equal for both actions 1 → 2 and 2 → 1. Then we can say that in an equation like eq. (14) when we put a retardation in the form

$$\frac{\alpha}{r} \longrightarrow \frac{\alpha}{r'},$$

we have to put also

$$\frac{\partial^2}{\partial \vec{r}^{\,2}} \longrightarrow \frac{\partial^2}{\partial \vec{r}^{\,\prime\,2}} .$$

It means that the variable r also acquires a "generalization" when we consider any relativistic interaction!

We note also that the relation between the velocities allows us to take into account the magnetic part of the interaction symmetrically.

We have also to discuss relativistic properties of our equation. It is seen that we cannot make a relativistic transformation in it (we cannot choose another reference frame). It is because of the equal-time formalism, that we choose, and of the problem of square roots of operators. But we can say that the equation is the relativistic one in a "weak sense". This means that we can make any relativistic transformation in the solution of the equation! These transformations would not change the form of the solution. That change of reference frames has been made above in the limit case of one particle being at rest.

We could return to the first steps of the derivation of the equation and avoid the equal-time formalism. We can take two time variables:

$$\left(\frac{1}{i}\frac{\partial}{\partial t_1} + \frac{1}{i}\frac{\partial}{\partial t_2}\right)\Psi(t_1, t_2, \vec{x}_1, \vec{x}_2) = \hat{H}\Psi(t_1, t_2, \vec{x}_1, \vec{x}_2) . \qquad (41)$$

Then we would repeat all the steps of the derivation. There would appear the problem of the double counting for the Coulomb potential. It would be solved in just the same way as for the magnetic one. So, if we still worked in our frame of reference, we would obtain just the same results.

So, our equation has really relativistic properties: It keeps exactly the relativistic kinematics and the relativistic properties of the electromagnetic interaction. The reference frame, that we have chosen, appears to be more suitable for relativistic interactions of two particles with different masses than the centre-of-mass system.

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