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HOW TO BUILD ANALOGUES OF THE BESSEL-CLIFFORD EXPANSIONS FOR THE SUM OF THE REPULSIVE COULOMB POTENTIAL AND CENTRAL POTENTIAL DECREASING MORE RAPIDLY THAN THE CENTRIFUGAL ONE?

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1 Introduction

In nonrelativistic quantum mechanics [1] the Coulomb functions [2] $F_l(\rho,\eta)$ and $G_l(\rho,\eta)$ with $\eta > 0$ and $l = 0,1,\ldots$ are well-known as exact regular and irregular solutions of the Schrödinger scattering problem with the repulsive Coulomb potential $V_c = 2\eta/\rho$, the angular momentum l and the linear momentum $k = \rho/r = 1/2\eta R$, where r is the distance and R is the Bohr radius. The structure of the Coulomb functions is sufficiently complex and is described by different representations [2,3] in the dependence on the relations between the argument ρ, the parameter η and the index l. One of these representation is the Bessel--Clifford expansion.

Introducing the dimensionless argument $x \equiv r/R$ and the dimensionless parameter $q \equiv kR$ this expansion for F_l and G_l can be written as

$$\begin{Bmatrix} F_l(\rho,\eta) \\ G_l(\rho,\eta) \end{Bmatrix} = \begin{Bmatrix} qC_l(q) \\ C_l^{-1}(q) \end{Bmatrix} \sum_{n=0}^{\infty} q^{2n} \begin{Bmatrix} f_{ln}(x) \\ g_{ln}(x) \end{Bmatrix} . \qquad (1)$$

Here C_l is the known Coulomb factor,

$$C_l(q) = (2q)^l \, |\Gamma(l + 1 + i/2q)| \, exp\,(-\pi/4q) \quad , \qquad (2a)$$

where Γ is the Gamma-function [2], and all the functions f_{ln} and g_{ln} are the finite sums

$$\begin{Bmatrix} f_{ln}(x) \\ g_{ln}(x) \end{Bmatrix} = 2^{-2n} \sum_{m=2n}^{3n} a_{nm} \, z^{m+1} \begin{Bmatrix} 2^{-1} \; I_{2l+m+1}(z) \\ (-1)^{-m} \; K_{2l+m+1}(z) \end{Bmatrix} \qquad (2b)$$

containing the modified Bessel functions [2] $I_n(z)$ and $K_n(z)$ of the variable $z = 2\,x^{1/2}$ and the coefficients a_{nm} obeying the recursive chains ($m = 2n,\ldots,3n$ for each $n = 1,2,\ldots$) of equations

$$2m\, a_{nm} + 2(\, 2l + m \,)\, a_{n-1,m-2} + a_{n-1,m-3} = 0 \qquad (2c)$$

in which, by definition, $a_{00} = 1$ and $a_{nm} = 0$, if $n > 0$ and $m < 2n$ or $m > 3n$.

According to (2), both the expansions (1) contain only the known special functions and the numerical coefficients which can easily be calculated. Therefore, the construction of the finite ($M < \infty$) sums $S^{(M)} = F_l^{(M)}$, $G_l^{(M)}$ of the series (1) decomposed as

$$S = S^{(M)} + {}^{(M)}S \tag{3}$$

is a quite simple problem. On the other hand, it is known [2] that the estimates

$$\left\{ \begin{matrix} {}^{(M)}F_l(\rho,\eta) \\ {}^{(M)}G_l(\rho,\eta) \end{matrix} \right\} = O(q^{2M+2} \left\{ \begin{matrix} qC_l(q) \\ C_l^{-1}(q) \end{matrix} \right\}) \tag{4}$$

of the second terms, i.e. ${}^{(M)}S \equiv {}^{(M)}F_l$, ${}^{(M)}G_l$, of the decompositions (3) are uniform in ρ if

$$q \to 0 , \quad \rho \ll \rho_{cl} \equiv \eta (1 + (1 + l(l+1)/\eta)^{1/2}) . \tag{5}$$

In this case, by virtue of the above facts, the representation (1) and (3) are extremely appropriate for the approximation $S \cong S^{(M)}$ of the functions $S = F_l$, G_l with the absolute accuracy (4). This approximation is the basis, for example, for both the effective algorithms [3] for the calculation of the Coulomb functions in the case (5) and important analytical investigations of the low-energy ($q \to 0$) behaviour of different functions characterising elastic [4] and inelastic [5] collisions of two quantum mechanical objects charged similarly in sign. Generally, an effective interaction between such objects is not the pure Coulomb one. For this reason, the possible generalizations of the expansions (1) are certainly interesting from both the mathematical and physical points of view. Some examples illuminating this statement will be given in the conclusion of the present work.

The main aim of this work is a short description of the semi analytical method applying which one can very easily construct the analogues of the expansions (1) for the regular u_l^+ and irregular u_l^- solutions of the Schrödinger scattering problem with the potential $V_c + V$, where V is a central potential obeying the sufficiently general condition

$$x V(x) \in \mathbb{C}^0_{(0,\infty)} \cap \mathbb{L}^1_{[0,\infty]} \tag{6}$$

In the appropriate system of units ($\hbar = 2\mu = 1$) this problem reads as the equations

$$(\partial_x^2 - l(l+1)x^{-2} - V_c(x) - V(x) + q^2) u_l^{\pm}(x,q) = 0 , \tag{7a}$$

given on $\mathbb{R}^+ \equiv \{ x: 0 \le x \le \infty \}$ and added by the following asymptotical boundary conditions:

$$u_l^{\pm}(x,q) = O(x^{\pm(l+1/2)+1/2}) , \quad x \to 0 , \tag{7b}$$

and

$$u_l^{\pm}(x,q) \to \sin(\theta_{cl}(\rho,\eta) + (1 \mp 1)\pi/2 + \delta_l(q)), \ x \to \infty , \tag{7c}$$

where, by definition,

$$\theta_{cl}(\rho,\eta) \equiv \rho - \eta \ ln2\rho + \delta_{cl}(q) - \pi l/2 , \quad \rho \equiv xq , \quad \eta \equiv 1/2q$$

and δ_l is the phase-shift produced by the potential V in addition to the Coulomb phase-shift δ_{cl} .

In the following, the index l is omitted whenever possible and if ρ, η, x, q, b, n and M are not defined, then it is implied that $\rho = xq$, $\eta = 1/2q$, $x \in \mathbb{R}^+$, $0 < q < \infty$, b is an arbitrary but fixed positive number, n= 0,1,..., and, finally, M is an arbitrary but fixed and finite natural number.

2 The Method

There are four ideas forming the basis of the suggested method. The first one is to use F and G as known and, in defining meaning, basis functions. This seems to be natural and comfortable, because F and G coincide with the solutions u_l^+ and u_l^- of the problem (7) in the trivial case, when $V \equiv 0$, and because all the properties of F and G are well-known. The formulae (1-5) and the Wronskian identity [2],

$$G(\rho,\eta) \ \partial_x F(\rho,\eta) + F(\rho,\eta) \ \partial_x G(x,q) \equiv q \ . \tag{8}$$

are the key ones for the described method.

The second idea is to apply the variable phase approach [6,7] in its linear version (which is in essence equivalent to the variable constants method [8,9]) to reformulate the problem (7) in the form more adopted for all further analysis.

To this end, the phase functions $c_l^{\pm}$ and $s_l^{\pm}$ are introduced. In essence, they are the variable constants satisfying, by definition, the Lagrange identities

$$F(\rho,\eta)\ \partial_x c^{\pm}(x,q) + G(\rho,\eta)\ \partial_x s^{\pm}(x,q) \equiv 0 \quad . \tag{9}$$

Next, u^+ and afterwards u^- are looked for

$$u^{\pm}(x,q) = N^{\pm}(q)\ U^{\pm}(x,q) + \left\{ \begin{matrix} 0 \\ \alpha(q)\ u^+(x,q) \end{matrix} \right\} \quad , \tag{10a}$$

where

$$U^{\pm}(x,q) \equiv c^{\pm}(x,q)\ F(\rho,\eta) + s^{\pm}(x,q)\ G(\rho,\eta) \quad , \tag{10b}$$

and $N^{\pm}$ and α are the factors providing the asymptotics (7c). By a usual trick [6-9] based on the substitution of $u^{\pm}$in the form (10) into (7a) and application of (8) and (9) one gets two systems (the first one for c^+ , s^+ and the second one for c^-, s^-) of ordinary first order differential equations:

$$\partial_x \left\{ \begin{matrix} c^{\pm}(x,q) \\ s^{\pm}(x,q) \end{matrix} \right\} = q^{-1}\ V(x)\ U^{\pm}(x,q) \left\{ \begin{matrix} G(\rho,\eta) \\ -\ F(\rho,\eta) \end{matrix} \right\} . \tag{11a}$$

In the vicinity of zero ($x \to 0$) the appropriate solutions of eqs.(11a) must have the following asymptotics:

$$\left\{ \begin{matrix} c^+(x,q) \\ s^+(x,q) \end{matrix} \right\} \to \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} + q^{-1} \int_0^x V(t)\ F(\rho,\eta) \left\{ \begin{matrix} G(\rho,\eta) \\ -\ F(\rho,\eta) \end{matrix} \right\} dt \tag{11b}$$

and

$$c^-(x,q) \to c^-(x_0,q) + q^{-1} \int_{x_0}^x V(t)\ G^2(\rho,\eta)\ dt \quad ,$$

$$s^-(x,q) \to 1 - q^{-1} \int_0^x V(t)\ F(\rho,\eta)\ G(\rho,\eta)\ dt \quad , \tag{11c}$$

where $\rho = tq$, if $V(x)\ G^2(\rho,\eta) \in \mathbb{L}^1_{[0,b]}$, then $x_0 = 0$ and $c^-(0,q) = 0$, in the opposite case x_0 is an arbitrary but fixed parameter obeying the inequalities $x_0 > x$ and $x_0 q \ll 1$. The asymptotics (11b) and (11c) are chosen as the boundary conditions for eqs.(11a), because just they ensure both the required asymptotics (7b) of $u^{\pm}$ and physically natural limit relations: $u^+ \to F$ and $u^- \to G$ when $V(x) \to 0$ for all x . These facts may readily be verified by using (6), (10) and (11).

Applying traditional methods [8,9] it is easy to convince oneself that the problems (11) with the conditions (6) have unique solutions satisfying the following relations:

$$|c^{\pm}(x,q)| + |s^{\pm}(x,q)| \neq 0 \quad , \tag{12a}$$

$$c^+,\ s^{\pm} \in C^0_{[0,\infty]} \quad , \qquad c^- \in C^0_{(0,\infty)} \quad , \tag{12b}$$

$$w(x,q) \equiv c^+(x,q)\ s^-(x,q) - c^-(x,q)\ s^+(x,q) \equiv 1 \ . \tag{12c}$$

For example, the identity (12c) holds because, due to (11b) and (11c), $w(0,q) = 1$, and by using (11a) one has $\delta_x w(x,q) = 0$ for all $x > 0$.

There is an essential point that should be explained in more detail. The point is that under assumption (6) the function c^- (in contrast with c^+ and $s^{\pm}$) may have irregular asymptotics (see (11c)),

$$c^-_l(x,q) \to (\ q^{2l+1}\ C^2_l(q)\ )^{-1} \int_{x_0}^x t^{-2l}\ V(t)\ dt \quad ,$$

in the origin. In this case, $c^-$and $s^-$are uniquely and correctly defined by the known recipe [9,10] based on the shift of the boundary conditions from zero to the point x_0 close to zero in the above-described meaning. This recipe is applied as follows.

The values of c^+, s^+ and s^- at $x = x_0$ are defined, respectively, by solving the problem (11a), (11b) for c^+, s^+ and by asymptotics (11c) for s^-. These values are substituted into the identity (12c) written at $x = x_0$. The obtained equation is resolved with respect to $c^-(x_0,q)$. Now, when $c^-(x_0,q)$ and $s^-(x_0,q)$ are known, the searched functions c^- and s^- are uniquely represented by explicit formulae (11c) when $x \le x_0$ and for $x \ge x_0$ they are defined as the solution of eqs. (11a) with the boundary conditions at the point $x_0 \neq 0$. This problem has a unique solution in the $C^0_{[x_0,\infty]}$- class of functions, because [8,9] in the region $x \ge x_0 > 0$ the functions F , G and V containing in (11a) are continuous.

By virtue of (11b), (11c), (12) and the above-described construction of c^- and s^- in the case $V(x)\ G^2(\rho,\eta) \notin \mathbb{L}^1_{[0,b]}$, the functions

$$\delta(x,q) \equiv \arctan\ (\ s^+(x,q)\ /\ c^+(x,q)\ ) \quad , \tag{13a}$$

$$N^{\pm}(x,q) \equiv (\ (\ c^+(x,q)\ )^2 + (\ s^+(x,q)\ )^2\ )^{\mp 1/2} \ , \tag{13b}$$

and

$$\alpha(x,q) \equiv - c^{+}(x,q)\, c^{-}(x,q) - s^{+}(x,q)\, s^{-}(x,q) \; , \qquad (13c)$$

are single-valued and finite for all x . In accordance with the spirit of the variable phase approach, this allows one to correctly define δ from (7c) and $N^{\pm}$, α from (10a) as the limits

$$A(q) \equiv \lim_{x \to \infty} A(x,q) \qquad (14)$$

of the corresponding functions $A(x,q)$ introduced by (13). Under this definition, $u^{\pm}$ defined by (10a) satisfies (7c). To see this, one has to replace F and G in (10) by their known asymptotics as $x \to \infty$ (they are given by (7c) with $\delta_{cl} \equiv 0$) and then take into account (12), (13) and (14).

It is worth stressing that each function $A = c^{\pm}$, $s^{\pm}$ and $A = \delta$, $N^{\pm}$, α defined by (11) and (13) has a pleasant property,

$$A(x,q) \equiv A(b,q) \; , \; x \geq b \; , \text{ if } V(x) \equiv 0 \text{ for all } x \geq b, \qquad (15)$$

and an apparent physical meaning: $c^{\pm}(x,q)$ and $s^{\pm}(x,q)$ taken at some point $x = b$ are obviously the amplitudes with which F and G are present at $x = b$ in the wave-functions (10b) unnormalized to per unit density at infinity while, due to (13-15), $\delta(x,q)$ and $N^{\pm}(x,q)$, $\alpha(x,q)$ taken at $x = b$ are the phase shift δ and the factors $N^{\pm}$, α for the potential V cut-off at the point $x = b$.

So, the second idea is accomplished: the solving of the original problem (7) for u^{+} and u^{-} is reduced to the solving of the problem (11) for c^{+} , s^{+} and afterwards for c^{-} , s^{-} , subsequent calculation of the limits (14) of the functions (13), and finally, the construction of the relevant functions $u^{\pm}$ via the formulae (10). As a result of a reformulation like that, the new functions $c^{\pm}$and $s^{\pm}$are introduced instead of $u^{\pm}$. These functions have an apparent physical meaning and obey the simplest equations which may be analyzed in an elementary way by the known asymptotical methods [11,12]. One of them is the method of construction of the solution to the linear differential problem with a parameter as a series in which the argument and parameter are separated from each other.

The third idea consists in application of this method to the problems (II) and is accomplished as follows.

By substituting F, G as the known series (I) and $c^{\pm}$, $s^{\pm}$ as the searched series

$$\left\{ \begin{matrix} c^{\pm}(x,q) \\ s^{\pm}(x,q) \end{matrix} \right\} = \left\{ \begin{matrix} (\, qC^2(q)\,)^{(-1\pm1)/2} \\ (\, qC^2(q)\,)^{(\;1\pm1)/2} \end{matrix} \right\} \sum_{n=0}^{\infty} q^{2n} \left\{ \begin{matrix} c_n^{\pm}(x) \\ s_n^{\pm}(x) \end{matrix} \right\} \qquad (16)$$

into (10b), (11a) and (11b), (11c), one gets the representation

$$U^{\pm}(x,q) = q^{(1\pm1)/2}\, C^{\pm1}(q) \sum_{n=0}^{\infty} q^{2n}\, U_n^{\pm}(x) \quad , \qquad (17a)$$

$$U_n^{\pm}(x) \equiv \sum_{m'+m=n} (\, c_{m'}^{\pm}(x)\, f_m(x) + s_{m'}^{\pm}(x)\, g_m(x)\,) \quad , \qquad (17b)$$

for the functions $U^{\pm}$, the infinite set of equations

$$\partial_x \left\{ \begin{matrix} c_n^{\pm}(x) \\ s_n^{\pm}(x) \end{matrix} \right\} = V(x) \sum_{m'+m=n} U_{m'}^{\pm}(x) \left\{ \begin{matrix} g_m(x) \\ - f_m(x) \end{matrix} \right\} \qquad (18a)$$

for new unknown functions c_n^{+} , s_n^{+} and c_n^{-} , s_n^{-} , $n = 0,1,\ldots,$ and the explicit formulae

$$\left\{ \begin{matrix} c_n^{+}(x) \\ s_n^{+}(x) \end{matrix} \right\} \to \left\{ \begin{matrix} \delta_{n0} \\ 0 \end{matrix} \right\} + \sum_{m'+m=n} \int_0^x V(t)\, f_{m'}(t) \left\{ \begin{matrix} g_m(t) \\ - f_m(t) \end{matrix} \right\} dt \quad (18b)$$

and

$$c_n^{-}(x) \to c^{-}(x_0) + \sum_{m'+m=n} \int_{x_0}^x V(t)\, g_{m'}(t)\, g_m(t)\, dt \; ,$$

$$(18c)$$

$$s_n^{-}(x) \to \delta_{n0} - \sum_{m'+m=n} \int_0^x V(t)\, f_{m'}(t)\, g_m(t)\, dt$$

describing the asymptotics of these functions as $x \to 0$. In (18c) δ_{nm} is the Kronecker symbol [2], $x_0 = 0$ and $c_n^{-}(0) = 0$ if $x^{-2l}V(x) \in L^1_{[0,b]}$ and in the opposite case x_0 is an arbitrary but fixed parameter obeying the inequalities $x < x_0 \ll 1$.

Applying known theorems [8-10] one can show that the problem (18) for c^+_n , s^+_n (or for c^-_n , s^-_n) in the case $c^-_n(0) = 0$ has a unique solution belonging to the $C^0_{[0,\infty)}$-class of functions. In the case $|c^-_n(0)| = \infty$ the appropriate and unique set of the functions c^-_n and s^-_n , $n = 0,1,\ldots,$ is built by using the special recipe. In principle, this recipe is analogous to the above-mentioned one for construction of c^- and s^- and is used as follows.

The values c^+_n , s^+_n and s^-_n , $n = 0,1,\ldots,$ at $x = x_0$ are defined, respectively, by solving the problem (18a), (18b) for c^+_n , s^+_n and by the asymptotics (18c) for s^-_n . These values are substituted into the relations

$$\sum_{m'+m=n} (c^+_{m'}(x)\, s^-_m(x) + c^-_{m'}(x)\, s^+_m(x)) \equiv \delta_{n0} , \quad n = 0,1,\ldots,$$

written at $x = x_0$ and derived by inserting the series (16) into the identity (12c) and subsequent separating of q. As a result of this substitution, one gets the equations for c^-_n, $n = 0,1,\ldots$. Now, when $c^-_n(x_0)$ and $s^-_n(x_0)$ are found, the functions c^-_n and s^-_n are represented by the integrals (18c) when $x \le x_0$ and for $x \ge x_0$ they are defined as the solution of eqs.(18a) with the boundary conditions at $x = x_0 > 0$.

It is useful to discuss the structure of the total problem (18) and then recall the consequences caused by it. As one sees, each ($n = 0,1,\ldots$) problem (18) for c^+_n , s^+_n or for c^-_n , s^-_n contains neither q nor $c^\pm_m$, $s^\pm_m$ with $m > n$, moreover, eqs.(18a) for $c^\pm_n$, $s^\pm_n$ and $c^\pm_m$, $s^\pm_m$, with $n \ne m$ differ from each other only by inhomogeneous terms and the corresponding boundary conditions (18b) or (18c) are simple enough. Due to this structure of the total problem (18), some analytical properties of $c^\pm_n$ and $s^\pm_n$, $n = 0,1,\ldots,$ (for example, their asymptotics as $x \to \infty$) may be easily established by induction while calculation of $c^\pm_n$ and $s^\pm_n$ with $n = 0,1,\ldots,M < \infty$, is reduced to the subsequent ($n = 0,1,\ldots M$) integration of the simplest differential problems, namely, coupled pairs of eqs. (18a) for c^+_n and s^+_n , or for c^-_n and s^-_n written in the increasing order of the index n and added by the corresponding boundary conditions (18b) or

(18c). Hence, both the analytical and numerical investigations of the finite sums of the expansions (16) are simple problems.

Now, let all the functions $c^\pm_n$ and $s^\pm_n$ be found by solving the total problem (18) and the factors $N^\pm$ and α from (10a) be calculated as the limits (14) of the corresponding functions (13) expressed in terms of the found solutions $c^\pm$ and $s^\pm$ of the problems (11). Then, in order to achieve the main aim of the present work, it is sufficient to substitute $U^\pm$ in the form (17a) into (10) and thus to get the desirable expansions:

$$u^\pm_l(x,q) = q^{(1\pm1)/2} (C_l(q)\, N^+_l(q))^{\pm1} \times \sum_{n=0}^{\infty} q^{2n} \left(U^\pm_{ln}(x) + \left\{ \begin{matrix} 0 \\ \beta_l(q)\, U^+_{ln}(x) \end{matrix} \right\} \right) \qquad (19)$$

in which x and q are separated from each other and, by definition,

$$\beta_l(q) \equiv q\, \alpha_l(q) (C_l(q)\, N^+_l(q))^2 . \qquad (20)$$

It is interesting to find the limits of all the functions containing in (19) as $V \to 0$ and to compare the structure of the expansions (1) and (19).

By consequent using of (11), (18), (13), (14), (17), (19) and (20), one can prove two statements. The first one means that for any possible x, q, l and n one has

$$c^+_l, s^-_l \to 1;\ s^+_l,\ s^+_{ln} \to 0;\ c^+_{ln},\ s^-_{ln} \to \delta_{n0};\ N^\pm_l \to 1;\ U^+_l \to f_{ln};\ u^+_l \to F_l$$

and the series (19) for u^+_l transforms into the series (1) for F_l when V obeys the condition (6) and vanishes for all x in an arbitrary way. The second statement means that again for any possible x, q, l and n one has

$$c^-_l,\ c^-_{ln} \to 0 ;\quad \alpha_l ,\ \beta_l \to 0 ;\quad U^-_l \to g_{ln} ;\quad u^-_l \to G_l$$

and the series (19) for u^-_l transforms into the series (1) for G_l when V obeys the condition (6) and vanishes for all x in such a way that $x^{-2l}V(x)$ is the integrable function in the vicinity of zero.

The representations (1) and (19) for F_l and u_l^+ are completely identical in structure: both of them are the products of the factors $qC_l(q)$ and $qC_l(q)\ N_l^+(q)$ (depending only on q) and the infinite series in even powers of q and the functions $f_{ln}(x)$ and $U_{ln}^+\ (x)$ (depending only on x). Therefore, $qC_l(q)\ N_l^+(q)$ and U_{ln}^+ are the analogues of $qC_l(q)$ and f_{ln} .

The representation (19) for u_l^- has a more complex structure than the representation (1) for G_l . Indeed, although x and q are separated from each other in both these expansions, all the terms $q^{2n}\ (\ U_{ln}^- + \beta_l\ U_{ln}^+\ )$ of the expansion (19) for u_l^- , in contrast with all the terms $q^{2n} g_{ln}$ of the expansion (1) for G_l , contain an additional q-dependence which is described by the factor β_l defined by (20). In general, $\beta_l(q) \neq 0$, therefore, $(\ C_l\ N_l^+\ )^{-1}$, β_l and $U_{ln}^{\pm}$ from the expansion (19) for u_l^- have no analogues in the expansion (1) for G_l.

Clearly, for the sake of completeness, it is worth to display the conditions which are sufficient for the approximation $S \cong S^{(M)}$ of the series $S = u^{\pm}$, i.e. the series (19), by their finite sums $S^{(M)} = u^{(M)}$ and also to estimate the residual terms ${}^{(M)}S \equiv S - S^{(M)} \equiv {}^{(M)}u^{\pm}$ of such an approximation. One can do this by realizing the fourth idea of the suggested method as follows.

In (10) and (11) each function $S = F, G, c^{\pm}, s^{\pm}$ is replaced by its decomposition (3) in which $S^{(M)}$ is defined as a finite sum of the corresponding series (1) or (16). After simple transformations ${}^{(M)}u^{\pm}$ are represented as

$$ {}^{(M)}u^{\pm} = N^{\pm}\Big(\ {}^{(M)}F\ c^{\pm(M)} + {}^{(M)}G\ s^{\pm(M)} + {}^{(M)}c^{\pm}\ F + {}^{(M)}s^{\pm}\ G + q^{(1\pm1)/2}\ C^{\pm1} \sum_{n=M+1}^{2M+2} q^{2n}\ U_n^{\pm}\ \Big) + \begin{Bmatrix} 0 \\ \alpha\ {}^{(M)}u^{+} \end{Bmatrix} \qquad (21) $$

and the problem (11) for c^+, s^+ (or for c^-, s^-) is reduced to two mutually coupled problems: to the problem (18) for c_n^+, s_n^+ (or for c_n^- , s_n^-) with n = 0,1,...,M and to the problem for ${}^{(M)}c^+$, ${}^{(M)}s^+$ (or for ${}^{(M)}c^-$, ${}^{(M)}s^-$) . The latter problem,

being written in a more compact way, has a form of the system

$$ \partial_x \begin{Bmatrix} {}^{(M)}c^{\pm} \\ {}^{(M)}s^{\pm} \end{Bmatrix} = q^{-1}\ V\ U^{\pm} \begin{Bmatrix} G \\ -F \end{Bmatrix} - \partial_x \begin{Bmatrix} c^{\pm(M)} \\ s^{\pm(M)} \end{Bmatrix} \qquad (22a) $$

with the corresponding boundary conditions

$$ {}^{(M)}S = S - S^{(M)} \ ; \quad S \equiv c^{\pm}\ ,\ s^{\pm} \ ; \quad x \to 0 \ , \qquad (22b) $$

that may be disentangled with the help of (11b), (11c) and (18b), (18c).

Further, both the problems (22) are investigated under the conditions (5) by the known asymptotical method [11,12].

For this aim, in (22) F, G, and $c^{\pm}$, $s^{\pm}$ are again represented as the corresponding decomposition (3); $F^{(M)}$, $G^{(M)}$ and $c^{\pm(M)}$, $s^{\pm(M)}$ are replaced by the sums of the first (M+1) terms of the corresponding series (1) and (16); ${}^{(M)}F$ and ${}^{(M)}G$ are estimated with the help of (4); the inequalities $|c_n^{\pm}|$, $|s_n^{\pm}| < \infty$ valid on conditions (6), $0 < x < \infty$ and $n \le M < \infty$ are taken into account and, finally, the new unknown functions $y_i^{\pm}$ with i = 1,2, are introduced by the relations

$$ \begin{Bmatrix} {}^{(M)}c^{\pm}(x,q) \\ {}^{(M)}s^{\pm}(x,q) \end{Bmatrix} = q^{2M+2} \begin{Bmatrix} (qC^2(q))^{(-1\pm1)/2}\ y_1^{\pm}(x,q) \\ (qC^2(q))^{(\ 1\pm1)/2}\ y_2^{\pm}(x,q) \end{Bmatrix} . \qquad (23) $$

As a result, the problems (22) are reduced to the asymptotical ($q \to 0$, $xq \ll \rho_c$) equations

$$ \partial_x \begin{Bmatrix} y_1^{\pm}(x,q) \\ y_2^{\pm}(x,q) \end{Bmatrix} = V\ (\ f_0\ y_1^{\pm} + g_0\ y_2^{\pm}\ ) \begin{Bmatrix} g_0 \\ -f_0 \end{Bmatrix} + V \begin{Bmatrix} z_1^{\pm} \\ z_2^{\pm} \end{Bmatrix} + O(\ q^2(\ 1 + y_1^{\pm} + y_2^{\pm}\ )\ ) \qquad (24a) $$

with the functions

$$ \begin{Bmatrix} z_1^{\pm} \\ z_2^{\pm} \end{Bmatrix} \equiv (\ f_{M+1}\ c_0^{\pm} + g_{M+1}\ s_0^{\pm}\ ) \begin{Bmatrix} g_0 \\ -f_0 \end{Bmatrix} + \sum_{m'+m=M+1} U_{m'}^{\pm} \begin{Bmatrix} g_m \\ -f_m \end{Bmatrix} \qquad (24b) $$

and the boundary conditions

$$\left\{ \begin{matrix} y_1^+(x,q) \\ y_2^+(x,q) \end{matrix} \right\} \rightarrow \sum_{m'+m=M+1} \int_0^x V(t)\, f_{m'}(t) \left\{ \begin{matrix} g_m(t) \\ -f_m(t) \end{matrix} \right\} dt + O(q^2), \quad (24c)$$

and

$$y_1^-(x,q) \rightarrow y_1^-(x_0,q) + \sum_{m'+m=M+1} \int_{x_0}^x V(t)\, g_{m'}(t)\, g_m(t)\, dt + O(q^2), \quad (24d)$$

$$y_2^-(x,q) \rightarrow -\sum_{m'+m=M+1} \int_0^x V(t)\, f_{m'}(t)\, f_m(t)\, dt + O(q^2) .$$

In (24c) and (24d) $x \rightarrow 0$, $x_0 = 0$ and $y_1^-(0,q) = 0$ if $x^{-2l}V(x) \in L^1_{[0,b]}$ and in the other case x_0 is an arbitrary but fixed parameter obeying the inequalities $x < x_0 \ll 1$.

As a next step, it is proven that the solutions of the obtained problems (24) have the properties

$$| y_i^\pm(x,q) | < \infty \; , \; i = 1,2 \; , \quad 0 < xq \ll \rho_c \; , \quad (25)$$

and

$$\lim_{q \rightarrow 0} y_i^\pm(x,q) = y_i^\pm(x,0) \neq 0 \; , \; i = 1,2 \; , \quad 0 \leq x < \infty ,$$

because V obeys the condition (6) and all the functions f_n , g_n and $z_i^\pm$ defined by (2) and (24b) are finite when $0 < x < \infty$.

The final result,

$$\left\{ \begin{matrix} {}^{(M)}c^\pm(x,q) \\ {}^{(M)}s^\pm(x,q) \end{matrix} \right\} = O\left(q^{2M+2} \left\{ \begin{matrix} (qC^2(q))^{(-1\pm1)/2} \\ (qC^2(q))^{(1\pm1)/2} \end{matrix} \right\} \right) , \quad (26)$$

of the above, briefly described, investigation of the problems (22) is obtained with the help of (23) and (25) and is valid under the conditions (5).

Evidently, due to formulae (4), (19), (21), (26) and the identities $u^\pm \equiv u^{\pm(M)} + {}^{(M)}u^\pm$, the conditions (5) are sufficient for the approximation of $u^\pm$ by $u^{\pm(M)}$ with the absolute accuracy

$$| {}^{(M)}u^\pm(x,q) | = O\left(q^{2M+2+(1\pm1)/2}\, C^{\pm1}(q) \right) . \quad (27)$$

It should be emphasized that owing to (4) and (27), ${}^{(M)}F = O({}^{(M)}u^+)$ and ${}^{(M)}G = O({}^{(M)}u^-)$ if the conditions (5) are fulfilled. This observation reveals once more the analogy

between the Bessel-Clifford expansions (1) and more general expansions (19).

As is shown before, the construction of their finite sums $u^{\pm(M)}$ is actually reduced to the solution of the simple differential problems (11) and (18). This construction may be performed in the following order:

1) the calculation of C , f_n and g_n with $n = 0,1,\ldots,M$ via the formulae (2);

2) the solution of the recurrence chain of (M+1) problems (18) for c_n^+ , s_n^+ and afterwards for c_n^-, s_n^- with $n = 0,1,\ldots,M$;

3) the construction of the functions $U_n^\pm$ defined by (17b);

4) the solution of the problem (11) for c^+, s^+ and then for c^-, s^-;

5) the definition of $N^\pm$, α and β by using (13), (14) and (20);

6) the construction of $u^{\pm(M)}$ as sums of the first (M+1) terms of the expansions (19).

Evidently, the practical realization of this algorithm does not have any special difficulties.

3 Conclusion

In conclusion it is necessary to clarify (as it was promised) to what extent the obtained results are useful. The above algorithm is an effective way for practical calculations of the sums $u^{\pm(M)}$ approximating the wave-functions $u^\pm$ under conditions (5). The developed method seems to be perspective also for analytical analysis of the low--energy expansions of all the functions associated with the problem (7).

There are two examples apparently demonstrating the latter statement and, moreover, showing that under the well--defined conditions the phase-shift δ and the factors $N^\pm$, α and β may be found without the solution of the problems (11).

<u>The first example</u>. Let $0 < b < \infty$, $q \rightarrow 0$ and $V(x) \equiv 0$ at $x \geq b$. Then, $bq \ll \rho_c$ for sufficiently small q, the

conditions (5) are fulfilled and the estimates (26) are valid at $x \le b$. These estimates take place also at $x > b$, because the functions ${}^{(M)}c^{\pm}$ and ${}^{(M)}s^{\pm}$ have the property (15) caused by relations (23), equations (24) and the identity $V(x) \equiv 0$ at $x \ge b$. Thus, in the considered case the estimates (26) are uniform on $\mathbb{R}^{+}$. Using this important fact, the decompositions (3) for $c^{\pm}$ and $s^{\pm}$ and the definitions (13), (14) and (20) one can readily get the desirable low-energy expansions of the phase shift δ and the factors $N^{\pm}$, α and β considered as functions of q. For the functions

$$A(q) \equiv qC^2(q)\ cotan\ \delta(q)\ ,\ N^{+}(q),\quad qC^2(q)\ \alpha(q),\quad \beta(q) \qquad (28)$$

these expansions can be written by means of the same formula

$$A(q) = \sum_{n=0}^{M} A_n q^{2n} + O(q^{2M+2}) \qquad (29)$$

in which all the coefficients A_n, $n = 0,1,\ldots,M$, are simple algebraic combinations of the functions $c_m^{\pm}$ and s_m^{+}, $m = 0,1,\ldots,M$, taken at $x = b$.

For instance:

for $A(q) = qC^2(q)\ cotan\ \delta(q)$

$$A_0 = c_0^{+}/s_0^{+}\ ,\ A_1 = (c_1^{+} - A_0 s_1^{+})/s_0^{+}\ ,\ A_2 = (c_2^{+} - A_0 s_2^{+} - A_1^{+} s_1^{+})/s_0^{+}\ ;(30a)$$

for $A(q) = N^{+}(q)$, $A_n = N_n^{+}$ and

$$N_0^{+} = 1/c_0^{+}\ ,\quad N_1^{+} = -c_1^{+}(N_0^{+})^2\ ,\ N_2^{+} = -(c_1^{+} N_1^{+} + c_2^{+} N_0^{+})\ N_0^{+}\ ;(30b)$$

for $A(q) = qC^2(q)\ \alpha(q)$, $A_n = \alpha_n$ and

$$\alpha_n = \sum_{m'+m=n} c_{m'}^{+} c_m^{-}\ ,\qquad n = 0,1,\ldots,M\ ; \qquad (30c)$$

and, finally, for $A(q) = \beta(q)$

$$A_n = \sum_{m''+m'+m=n} \alpha_{m''} N_{m'}^{+} N_m^{+}\ ,\quad n = 0,1,\ldots,M\ , \ (30d)$$

where N_n^{+} and α_n are given by (30b) and (30c).

The second example. Let $q \to 0$, $V(x) \not\equiv 0$ for any x but

$$x^{3M+4} \exp(4x^{1/2}) V(x) \in L^1_{[b,\infty]}\ .$$

Then, the low-energy asymptotics of the functions (28) have also the forms (29) with the first coefficients A_n given by the corresponding formulae (30) in which all the functions $c_n^{\pm}$ and s_n^{+} are now taken at $x \pm \infty$.

The proof of this statement requires a more detailed analysis of the problems (11) and (18): therefore, it is postponed to the future work based on the present one.

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