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WEAK INTERACTIONS IN A BILOCAL CHIRAL THEORY
I. BOUND STATES WAVE FUNCTIONS FOR GIVEN ANGULAR MOMENTUM

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1 Introduction

One of the perspective directions of investigations in particle physics is the study of weak decay processes with help of a new class of experimental techniques, known as $c-\tau$ resp. B - factories (see, e.g., recent proposals, concerning this topics [1]).

The aim of this activities is to answer a lot of principal questions partly beyond the standard model [2] and to perform spectroscopical investigations in charmonium and B-meson physics on a level of accuracy that is to a marked degree in excess of the accuracy of conventional approaches.Besides others, this requires theoretical work in following directions:

description of mesons, resp. quarkonia, in this energy region with maximum degree of sophistication,including, e.g., the dependence of the potential shape and on the angular momentum structure;

the analysis of leptonic and non-leptonic decay processes,including the weak mixing processes of the different quark generations;

the evaluation of higher order loop diagrams.

We have performed a cycle of investigations in this direction and represent our results in a series of publications. The first result included into this paper concerns the derivation of bound state wave functions for given angular momentum in the bilocal approach.It includes the derivation of the Schwinger-Dyson equation, of the Bethe-Salpeter equation as well as the detailed description of the angular momentum structure of Bethe-Salpeter equations for the case of pseudoscalar mesons.

2 Bound states wave functions for given angular momentum in bilocal approach

2.1 Schwinger - Dyson equation

As denoted in the introduction we have to formulate the theory of hadrons as relativistic bound states in QCD.

One of the best ways to get this theory of hadrons is the generalization of the approach suggested by Heisenberg and Pauli in their papers [3] devoted to the first quantization of electrodynamics. The main idea of these papers is to quantize only the physical degrees of freedom (two transversal fields). According to this approach, the Coulomb potential in QED or the J/ψ potential in QCD are relativistic covariant and depend on the time - axis (η_μ) parallel to the eigenvector of the bound state total momentum operator (in other words the fields are always moving together with the bound states formed by the fields).

In this case, it is possible to prove that bilocal fields of atoms or hadrons are the irreducible representations of the Poincare group and one can describe very subtle effects of chiral symmetry breaking [4].

In papers [4, 5] it was shown that by this way we get the nonlocal hadronization of QCD with the rising J/ψ - potential which contains both the local chiral Lagrangian for light quarkonia and the nonrelativistic potential spectroscopy for heavy quarkonia, so that this model represents the bilocal generalization of chiral theory.

It is widely believed that the potential models are related only to the nonrelativistic approximation. This opinion, however, is based on the experience of solving scattering and dissociation problems in QED, where the Coulomb propagator corresponding to transversal photon exchange, converts into the relativistic invariant Feynman propagator up to the longitudinal part vanishing on the mass shell. Such a "relativistic method" of the potential model is incorrect for the description of bound states where elementary particles are off their mass shells and the longitudinal part of the propagator differs from zero.

From the experience of the description of atoms in QED one knows that the bound state is formed by an instantaneous interaction (Coulomb) potential (with a singularity on the time axis) whereas the transversal photon exchange plays the role of a correction and the relativistic description of the spectrum of a moving atom is done by means of the Coulomb potential moving together with the atom.

Generalizing this approach one gets[4, 5] :

$$W_{eff}[\psi,\bar\psi] = \int dx[\bar\psi(x)(i\partial\!\!\!/ - m^0)\psi(x) + \\ + \frac{1}{2}\int dy(\psi(y)\bar\psi(x))\mathcal{K}^{(\eta)}(z^\perp \mid X)(\psi(x)\bar\psi(y))]. \quad (1)$$

Here $\partial\!\!\!/ \equiv \partial^\mu\gamma_\mu$, $\mathcal{K}^{(\eta)}$ is the kernel

$$\mathcal{K}^{(\eta)}(z^\perp \mid X) = \eta\!\!\!/ V(z^\perp)\delta(z\cdot\eta)\eta\!\!\!/, \ (\eta\!\!\!/ = \eta^\mu\gamma_\mu), \quad (2)$$

where z and X are the relative and total coordinates defined, respectively, as

$$z = x - y, \ X = \frac{x+y}{2}, \quad (3)$$

and $V(z^\perp)$ is the potential depending on the transversal (with respect to the time - axis η) component of the relative coordinate, $z^\perp_\mu = z_\mu - \eta_\mu(z\cdot\eta)$ (a consistent construction of the gauge theories with such properties has been proposed recently [6]).

The next question is how to choose the time - axis in (1) for describing the bound state. It has been suggested in ref. [9] to take in this case as a time - axis the unit vector which is proportional to the eigenvector of the bound state total momentum operator, i.e.:

$$\eta_\mu \sim \mathcal{P}_\mu . \quad (4)$$

When this requirement is satisfied the bound state wave functions automatically belong to the irreducible representation of the Poincare group [9]. The expression (1) with the kernel (2) and with the time - axis defined by (4) represents a relativistic covariant action.

The next task is to introduce a bilocal meson field. In a path integral formulation this is achieved by means of the Legendre transformation [5],[9] - [11]

$$\begin{aligned} \frac{1}{2}\int dxdy(\psi(y)\bar\psi(x))\mathcal{K}(x,y)(\psi(x)\bar\psi(y)) = \\ = -\frac{1}{2}\int dxdy\mathcal{M}(x,y)\mathcal{K}^{-1}(x,y)\mathcal{M}(x,y) + \\ + \int dxdy(\psi(x)\bar\psi(y))\mathcal{M}(x,y) \end{aligned} \quad (5)$$

where $\mathcal{K}^{-1}$ is the inverse of the kernel (2). Following refs. [8], [10]- [11] we introduce the short - hand notation

$$\begin{aligned} \int dx\bar\psi(x)(i\partial\!\!\!/ - m^0)\psi(x) &\equiv \int dxdy\psi(y)\bar\psi(x)(i\partial\!\!\!/ - m^0)\delta(x-y) \\ &= (\psi\bar\psi, -G_0^{-1}) ; \\ \int dxdy(\psi(x)\bar\psi(y))\mathcal{M}(x,y) &= (\psi\bar\psi, \mathcal{M}). \end{aligned}$$

Then, from action (1) we get

$$W_{eff}[\mathcal{M}] = (\psi\bar\psi, (-G_0^{-1} + \mathcal{M})) - \frac{1}{2}(\mathcal{M}, \mathcal{K}^{-1}\mathcal{M}). \quad (6)$$

After integration over N_c fermion fields and normal ordering, this action takes the form

$$W_{eff}[\mathcal{M}] = -\frac{1}{2}N_c(\mathcal{M}, \mathcal{K}^{-1}\mathcal{M}) - iN_c\sum_{n=1}^{\infty}\frac{1}{n}\Phi^n. \quad (7)$$

Here $\Phi \equiv G_0\mathcal{M}, \Phi^2, \Phi^3$ etc. mean the following expressions

$$\begin{aligned} \Phi(x,y) &= \int dzG_0(x,z)\mathcal{M}(z,y), \\ \Phi^2 &= \int dxdy\Phi(x,y)\Phi(y,x), \\ \Phi^3 &= \int dxdydz\Phi(x,y)\Phi(y,z)\Phi(z,x) \text{ ,etc} \end{aligned} \quad (8)$$

Only the contributions with inner fermionic lines (but no scattering and dissociation channel contribution) are included in the effective action since we are interested only in the bound states description.

The requirement for the choice of the time - axis in bilocal dynamics is now equivalent to the condition [9]

$$z_\mu \cdot i\frac{\partial\mathcal{M}(z,X)}{\partial X_\mu} = 0 \quad (9)$$

where z_μ and X_μ are relative and total coordinates.

To perform the path integral over the meson field we apply the saddle point method determining the minimum of the action (7)

$$N_c^{-1}\frac{\delta W_Q(\mathcal{M})}{\delta \mathcal{M}} == -\mathcal{K}^{-1}\mathcal{M} - i\sum_{n=1}^{\infty} G_0(\mathcal{M}G_0)^n \equiv -\mathcal{K}^{-1}\mathcal{M} + \frac{i}{G_0^{-1}-\mathcal{M}} = 0. \quad (10)$$

We denote the corresponding classical solution for the bilocal field by $\Sigma(x-y)$. It depends only on the difference $x-y$ because of translation invariance of vacuum solutions.

The next step is the expansion of the action (1) around the point of minimum $\mathcal{M} = \Sigma + \mathcal{M}'$,

$$\begin{aligned} W_Q(\Sigma+\mathcal{M}') &= W_Q(\Sigma) + N_c[-\frac{1}{2}\mathcal{M}'\mathcal{K}^{-1}\mathcal{M}' - \frac{i}{2}(G_\Sigma \mathcal{M}')^2] + \\ &- iN_c\sum_{n=3}^{\infty}\frac{1}{n}(G_\Sigma\mathcal{M}')^n, \;\; (G_\Sigma = (G_0^{-1}-\Sigma)^{-1}), \end{aligned} \quad (11)$$

and the representation of the small fluctuations $\mathcal{M}'$ as a sum over the complete set of classical solutions Γ,

$$\frac{\delta^2 W_Q(\Sigma+\mathcal{M}')}{\delta\mathcal{M}^2}|_{\mathcal{M}'=0}\cdot\Gamma = 0. \quad (12)$$

Using the definitions (8) and (11) we obtain the standard form of equations (10) and (12):

$$\Sigma(x-y) = m^0\delta^{(4)}(x-y) + i\mathcal{K}(x,y)G_\Sigma(x-y), \quad (13)$$

$$\Gamma = -i\mathcal{K}(x,y)\int dz_1dz_2 G_\Sigma(x-z_1)\Gamma(z_1,z_2)G_\Sigma(z_2-y) \quad (14)$$

which are, respectively, the Schwinger - Dyson (SD) and Bethe - Salpeter (BS) equations.

In momentum space

$$\begin{aligned} \Sigma(k) &= \int dx\Sigma(x)e^{ikx}, \\ \Gamma(k|\mathcal{P}) &= \int dxdy e^{i\frac{x+y}{2}\mathcal{P}} e^{i(x-y)k}\Gamma(x,y) \end{aligned}$$

we obtain for the mass operator $\Sigma(k)$ and the vertex function $\Gamma(k|\mathcal{P})$

$$\Sigma(k) = m^0 + i\int\frac{dq}{(2\pi)^4}V(k^\perp - q^\perp)\not\eta G_\Sigma(q)\not\eta, \quad (15)$$

$$\Gamma(k,\mathcal{P}) = -i\int\frac{dq}{(2\pi)^4}V(k^\perp-q^\perp)\not\eta[G_\Sigma(q+\frac{\mathcal{P}}{2})\Gamma(q|\mathcal{P})G_\Sigma(q-\frac{\mathcal{P}}{2})]\not\eta \quad (16)$$

where $G_\Sigma(q) = (\not q - \Sigma(q))^{-1}$, $V(k^\perp)$ means the Fourier transform of the potential, $k^\perp_\mu = k_\mu - \eta_\mu(k\cdot\eta)$ is the transversal with respect to η_μ relative momentum, $\mathcal{P}_\mu$ is the total momentum.

We may integrate in (15) and (16) over the longitudinal momentum $q_0 = (q\cdot\eta)$ using the representation

$$\Sigma_a(q) = \not q^\perp + E_a(q^\perp)S_a^{-2}(q^\perp) \quad (17)$$

for the self - energy with

$$S_a^{-2}(q^\perp) = exp\{-\hat{\not q}^\perp 2v_a(q^\perp)\}, \;\; \hat q^\perp_\mu = q^\perp_\mu/|q^\perp| \quad (18)$$

where S_a is the Foldy - Wouthuysen type transformation matrix with the parameter v_a.

Then one has

$$G_{\Sigma_a} = [\frac{\Lambda^{(\eta)}_{(+)a}(q^\perp)}{q_0 - E_a(q^\perp)+i\epsilon} + \frac{\Lambda^{(\eta)}_{(-)a}(q^\perp)}{q_0 + E_a(q^\perp)+i\epsilon}]\not\eta \quad (19)$$

where

$$\Lambda^{(\eta)}_{(\pm)a}(q^\perp) = S_a(q^\perp)\Lambda^{(\eta)}_{(\pm)}(0)S_a^{-1}(q^\perp), \;\; \Lambda^{(\eta)}_{(\pm)}(0) = (1\pm\not\eta)/2 \quad (20)$$

are the operators projecting on states with positive ($+E_a$) and negative ($-E_a$) energies.

As a result, we obtain the following equations for the one - particle energy E and the angle v:

$$E_a(k^\perp)\cos 2v(k^\perp) = m_a^0 + \frac{1}{2}\int\frac{d^3q^\perp}{(2\pi)^3}V(k^\perp-q^\perp)\cos 2v(q^\perp) \quad (21)$$

$$E_a(k^\perp)\sin 2v(k^\perp) = |k^\perp| + \frac{1}{2}\int\frac{d^3q^\perp}{(2\pi)^3}V(k^\perp-q^\perp)|k^\perp\cdot q^\perp|\sin 2v(q^\perp) \quad (22)$$

2.2 Bethe - Salpeter equation

The Bethe - Salpeter equation has the form

$$\Gamma(q|\mathcal{P}) = -i\int\frac{dp}{(2\pi)^4}V(\mathrm{q}-\mathrm{p})\not\eta G_1(p+\frac{\mathcal{P}}{2})\Gamma(p|\mathcal{P})G_2(p-\frac{\mathcal{P}}{2})\not\eta. \quad (23)$$

The quantity $\mathcal{P}$ is the total momentum of the bound state, and $\mathcal{P} = (\mathcal{P}_0, \vec{\mathcal{P}})$; $\mathcal{P}_0 = E_H = \sqrt{\vec{\mathcal{P}}^2 + M_H^2}$. The factor $\not\eta$ appearing in this equation for the effective interaction in the case of moving bound states $\vec{\mathcal{P}} \neq 0$ has been evaluated in accordance with (4) by taking η in the direction of $\mathcal{P}$: $\eta_\mu = \mathcal{P}_\mu/\sqrt{\mathcal{P}^2}$

In the rest frame the Bethe - Salpeter equation in the terms of Γ (vertex functions) has the form

$$\Gamma(q) = -i\int\frac{dp}{(2\pi)^4}V(\mathbf{q}-\mathbf{p})\gamma_0 G_1(p+\frac{M_H}{2})\Gamma(p)G_2(p-\frac{M_H}{2})\gamma_0 \quad (24)$$

The Green functions are given by

$$G_1(p+\frac{M_H}{2}) = (\frac{\Lambda_+}{p_0+(M_H/2)-E_1+i\epsilon}+\frac{\Lambda_-}{p_0+(M_H/2)+E_1-i\epsilon})\gamma_0 ,$$

$$G_2(p-\frac{M_H}{2}) = \gamma_0(\frac{\bar\Lambda_+}{p_0-(M_H/2)-E_2+i\epsilon}+\frac{\bar\Lambda_-}{p_0-(M_H/2)+E_2-i\epsilon}) , \quad (25)$$

where E_i are the solutions of the equations (21)-(22). Integrating the Bethe - Salpeter equation over p_0 we obtain

$$\Gamma(q) = \int\frac{d\mathbf{p}}{(2\pi)^3}V(\mathbf{q}-\mathbf{p})\gamma_0\{\frac{\Pi_{+-}}{E-M_H}+\frac{\Pi_{-+}}{E+M_H}\}\gamma_0 , \quad (26)$$

$$\Pi_{\pm\mp} = \Lambda_\pm^{(1)}\gamma_0\Gamma\gamma_0\bar\Lambda_\mp^{(2)} . \quad (27)$$

In the rest frame Γ has the form

$$\Gamma_l = 1\cdot S_l + \gamma_5\cdot\Gamma_l^p + \gamma_i^v\cdot\Gamma_l^{vi} + \gamma_i^a\cdot\Gamma_l^{ai} , \quad (28)$$

where $\gamma^s = 1$, $\gamma^p = \gamma_5$, $\gamma_i^v = \gamma_i$, $\gamma_i^a = \gamma_i\gamma_5$ for scalar, pseudoscalar, vector and axial - vector (s,p,v,a) bound states, respectively ($l = 1,2$; $i = 1,2,3$).
For what follows it is favourable to write Γ in the sum form

$$\Gamma = \sum_I(\Gamma_1^I+\gamma_0\Gamma_2^I)\gamma^I , \qquad I = s,p,v,a \quad (29)$$

The expression $(\gamma_0\Gamma\gamma_0)$ in (26) is equal to

$$(\gamma_0\Gamma\gamma_0) = [I]\sum_I(\Gamma_1^I+\gamma_0\Gamma_2^I)\gamma^I , \qquad \gamma_0\gamma^I = [I]\gamma^I\gamma_0 \quad (30)$$

$[I] = (1,-1,-1,1)$ for (s,p,v,a), respectively.
We can express $\Pi_{\pm\mp}$ given in formula(26) by the operator S introduced with help of (18)

$$\Pi_{\pm\mp} = \frac{1}{4}[I]\{[\Gamma_1^I(\gamma^I-[I]S_1^{-2}\gamma^I S_2^{-2})\pm\Gamma_2^I(S_1^{-2}\gamma^I-[I]\gamma^I S_2^{-2})] \pm \gamma_0[\Gamma_1^I(S_1^2\gamma^I-[I]\gamma^I S_2^{-2})\pm\Gamma_2^I(\gamma^I-[I]S_1^2\gamma^I S_2^{-2})]\} ; \quad (31)$$

Multiplying the Bethe - Salpeter equation step by step by $\bar\gamma^K = (1,\gamma_5,\gamma_i,\gamma_i\gamma_5)$ and $\bar\gamma^K\gamma_0$ and taking the traces having in mind that

$$\frac{1}{4}\mathrm{tr}(\bar\gamma^K\gamma^I) = \delta^{KI} = \begin{cases}(1,1,\delta_{ij},\delta_{ij}) & \text{for (s,p,v,a)}\\ 0 & K\neq I\end{cases} ,$$

we obtain the expressions for traces

$$\frac{1}{4}\mathrm{tr}(\bar\gamma^K S_1^{-2}\gamma^I S_2^{-2}) = \bar s_1 s_2\frac{1}{4}\mathrm{tr}(\bar\gamma^K\gamma^I) + c_1c_2\frac{1}{4}\mathrm{tr}(\bar\gamma^K\hat p\gamma^I\hat p) ,$$

$$\frac{1}{4}\mathrm{tr}(\bar\gamma^K S_1^{2}\gamma^I S_2^{-2}) = s_1 s_2\frac{1}{4}\mathrm{tr}(\bar\gamma^K\gamma^I) - c_1c_2\frac{1}{4}\mathrm{tr}(\bar\gamma^K\hat p\gamma^I\hat p) ,$$

$$\frac{1}{4}\mathrm{tr}(\bar\gamma^K\hat p\gamma^I\hat p) = [t_I]\cdot\delta^{KI} ,$$

with the definitions

$$[t_I] = (-1,1,\frac{1}{3},-\frac{1}{3}) \quad \text{for (s,p,v,a)} .$$

Using the relations

$$1\mp[I]s_1s_2-[I][t_I]c_1c_2 = \frac{1}{E}\{(m_1\mp[I]m_2)(\frac{m_1}{E_1}\mp[I]\frac{m_2}{E_2}) + (1-[I][t_I])\mathbf{p}^2(\frac{1}{E_1}+\frac{1}{E_2})\} ,$$

$$1\mp[I]s_1s_2+[I][t_I]c_1c_2 = \frac{1}{E}\{(m_1\mp[I]m_2)(\frac{m_1}{E_1}\mp[I]\frac{m_2}{E_2}) + (1+[I][t_I])\mathbf{p}^2(\frac{1}{E_1}+\frac{1}{E_2})\} ,$$

which contain the current quark masses m_i (for the heavy quarkonia $m_i = m_i^0$ [4, 5]), we obtain the Bethe - Salpeter equation for the vertex functions $\Gamma_{1,2}^I$

$$\begin{aligned}\Gamma_1^I(q) &= \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}V(\mathbf{q}-\mathbf{p})\{\frac{1}{E^2-M^2}\cdot[(m_1-[I]m_2)(\frac{m_1}{E_1}-\frac{m_2}{E_2}) + (1-[I][t_I])\mathbf{p}^2(\frac{1}{E_1}+\frac{1}{E_2})]\Gamma_1^I(p) + \frac{M}{E^2-M^2}(\frac{m_1}{E_1}-[I]\frac{m_2}{E_2})\Gamma_2^I(p)\}\\ \Gamma_2^I(q) &= \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}V(\mathbf{q}-\mathbf{p})\{\frac{M}{E^2-M^2}(\frac{m_1}{E_1}-[I]\frac{m_2}{E_2})\Gamma_1^I(p) + \frac{1}{E^2-M^2}\cdot[(m_1-[I]m_2)(\frac{m_1}{E_1}-\frac{m_2}{E_2}) + (1+[I][t_I])\mathbf{p}^2(\frac{1}{E_1}+\frac{1}{E_2})]\Gamma_2^I(p)\} .\end{aligned} \quad (32)$$

Sometimes it is favourable not to work with the vertex function Γ but with the Bethe - Salpeter wave function Ψ. In the rest frame both these quantities are connected with each other by the relation

$$\Gamma(q) = \int \frac{d\mathbf{p}}{(2\pi)^3} V(\mathbf{q}-\mathbf{p})\gamma_0 \Psi(p)\gamma_0 \ , \tag{33}$$

such that Ψ is defined as

$$\Psi(p) = \{\frac{\Pi_{+-}}{E-M} + \frac{\Pi_{-+}}{E+M}\} \ . \tag{34}$$

Now the quantities $\Pi_{\pm\mp}$ may be expressed as

$$\begin{aligned} \Pi_{\pm\mp} &= \Lambda^{(1)}_{\pm}\gamma_0\Gamma\gamma_0\bar{\Lambda}^{(2)}_{\mp} = \\ &= S_1^{-1}\Lambda^0_{\pm}S_1(\gamma_0\Gamma\gamma_0)S_2\Lambda^0_{\mp}S_2^{-1} = S_1^{-1} \overset{0}{\Pi}_{\pm\mp} S_2^{-1} = \\ &= S_1^{-1}\Lambda^0_{\pm}\gamma_0 S_1^{-1}\Gamma\gamma_0 S_1^{-1}\gamma_0\Lambda^0_{\mp}S_2^{-1} = -S_1^{-1}\Lambda^0_{\pm}(S_1^{-1}\Gamma S_2^{-1})\Lambda^0_{\mp}S_2^{-1} \ . \end{aligned} \tag{35}$$

$$\Lambda^0_{\pm} = \frac{1}{2}(1 \pm \gamma_0) \ , \ \Lambda^0_{\pm}\gamma_0 = \pm\Lambda^0_{\pm} \ , \ \gamma_0\Lambda^0_{\pm} = \pm\Lambda^0_{\pm} \ .$$

By help of these formulae we can write $\Psi(p), \Pi_{\pm\mp}$ as

$$\begin{aligned} \Psi(p) &= S_1^{-1}(p) \overset{0}{\Psi}(p) S_2^{-1}(p) \\ \Pi_{\pm\mp} &= S_1^{-1}(p) \overset{0}{\Pi}_{\pm\mp} S_2^{-1}(p) \end{aligned} \tag{36}$$

and

$$\overset{0}{\Pi}_{\pm\mp} = \Lambda^0_{\pm} \overset{0}{\Gamma} \Lambda^0_{\mp} \ , \tag{37}$$

where

$$\overset{0}{\Gamma} = S_1^{-1}\Gamma S_2^{-1} \ , \Gamma = S_1 \overset{0}{\Gamma} S_2 \ . \tag{38}$$

By help of the last relation we get

$$S_2(p) \overset{0}{\Gamma}(p) S_2(p) = -\int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\gamma_0 S_1^{-1}(q) \overset{0}{\Psi}(q) S_2^{-1}(q)\gamma_0$$

and

$$\begin{aligned} \overset{0}{\Gamma} &= -\int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})(S_1^{-1}(p)\gamma_0 S_1^{-1}(q)) \overset{0}{\Psi} (S_2^{-1}(q)\gamma_0 S_1^{-1}(p)) \equiv \\ &\equiv -\int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) S_1' \overset{0}{\Psi} S_2' \end{aligned} \tag{39}$$

and we obtain for the expressions on the r.h.s.

$$\overset{0}{\Pi}_{\pm\mp} = \Lambda^0_{\pm} \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})(S_1' \overset{0}{\Psi} S_2')\Lambda^0_{\mp} \tag{40}$$

and

$$\overset{0}{\Psi}(p) = \{\frac{\overset{0}{\Pi}_{+-}}{E-M} + \frac{\overset{0}{\Pi}_{-+}}{E+M}\} \ . \tag{41}$$

As a result we have the following two relations, which are similar to Schrödinger equations

$$\begin{aligned} \Lambda^0_{+} \overset{0}{\Psi} \Lambda^0_{-} &= \frac{1}{E-M} \overset{0}{\Pi}_{+-} \ ; \\ \Lambda^0_{-} \overset{0}{\Psi} \Lambda^0_{+} &= \frac{1}{E+M} \overset{0}{\Pi}_{-+} \ . \end{aligned}$$

We can rewrite this equations in the form

$$\begin{aligned} (E-M)\Lambda^0_{+} \overset{0}{\Psi} \Lambda^0_{-} &= \Pi_{+-} \ ; \\ (E+M)\Lambda^0_{-} \overset{0}{\Psi} \Lambda^0_{+} &= \Pi_{-+} \ . \end{aligned} \tag{42}$$

Let us introduce the wave function of bound states containing particles a and b, with help of the definition

$$\overset{0}{\Psi} = \overset{0}{\Psi}_1 + \gamma_0 \cdot \overset{0}{\Psi}_2 \ . \tag{43}$$

Expanding $\overset{0}{\Psi}_{1,2}$ over the full system $\gamma_5, \hat{e}^a(p), \hat{p}$

$$\overset{0}{\Psi} = \sum_{I=1}^{3} [\overset{0}{\Psi}_1{}^I + \gamma_0 \cdot \overset{0}{\Psi}_2{}^I]\gamma^I \ , \tag{44}$$

we obtain systems of coupled equations for $\overset{0}{\Psi}_{1,2}{}^I$. In this relation

$$\overset{0}{\Psi}_2{}^I = \overset{0}{L}_i \ , \ \overset{0}{\Psi}_2{}^I = \overset{0}{N}_i{}^a \ , \ \overset{0}{\Psi}_2{}^I = \overset{0}{\Sigma} \ , \ (i = 1,2) \tag{45}$$

the quantities $\overset{0}{L}_i, \overset{0}{N}_i{}^a, \overset{0}{\Sigma}$ correspond to pseudoscalar, vector and scalar particles, respectively. Let us multiply the expressions for the wave functions by γ^K and take the trace

$$\begin{aligned} M \overset{0}{\Psi}_2{}^I \cdot \frac{1}{4}\mathrm{tr}(\gamma^K\gamma^I) &= E \overset{0}{\Psi}_1{}^I \cdot \frac{1}{4}\mathrm{tr}(\gamma^K\gamma^I) - \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) T^{KJ}_{12} \overset{0}{\Psi}_1{}^J \ ; \\ M \overset{0}{\Psi}_1{}^I \cdot \frac{1}{4}\mathrm{tr}(\gamma^K\gamma^I) &= E \overset{0}{\Psi}_2{}^I \cdot \frac{1}{4}\mathrm{tr}(\gamma^K\gamma^I) - \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q}) T^{KJ}_{12} \overset{0}{\Psi}_1{}^J \ , \end{aligned} \tag{46}$$

where

$$K_{12}^{KJ} = \frac{1}{4}\mathrm{tr}(\gamma^K S_1' \gamma^J S_2') \; . \tag{47}$$

After calculation of traces we get

$$T_{12}^{KJ} = \mathrm{c}_p^{\alpha_K}\mathrm{c}_q^{\alpha_J}\mathrm{tr}(\gamma^J\gamma^K) - \mathrm{s}_p^{\alpha_K}\mathrm{s}_q^{\alpha_J}\mathrm{tr}(\gamma^J\gamma^K\hat{p}\hat{q}) \; , \tag{48}$$

where

$$\begin{aligned}
\mathrm{c}^{\pm\alpha_K} &= c_2c_1 \mp \alpha_K s_2 s_1 \; , \\
\mathrm{s}^{\pm\alpha_K} &= s_1c_2 \pm \alpha_K s_2 c_1 \; , \\
s_{1,2} &= \sin(\phi_{1,2}) \; , \; c_{1,2} = \cos(\phi_{1,2}) \; , \\
\alpha_K &= (-1,-1,1) \text{ for } \gamma^K = (\gamma^5, \hat{e}^a(p), \hat{p}) \; , \text{ respectively} \; .
\end{aligned}$$

As a result we obtain the systems for the wave functions of bound states :

1. Pseudoscalar particles

$$\gamma^K = \gamma^5 \; ; \; \alpha_K = -1; \; \alpha_J = 1(\gamma_J = \gamma^5) \; .$$

$$\begin{aligned}
M \overset{0}{L}_2 &= E \overset{0}{L}_1 - \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})(\mathrm{c}_p^-\mathrm{c}_q^- - \xi \mathrm{s}_p^-\mathrm{s}_q^-) \overset{0}{L}_1 \; ; \\
M \overset{0}{L}_1 &= E \overset{0}{L}_2 - \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})(\mathrm{c}_p^+\mathrm{c}_q^+ - \xi \mathrm{s}_p^+\mathrm{s}_q^+) \overset{0}{L}_2 \; .
\end{aligned} \tag{49}$$

2.Vector particles

$$\gamma^K = \hat{e}^a(p) \; ; \; \alpha_K = -1 \; ; \; \alpha_J = 1(\gamma_J = \hat{q}) \text{ and } \alpha_J = -1(\gamma_J = \hat{e}^b(q)) \; .$$

$$\begin{aligned}
M \overset{0}{N}_2{}^a &= E \overset{0}{N}_1{}^a + \\
&+ \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\mathrm{c}_p^-\mathrm{c}_q^-\underline{\delta}^{ab} - \mathrm{s}_p^-\mathrm{s}_q^-(\underline{\delta}^{ab}\xi - \eta^a\underline{\eta}^b)) \overset{0}{N}_1{}^b + (\eta_a \mathrm{c}_p^-\mathrm{c}_q^+) \overset{0}{\Sigma}_1\} \; ; \\
M \overset{0}{N}_1{}^a &= E \overset{0}{N}_2{}^a + \\
&+ \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\mathrm{c}_p^+\mathrm{c}_q^+\underline{\delta}^{ab} - \mathrm{s}_p^+\mathrm{s}_q^+(\underline{\delta}^{ab}\xi - \eta^a\underline{\eta}^b)) \overset{0}{N}_2{}^b + (\eta_a \mathrm{c}_p^+\mathrm{c}_q^-) \overset{0}{\Sigma}_2\} \; .
\end{aligned} \tag{50}$$

$$\eta^a = \hat{q}_i\hat{e}_i^a(p) \; , \; \underline{\eta}^a = \hat{p}_i\hat{e}_i^a(q) \; , \; \underline{\delta}^{ab} = \hat{e}_i^a(q)\hat{e}_i^b(p) \; .$$

3. Scalar particles

$$\gamma^K = \hat{p}_i \; ; \; \alpha_K = 1 \; ; \; \alpha_J = 1(\gamma_J = \hat{q}) \text{ and } \alpha_J = -1(\gamma_J = \hat{e}^b(q)) \; .$$

$$\begin{aligned}
M \overset{0}{\Sigma}_2 &= E \overset{0}{\Sigma}_1 + \\
&+ \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\xi \mathrm{c}_p^+\mathrm{c}_q^+ - \mathrm{s}_p^+\mathrm{s}_q^+) \overset{0}{\Sigma}_1 + (\underline{\eta}_b \mathrm{c}_p^-\mathrm{c}_q^+) \overset{0}{N}_1{}^b\} \; ; \\
M \overset{0}{\Sigma}_1 &= E \overset{0}{\Sigma}_2 + \\
&+ \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}-\mathbf{q})\{(\xi \mathrm{c}_p^-\mathrm{c}_q^- - \mathrm{s}_p^-\mathrm{s}_q^-) \overset{0}{\Sigma}_2 + (\underline{\eta}_b \mathrm{c}_p^+\mathrm{c}_q^-) \overset{0}{N}_2{}^b\} \; .
\end{aligned} \tag{51}$$

$$\xi = \hat{p}_i \cdot \hat{q}_i \; , \; \underline{\eta}_b = \hat{p}_i \cdot \hat{e}_i^b \; .$$

These relations have a very compact form for bound state wave functions.

2.3 Angular momentum structure of Bethe-Salpeter equations

After deriving the sets of equations (49)-(51) we will consider their angular momentum structure.

At beginning, let us fix the shape of the potential, appearing in equations (49)-(51). It is interesting to note that the simple polynomial ansatz

$$V(\mathbf{p}-\mathbf{q}) = \mathcal{E}|\mathbf{p}-\mathbf{q}|^n \; , \tag{52}$$

where $\mathcal{E}$- is a strength parameter, is quite useful. Really, the parametrization (52) includes the following potential shapes [12]:

$$\begin{aligned}
\text{Coulomb potential: } V(r) &= \left(-\frac{4}{3}\cdot\frac{\alpha_s}{r}\right), \; \alpha_s = \frac{g_s^2}{4\pi} \Longrightarrow V(\mathbf{p}) = -\frac{4}{3}\alpha_s \cdot \frac{4\pi}{\mathbf{p}^2} \; , \\
\mathcal{E} &= -(4\pi)\frac{4}{3}\alpha_s \; , \; n = -2 \; ; \\
\text{Linear potential: } V(r) &= a \cdot r \; , \; a - \text{parameter} \; , \Longrightarrow V(\mathbf{p}) = -\frac{(8\pi)a}{\mathbf{p}^4} \; , \\
\mathcal{E} &= -(8\pi)a \; , \; n = -4 \; ;
\end{aligned} \tag{53}$$

$$\begin{aligned}
\text{Delta-potential: } V(r) &= b \cdot \delta(\mathbf{r}) \; , \; b - \text{parameter} \; , \Longrightarrow V(\mathbf{p}) = b \; , \\
\mathcal{E} &= b \; , \; n = 0 \; .
\end{aligned}$$

However, for the oscillator potential we have the form

$$V(r) = -V_0 r^2\,,\ V_0 - \text{parameter}\,,\ \Longrightarrow V(\mathbf{p}) = V_0(2\pi)^3\Delta_{\mathbf{p}}\delta(\mathbf{p})\,, \tag{54}$$

which is different from (52). Now we discuss the model with the potential (52). The bound state wave functions for the oscillator potential are discussed in [13].

Let us expand the wave functions in (49) over spherical harmonics:

$$L_{\binom{2}{1}}(p) = \frac{1}{p}(N_{\binom{2}{1}}(p))_{l_1}\mathcal{Y}_{l_1m_1}(\hat{p})\ ; \tag{55}$$

and use for the decomposition of $V(\mathbf{p}-\mathbf{q})$ the expression

$$|\mathbf{p}-\mathbf{q}|^n = \sum_{l'=0}^{\infty} a^n_{l'}(p,q)\frac{4\pi}{2l'+1}(\vec{\mathcal{Y}}_{l'}\cdot\vec{\mathcal{Y}}_{l'}) = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{m'=l'}\frac{4\pi}{2l'+1}a^n_{l'}(p,q)(\mathcal{Y}^*_{l'm'}(\hat{q})\cdot\mathcal{Y}_{l'm'}(\hat{p}))\,. \tag{56}$$

In this relation

$$a^n_l = \frac{(-\frac{n}{2})_l}{(\frac{1}{2})_l}\cdot\frac{(pq)^l}{(p^2+q^2)^{l-\frac{n}{2}}}\cdot F(\frac{l}{2}-\frac{n}{4},\frac{l}{2}-\frac{n}{4}+\frac{1}{2};l+\frac{3}{2};(\frac{2pq}{p^2+q^2})^2) \tag{57}$$

and F denotes the corresponding hypergeometric functions. Similarly, for the parameter ξ, entering formula (49), we obtain

$$\xi = \frac{4\pi}{3}(\vec{\mathcal{Y}}_1\cdot\vec{\mathcal{Y}}_1) = \frac{4\pi}{3}\sum_{\mu=-1}^{\mu=1}\mathcal{Y}^*_{1\mu}(\hat{p})\cdot\mathcal{Y}_{1\mu}(\hat{q}) = \frac{4\pi}{3}\sum_{\mu=-1}^{\mu=1}\mathcal{Y}^*_{1\mu}(\hat{q})\cdot\mathcal{Y}_{1\mu}(\hat{p})\,. \tag{58}$$

By using the relations

$$\int d\Omega_q \mathcal{Y}^*_{l'm'}(\hat{q})\mathcal{Y}_{l_2m_2}(\hat{q}) = \delta_{l'l_2}\delta_{m'm_2}\ ;$$

$$\int d\Omega_q \mathcal{Y}^*_{l'm'}(\hat{q})\mathcal{Y}^*_{1\mu}(\hat{q})\mathcal{Y}_{l_2m_2}(\hat{q}) = \int d\Omega_q(-1)^{m'}(-1)^{\mu}\mathcal{Y}_{l'-m'}(\hat{q})\mathcal{Y}_{1-\mu}(\hat{q})\mathcal{Y}_{l_2m_2}(\hat{q}) = (-1)^{m'}(-1)^{\mu}\sqrt{\frac{(2l'+1)3(2l_2+1)}{4\pi}}\begin{pmatrix} l' & 1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l_2\\ -m' & -\mu & m_2\end{pmatrix};$$

$$\mathcal{Y}_{l'm'}\mathcal{Y}_{1\mu} = \sqrt{\frac{(2l'+1)3}{4\pi}}\sum_{l,m}\sqrt{2l+1}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ m' & \mu & m\end{pmatrix}(-1)^m\mathcal{Y}_{l-m}\ ;$$

we obtain from equation (49)

$$M[n_{\binom{2}{1}}(p)]_l = E[n_{\binom{1}{2}}(p)]_l - \frac{4\pi p}{2l+1}\int\frac{qdq}{(2\pi)^3}\{(c^{\mp}_p c^{\mp}_q)a^n_l(p,q)[n_{\binom{1}{2}}(q)]_l - (s^{\mp}_p s^{\mp}_q)\mathcal{F}_l\} \tag{59}$$

where

$$\mathcal{F}_l = \sum_{m=-l}^{m=l}\sum_{l',m'}\sum_{l_2,m_2}\sum_{m''=-1}^{1}(2l+1)\sqrt{(2l_2+1)(2l+1)}(-1)^{m+m'+m''}(-1)^{l'+1+l_2}\cdot \begin{pmatrix} l' & 1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & 1 & l_2\\ m' & m'' & -m_2\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ m' & m'' & -m\end{pmatrix}\cdot a^n_{l'}(p,q)[n_{\binom{1}{2}}(q)]_{l_2}$$

As a result we have the equations for given angular momentum

L=0:

$$M[n_{\binom{2}{1}}(p)]_0 = E[n_{\binom{1}{2}}(p)]_0 - 4\pi p\int\frac{qdq}{(2\pi)^3}\{(c^{\mp}_p c^{\mp}_q)a^n_0(p,q) - \frac{1}{3}(s^{\mp}_p s^{\mp}_q)a^n_1(p,q)\}[n_{\binom{1}{2}}(q)]_0 \tag{60}$$

L=1:

$$M[n_{\binom{2}{1}}(p)]_1 = E[n_{\binom{1}{2}}(p)]_1 - \frac{4\pi p}{3}\int\frac{qdq}{(2\pi)^3}\{(c^{\mp}_p c^{\mp}_q)a^n_1(p,q) - (s^{\mp}_p s^{\mp}_q)(3a^n_0(p,q) + \frac{6}{5}a^n_2(p,q)\}[n_{\binom{1}{2}}(q)]_1 \tag{61}$$

L=2:

$$M[n_{\binom{2}{1}}(p)]_2 = E[n_{\binom{1}{2}}(p)]_2 - \frac{4\pi p}{5}\int\frac{qdq}{(2\pi)^3}\{(c^{\mp}_p c^{\mp}_q)a^n_2(p,q) - (s^{\mp}_p s^{\mp}_q)(\frac{2}{15}a^n_1(p,q) + \frac{3}{35}a^n_3(p,q)\}[n_{\binom{1}{2}}(q)]_2 \tag{62}$$

The coefficients a^n_l in these formulae are

a_l^n	Coulomb potential	Linear potential
a_0^n	$\frac{1}{(p^2+q^2)} \cdot F(\frac{1}{2}, 1; \frac{3}{2}; z^2)$	$\frac{1}{(p^2+q^2)^2} \cdot F(1, \frac{3}{2}; \frac{3}{2}; z^2)$
a_1^n	$2 \cdot \frac{(pq)}{(p^2+q^2)^2} \cdot F(1, \frac{3}{2}; \frac{5}{2}; z^2)$	$4 \cdot \frac{(pq)}{(p^2+q^2)^3} \cdot F(\frac{3}{2}, 2; \frac{5}{2}; z^2)$
a_2^n	$\frac{8}{3} \cdot \frac{(pq)^2}{(p^2+q^2)^3} \cdot F(\frac{3}{2}, 2; \frac{7}{2}; z^2)$	$8 \cdot \frac{(pq)^2}{(p^2+q^2)^4} \cdot F(2, \frac{5}{2}; \frac{7}{2}; z^2)$
a_3^n	$\frac{8}{5} \cdot \frac{(pq)^3}{(p^2+q^2)^4} \cdot F(2, \frac{5}{2}; \frac{9}{2}; z^2)$	$\frac{64}{5} \cdot \frac{(pq)^3}{(p^2+q^2)^5} \cdot F(\frac{5}{2}, 3; \frac{9}{2}; z^2)$
a_4^n	$\frac{64}{35} \cdot \frac{(pq)^4}{(p^2+q^2)^5} \cdot F(\frac{5}{2}, 3; \frac{11}{2}; z^2)$	$\frac{108}{7} \cdot \frac{(pq)^4}{(p^2+q^2)^6} \cdot F(3, \frac{7}{2}; \frac{11}{2}; z^2)$

and $z = (2pq)/(p^2 + q^2)$.

3 Conclusion

In the paper relativistic covariant equations for quarkonia in the bilocal approach are obtained. These equations can be used to find the solutions for the bound state functions for any given angular momentum. For example, we considered the expressions for pseudoscalar mesons.

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