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ON SOLITON TYPE SOLUTIONS OF ONE LINEAR TIME-DEPENDENT EQUATION AND THE ISHIMORI-II MODEL WITH TIME DEPENDENT BOUNDARY CONDITIONS

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1. In work [1] a method to construct a class of soliton type potentials and their eigenfunctions (E.F.) was developed for the linear time-dependent Schrödinger equation

$$(i\partial_t + \partial_x^2 + U(x,t))\psi(x,t,k) = 0\,. \tag{1}$$

In what follows we extend this method to include the equation

$$(-i\partial_t + \partial_x^2 + iU(x,t)\partial_x)\psi(x,t,k) \equiv L\psi(x,t,k) = 0 \tag{2}$$

and obtain some soliton-like potentials $U(x,t)$ and E.F. $\psi(x,t,k)$.

Consider for that the function

$$\psi(x,t,k) = P_N \exp ik(x+kt) \tag{3}$$

with

$$P_N = a_N k^N + ... + a_1 k + 1, \tag{4}$$

the set of N complex numbers κ_i (poles) and complex $N\times N$ constant matrix b_{ij}. The following theorem takes place: if the function $\psi(x,t,k)$ defined by [3] and [4] is subject to the conditions

$$\psi(\bar{\kappa}_i) = -\sum_{j=1}^{N} b_{ij}\psi(\kappa_j), \quad i,j = 1,\cdots,N \tag{5}$$

and

$$U(x,t) = 2i\partial_x ln a_N \tag{6}$$

then $\psi(x,t,k)$ is a solution of eq.(2). The proof nearly does not differ from that for eq.(1). Indeed, eqs. (5) can be rewritten in the form

$$\sum_{j=1}^{N} A_{ij}a_j = B_i \tag{7}$$

with

$$
\begin{aligned}
A_{ij} &= \bar{\kappa}_i^j + \sum_{l=1}^{N} b_{ij}\kappa_l^j \exp i(\theta_l - \bar{\theta}_i), \\
B_i &= -\left(1 + \sum_{l=1}^{N} b_{ij} \exp i(\theta_l - \bar{\theta}_i)\right), \qquad i,j = 1,\cdots,N, \\
\theta_i &= \kappa_i(x + \kappa_i t).
\end{aligned} \tag{8}
$$

So we have a system of N linear equations for the N unknown functions $a_i(x,t)$. Suppose $det A \neq 0$, then (7) define unequely these functions:

$$a_i = a_i^o . \tag{9}$$

We show the function (3) under (9) to be the E.F. of (2) with the potential defined by (6). By acting with the operator L on eq.(5) we have

$$L\psi(\bar{\kappa}_i) + \sum_{j=1}^{N} b_{ij} L\psi(\kappa_j) = 0$$

in which

$$L\psi(\bar{\kappa}_i) = e^{i\bar{\theta}_i}(\bar{\kappa}_i^{N+1}\tilde{a}_{N+1} + \bar{\kappa}_i^N \tilde{a}_N + \cdots + \bar{\kappa}_i \tilde{a}_1)$$

and due to (3) the coefficient $\tilde{a}_{N+1}$ vanishes, therefore

$$\sum_{l=1}^{N} A_{il}\tilde{a}_l \equiv \sum_{j,l=1}^{N} (\bar{\kappa}_j^l e^{i\bar{\theta}_j} + b_{ij}\kappa_j^l e^{i\theta_j})\tilde{a}_l = 0, \qquad i = \overline{1,N} .$$

We come to the homogeneous linear system with the same matrix A_{ij}, hence all the $\tilde{a}_i = 0$ and

$$L\psi = \tilde{P}_{N+1} e^{ik(x+kt)} = (\tilde{a}_{N+1}k^{N+1} + \cdots + \tilde{a}_1 k) e^{ik(x+kt)} = 0$$

Q.E.D.

The representation (3) of the E.F. is called polinomial. We introduce now a new one

$$\hat{\Psi} = \frac{P_N e^{ik(x+kt)}}{\prod_{i=1}^{N}(k - \kappa_i)} \equiv a_n \left\{1 + \sum_{j=1}^{N} \frac{r_j(x,t)}{k - \kappa_j}\right\} e^{ik(x+kt)} \tag{10}$$

which can be called a pole-type representation (normalization). Each of the representations will be of use below. It is easy to see that for the function $\hat{\Psi}(x,t,k)$ instead of (5) one can get

$$\hat{\Psi}(x,t,\bar{\kappa}_i) = -\sum_{j=1}^{N} c_{ij} res_{k=\kappa_j} \hat{\Psi}(x,t,k) \equiv -\sum_{j=1}^{N} c_{ij}\hat{\Psi}_j(x,t) \tag{11}$$

where

$$res_{k=\kappa_j}\hat{\Psi}(x,t,k) = a_N(x,t) r_j(x,t) e^{i\kappa_j(x+\kappa_j t)} . \tag{12}$$

Via dividing eq.(11) by $a_N(x,t)$ we come to the equation

$$\Psi(x,t,\bar{\kappa}_i) = -\sum_{j=1}^{N} c_{ij}\Psi_j(x,t) \tag{13}$$

with

$$
\begin{aligned}
\Psi(x,t,k) &= \frac{\hat{\Psi}(x,t,k)}{a_N(x,t)} = \left\{1 + \sum_{j=1}^{N} \frac{r_j(x,t)}{k - \kappa_j}\right\} e^{ik(x+kt)} \\
c_{ij} &= \frac{\prod_{l=1}^{N}{}'(\kappa_j - \kappa_l)}{\prod_{l=1}^{N}(\bar{\kappa}_i - \kappa_l)} b_{ij} .
\end{aligned} \tag{14}
$$

Bearing in mind that $P_N(k \to 0) = 1$ one has

$$a_N = \frac{(-1)^N}{\prod_{j=1}^{N}\kappa_j} \frac{1}{1 - \sum_{j=1}^{N}\frac{\Gamma_j}{\kappa_j}} \tag{15}$$

or by denoting $\Psi_0 = \Psi(x,t,k=0) = 1 - \sum \frac{r_j}{\kappa_j}$

$$a_N = \prod_{j=1}^{N}\left(\frac{-1}{\kappa_j}\right)\frac{1}{\Psi_0} = \frac{det A_1}{det A} \tag{16}$$

and

$$U = -2_i\partial_x \ln \Psi_0 \tag{17}$$

$$\hat{\Psi}(x,t,k) = \prod_{j=1}^{N}\left(\frac{-1}{\kappa_j}\right)\frac{1}{\Psi_0}\Psi(x,t,k) . \tag{18}$$

From (17) it follows that $U = \bar{U}$ when

$$|1 - \sum \frac{\Gamma_j}{\kappa_j}| = const . \tag{19}$$

We would stress that eq.(13) has the same form as in paper [1] dealing with eq.(1).

2. To demostrate how the above technique works we use the example of one pole case: $N = 1,\ \kappa = \alpha + i\beta, \quad b = b_1 + ib_2$. Here we have one equation

$$a = (\frac{-1}{\bar{\kappa}})\ \frac{1 + be^{i(\theta - \bar{\theta})}}{1 + b\frac{\kappa}{\bar{\kappa}}e^{i(\theta - \bar{\theta})}} \ . \tag{20}$$

In order to the potential $U(x,t)$ be real it is necessary $|a|$=const or

$$b\kappa = \overline{b\kappa} \tag{21}$$

or

$$b_1\beta + b_2\alpha = 0 \ . \tag{22}$$

By differentiating $\ln a$ (20) we arrive at

$$V = \frac{8|\alpha|\beta^2 sgn\alpha}{4\alpha^2\cosh^2\eta + \beta^2 e^{-2\eta}}, \qquad b_1 = e^{-2\beta x_0} > 0 \tag{23}$$

$$U = -\frac{8|\alpha|\beta^2 sgn\beta}{4\alpha^2\sinh^2\eta + \beta^2 e^{-2\eta}}, \qquad b_1 = -e^{-2\beta x_0} < 0 \tag{24}$$

$$\eta = \beta(x + 2\alpha t + x_0) \tag{25}$$

also we have

$$\Psi = \left(1 - \frac{k}{\bar{\kappa}}\ \frac{1 + be^{-2(\eta - \beta x_0)}}{1 + \bar{b}e^{-2(\eta + \beta x_0)}}\right) e^{ik(x + kt)} \ . \tag{26}$$

If one sets $k = \bar{\kappa}$ then

$$\Psi = 2i\frac{\sqrt{b_1}\beta e^{i\theta}}{2\alpha\cosh\eta + i\beta e^{-\eta}}, \qquad \theta = i\alpha x + i(\alpha^2 - \beta^2)t \ . \tag{27}$$

Fixing $k = \kappa$ we come to the same expression up to the constant factor.

It should be pointed out that a general form for the E.F. of time dependent eq.(2) is that of (26) with arbitrary complex number k, so the solution is a five real parameter $(\alpha, \beta . b_1, k_1, k_2)$ function.

3. We now utilize the solutions found in order to obtain such for the Ishimori-II model

$$\vec{S}_t(x,y,t) + \vec{S}\wedge(\vec{S}_{xx} + \vec{S}_{yy}) + \varphi_x\vec{S}_y + \varphi_y\vec{S}_x = 0 \tag{28}$$

$$\varphi_{xx} - \varphi_{yy} + 2\vec{S}(\vec{S}_x \wedge \vec{S}_y) = 0 \tag{29}$$

where $\vec{S} = (S_x, S_y, S_z), \quad \vec{S}^2 = 1 \quad and \quad \varphi(x,t)$ is a real function.

To treat eqs.(28),(29) we shall use the results of work [2] where solutions of the problem was reduced to solutions of two linear equations of type (2), namely

$$iX_t(\xi,t) + \frac{1}{2}X_{\xi\xi} + iU_2(\xi,t)X_\xi = 0 \tag{30}$$

$$iY_t(\eta,t) + \frac{1}{2}Y_{\eta\eta} - iU_1(\eta,t)Y_\eta = 0 \tag{31}$$

with $\xi = \frac{1}{2}(x+y), \quad \eta = \frac{1}{2}(y-x)$ and real potentials: $U_i = \bar{U}_i$.

The special class of solutions of (26) related to degenerate spectral data (factorized) are given by the formulas [2]: (N=1)

$$S_x + iS_y = 2XY\frac{1 + \bar{a}b}{|1 - ab|^2}, \qquad S_- = \bar{S}_+ \tag{32}$$

$$S_3 = -\left(1 + 2\frac{(a + \bar{a})(b + \bar{b})}{|1 - ab|^2}\right), \tag{33}$$

$$\varphi(\xi,\eta,t) = 2i\ln(det\Delta) + 2\partial_\xi^{-1}U_2(\xi',t) + 2\partial_\eta^{-1}U_1(\eta',t) \tag{34}$$

$$a = \int_{-\infty}^{\eta} d_y\bar{Y}(y,t)\partial_y Y(y,t) \tag{35}$$

$$b = -\int_{-\infty}^{\xi} d_x X(x,y)\partial_x\bar{X}(x,t) \tag{36}$$

$$\Delta = \frac{1 - \overline{ab}}{1 + ab} \ . \tag{37}$$

One can easily see from (30),(31) that

$$Y(x,t) = \bar{X}(x,-t) \ .$$

Consider the case $b_1 > 0, k = \kappa$. Then

$$X = \frac{e^{i\alpha_1 x + i(\beta_1^2 - \alpha_1^2)t}}{2\alpha_1 cosh z_1 + i\beta_1 e^{-z_1}}, \qquad z_1 = \beta_1(x - 2\alpha_1 t + x_0) \tag{38}$$

$$Y = \bar{X}(y,-t) = \frac{e^{-i\alpha_2 y + i(\beta_2^2 - \alpha_2^2)t}}{2\alpha_2\cosh z_2 - i\beta_2 e^{-z_2}}, \qquad z_2 = \beta_2(y + 2\alpha_2 t + y_0) \tag{39}$$

$$a = \frac{1}{2}\ \frac{1 - i\frac{\alpha_2}{\beta_2}(1 + e^{2z_2})}{4\alpha_2^2\cosh^2 z_2 + \beta_2^2 e^{-2z_2}} \tag{40}$$

$$b = -\frac{1}{2}\ \frac{1 - i\frac{\alpha_1}{\beta_1}(1 + e^{2z_1})}{4\alpha_1^2\cosh^2 z_1 + \beta_1^2 e^{-2z_1}} \ . \tag{41}$$

The solution (32)–(41) is a one soliton solution which moves with the velocity $\vec{v} = (2\alpha_1, -2\alpha_2)$. One can proceed to the moving coodinate frame to obtain the solution at rest.

More detailed study of such solutions along with those for $N = 2$ are supposed to present elsewhere.

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References

[1] B. A. Dubrovin et al. Exact Solutions of Time-Dependent Schrödinger Eq. With Self-Consistant Potentials. *Sov.J.Part.Nuclei*, **19(3)**, (1988), 252-269

[2] V. G. Dubrovsky and B. G. Konopelchenko. Coherent Structures for the Ishimori Equation. Localized Solutions with Stationary Boundaries. *Physica*, **D48**, (1991) 367-395.

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Маханьков В.Г. E4-92-208

Солитонные решения одного линейного нестационарного уравнения и связанной с ним модели Ишимори

Обобщен метод, развитый ранее в работе Б.Дубровина, И.Кричевера, Т.Маланюка и автора, для исследования решений нестационарного уравнения Шредингера на случай нестационарного деривативного уравнения Шредингера и связанной с ним двумерной спиновой модели Ишимори. На примере односолитонных состояний исследуется случай граничных условий, зависящих от времени.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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On Soliton Type Solutions of One Linear Time-Dependent Equation and the Ishimori-II Model with Time Dependent Boundary Conditions

The method of work [1] is extended to obtain soliton type solutions to linear time-dependent derivative Schrödinger equation

$$(-i\partial_t + \partial_x^2 + iU(x, t)\partial_x)\psi(x, t, k) = 0$$

and thereby a class of solutions to the Ishimori-II model. A new class of such solutions is given.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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