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дубна

E4-91-500

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QUARK-ANTIQUARK BOUND STATES

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1991

1 Introduction

One of the major outstanding problems in theoretical elementary particle physics is the question of the origin of confinement. As long as there is no satisfactory solution to this problem which allows us to calculate the properties of bound states of coloured constituents from first principles, that is—as has been nowadays generally accepted—quantum chromodynamics, one has to rely on somewhat more or less phenomenological approaches to bound states.

In this rather condensed survey we would like to have a quick look at some aspects of the treatment of quark–antiquark bound states from various points of view. Particular emphasis will be laid on the interrelationship of nonrelativistic and (at least semi-) relativistic approaches.

This brief review is organized as follows. In Section 2 we discuss the significance of the description of hadrons as bound states of (constituent) quarks by nonrelativistic potential models. To this end we derive the relativistic generalization of the quantum-mechanical virial theorem and use it to clarify the connection between the nonrelativistic and (semi-) relativistic treatment of bound states.

In Section 3 we present a new approach to the fermion–antifermion bound-state problem. This approach is based on the construction of an effective Hamiltonian which incorporates relativistic kinematics as well as an interaction potential, the perturbative part of which is derived from the quantum field theory describing the truly fundamental interaction between the bound-state constitutents.

We adopt this effective-Hamiltonian method in Section 4 for the description of hadrons as bound states of constituent quarks. However, we do not intend to attempt by the given prescription a satisfactory numerical fit of the experimentally observed hadron spectra. Rather, we try to check the proposed procedure on a very basic level, namely, by application to some selected questions, where we can obtain the solution to the resulting equation of motion by analytical computation. We shall find that our effective-Hamiltonian method is able to reproduce some general features of the empirical meson spectrum.

Finally, in Section 5 we comment on the relationship between the effective-Hamiltonian method and the Bethe–Salpeter formalism.

2 Reliability of Nonrelativistic Potential Models

An amazing circumstance in hadron spectroscopy is the description of hadrons as bound states of quarks by nonrelativistic [1,2] and (semi-) relativistic [3,4] potential models at an equally good level. The relativistic version of the quantum-mechanical virial theorem [5,6] may provide a tool to clarify this situation.

The static inter-quark potential $V(r)$, $r \equiv |\vec{x}|$, has to consist of at least two parts: At short distances it originates from one-gluon exchange (which gives an approximately Coulomb-like contribution),

$$\lim_{r\to 0} V(r) \simeq -\frac{4}{3}\frac{\alpha_s}{r} \quad , \qquad \alpha_s \equiv \frac{g_s^2}{4\pi} \quad , \tag{1}$$

where $\frac{4}{3}$ is the appropriate colour factor for mesons. At large distances it has to provide for confinement, $V(r) \to \infty$ for $r \to \infty$. Lattice gauge theories indicate that the rise of the potential is approximately linear,

$$\lim_{r\to\infty} V(r) \simeq ar \quad . \tag{2}$$

The superposition of these two parts is the funnel (or Cornell) potential [7,8,9]

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + ar \quad . \tag{3}$$

This form (Fig. 1) represents the prototype of all realistic "QCD-inspired" potential models.

Let us try to get an idea of the order of magnitude of the parameters α_s and a entering in the above potential. Our main assumption will be that light quarks feel predominantly the linear part of the potential, i.e., $V_{\text{eff}} = ar$.

From the scaling behaviour of the Schrödinger equation one obtains for the energy levels of the bound state[1]

$$E_n = \left(\frac{a^2}{2\mu}\right)^{\frac{1}{3}} \cdot \epsilon_n \quad \text{for} \quad V = ar \quad , \tag{4}$$

where μ is the reduced mass of the two-particle system and $-\epsilon_n$ are the zeros of the Airy function, $\epsilon_1 = 2.34, \epsilon_2 = 4.09, \ldots$. Thus the mass

[1]For a brief introduction to the scaled Schrödinger equation see, for instance, Ref. [2].

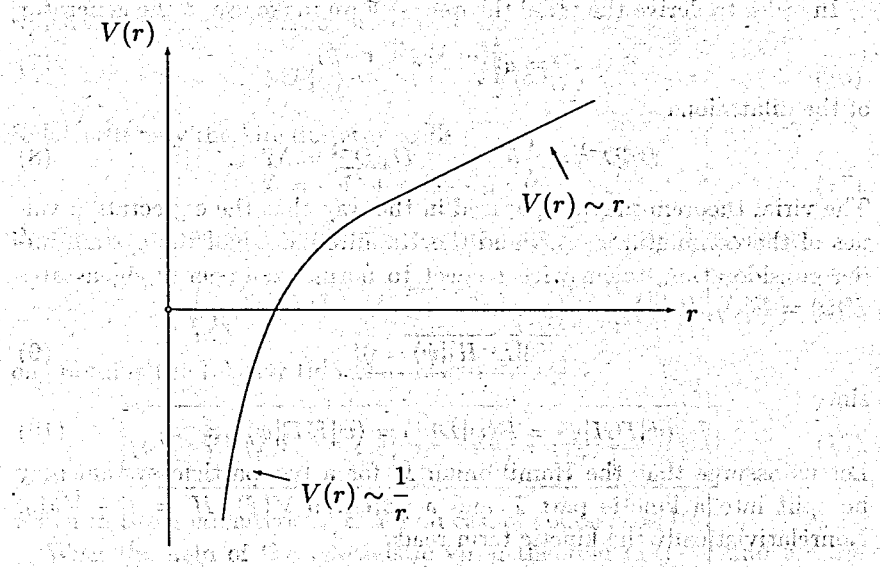

Figure 1: Funnel potential

difference between ground state and first radial excitation is given by

$$M' - M = \left(\frac{a^2}{2\mu}\right)^{\frac{1}{3}} \cdot (\epsilon_2 - \epsilon_1) = 1.75\left(\frac{a^2}{2\mu}\right)^{\frac{1}{3}} \quad . \tag{5}$$

With the constituent quark masses $m_u = m_d = 0.34$ GeV one finds from $M_{\pi(1300)} - M_\pi$ and $M_{\rho(1700)} - M_\rho$ for the slope of the linear potential $a = 0.27$ GeV^2.

From the spin-spin interaction entering in the generalized Breit–Fermi Hamiltonian one obtains under the above assumption for the mass-squared difference between corresponding spin-singlet and spin-triplet states[2]

$$M_{1^-}^2 - M_{0^-}^2 \simeq \frac{32}{9}\alpha_s a \quad . \tag{6}$$

From $M_{\rho^2} - M_{\pi^2}$ and $M_{K^*}^2 - M_K^2$ one gets $\alpha_s \simeq 0.6$ for the strong fine structure constant at an energy scale corresponding to light hadrons.

[2]For a brief introduction to the Breit–Fermi Hamiltonian see, for instance, Ref. [2].

In order to derive the virial theorem [10] we make use of the generator

$$D = e^{\frac{i}{2}(\ln\lambda)(\vec{p}\vec{x}+\vec{x}\vec{p})} \tag{7}$$

of the dilatations

$$D\vec{p}D^{-1} = \frac{1}{\lambda}\vec{p} \quad , \qquad D\vec{x}D^{-1} = \lambda\vec{x} \quad . \tag{8}$$

The virial theorem may be phrased in the way that the expectation values of the commutator of D and the Hamiltonian H of the system under consideration, taken with respect to normalized energy eigenstates $H|\psi\rangle = E|\psi\rangle$, $\langle\psi|\psi\rangle = 1$, vanish:

$$\boxed{\langle\psi|[D,H]|\psi\rangle = 0} \quad , \tag{9}$$

since

$$\langle\psi|DH|\psi\rangle = E\langle\psi|D|\psi\rangle = \langle\psi|HD|\psi\rangle \quad . \tag{10}$$

Let us assume that the Hamiltonian H for a two-particle system may be split into a kinetic part T and a potential $V(\vec{x})$, $H = T + V(\vec{x})$. Nonrelativistically the kinetic term reads

$$T = \frac{\vec{p}^{\,2}}{2\mu} \quad , \tag{11}$$

where $\mu = m_1 m_2/(m_1+m_2)$ denotes the reduced mass of the bound-state constituents with masses m_1 and m_2, respectively. Under the dilatations (8) the nonrelativistic Hamiltonian behaves like

$$DHD^{-1} = \frac{1}{\lambda^2}T + V(\lambda\vec{x}) \quad . \tag{12}$$

With the expansions in powers of $\ln\lambda$

$$\frac{1}{\lambda^2} \simeq 1 - 2\ln\lambda \tag{13}$$

and

$$V(\lambda\vec{x}) \simeq V(\vec{x}) + (\ln\lambda)\vec{x}\cdot\vec{\nabla}V(\vec{x}) \tag{14}$$

we compute the first derivative of $\langle\psi|HD|\psi\rangle = 0$ with respect to $\ln\lambda$ and obtain

$$\begin{aligned} &\lim_{\lambda\to 1}\frac{1}{\ln\lambda}\langle\psi|(DHD^{-1}-H)D|\psi\rangle \\ &= \lim_{\lambda\to 1}\langle\psi|[-2T+\vec{x}\cdot\vec{\nabla}V(\vec{x})]D|\psi\rangle \\ &= \langle\psi|[-2T+\vec{x}\cdot\vec{\nabla}V(\vec{x})]|\psi\rangle = 0 \quad , \end{aligned} \tag{15}$$

which is the nonrelativistic virial theorem:

$$\langle\psi|T|\psi\rangle = \frac{1}{2}\langle\psi|\vec{x}\cdot\vec{\nabla}V(\vec{x})|\psi\rangle \quad . \tag{16}$$

Relativistically the kinetic term reads

$$T = \sqrt{\vec{p}^{\,2}+m_1^2} + \sqrt{\vec{p}^{\,2}+m_2^2} \quad . \tag{17}$$

Performing the same steps as above and noticing the expansion

$$\sqrt{\frac{1}{\lambda^2}\vec{p}^{\,2}+m^2} \simeq \sqrt{\vec{p}^{\,2}+m^2} - \frac{\vec{p}^{\,2}}{\sqrt{\vec{p}^{\,2}+m^2}}\ln\lambda \tag{18}$$

one obtains the relativistic virial theorem [5,6]

$$\boxed{\langle\psi|\vec{x}\cdot\vec{\nabla}V(\vec{x})|\psi\rangle = \left\langle\psi\left|\frac{\vec{p}^{\,2}}{\sqrt{\vec{p}^{\,2}+m_1^2}} + \frac{\vec{p}^{\,2}}{\sqrt{\vec{p}^{\,2}+m_2^2}}\right|\psi\right\rangle} \quad , \tag{19}$$

which in the nonrelativistic case reduces, of course, to (16).

With the help of the relativistic virial theorem (19) we find for the relativistic energy eigenvalues

$$E \equiv \langle H\rangle = \langle\vec{x}\cdot\vec{\nabla}V\rangle + \langle V\rangle + \left\langle\frac{m_1^2}{\sqrt{\vec{p}^{\,2}+m_1^2}} + \frac{m_2^2}{\sqrt{\vec{p}^{\,2}+m_2^2}}\right\rangle \quad . \tag{20}$$

For the funnel potential (3), however, $rV'+V = 2ar$. Despite the presence of the Coulomb term in the potential it drops out in the expression for the energy! Nonrelativistically, the energy is given by

$$E = \frac{1}{2}\langle rV'(r)\rangle + \langle V(r)\rangle \quad . \tag{21}$$

In contrast to the relativistic case, due to the factor $\frac{1}{2}$ in front of $\langle rV'\rangle$ in the above relation, the cancellation of the Coulomb part is incomplete in the nonrelativistic version. Consequently, to the extend that the third term on the right-hand side of Eq. (20) may be neglected, the expectation value of the nonrelativistic Hamiltonian with a purely linear potential $V = ar$ is of formal resemblance to the expectation value of the relativistic Hamiltonian with the funnel potential (3). The corresponding eigenstates are, of course, different. Nevertheless, the above similarity may lead one astray to treat bound states of light constituents nonrelativistically, by employing only a linear potential.

There are some further, but less rigorous, hints why the description of bound states by the nonrelativistic Schrödinger formalism might not be complete nonsense.

First of all, according to $|\langle\mathcal{O}\rangle| \leq \sqrt{\langle\mathcal{O}^2\rangle}$ valid for any hermitian operator $\mathcal{O}$, the relativistic kinetic energy satisfies $\langle\sqrt{\vec{p}^2+m^2}\rangle \leq \sqrt{\langle\vec{p}^2\rangle+m^2}$. From this one finds for the expectation value of H

$$\begin{aligned}\langle H\rangle &= 2\left\langle\sqrt{\vec{p}^2+m^2}\right\rangle + \langle V\rangle \leq 2\sqrt{\langle\vec{p}^2\rangle+m^2} + \langle V\rangle \\ &= 2\frac{\langle\vec{p}^2\rangle+m^2}{\sqrt{\langle\vec{p}^2\rangle+m^2}} + \langle V\rangle = \left\langle 2\frac{\vec{p}^2+m^2}{\sqrt{\langle\vec{p}^2\rangle+m^2}} + V\right\rangle \quad . \end{aligned} \tag{22}$$

Consequently, the relativistic energy eigenvalue $E \equiv \langle H\rangle$ is bounded from above by

$$E \leq \left\langle 2\frac{\vec{p}^2+m^2}{\sqrt{\langle\vec{p}^2\rangle+m^2}} + V\right\rangle \quad . \tag{23}$$

The operator on the right-hand side of this inequality is formally of the same structure as the nonrelativistic Schrödinger Hamiltonian

$$H_{\rm NR} = 2\hat{m} + \frac{\vec{p}^2}{\hat{m}} + V_{\rm NR} \tag{24}$$

with, however, an effective mass $\hat{m} = \frac{1}{2}\sqrt{\langle\vec{p}^2\rangle+m^2}$ and the nonrelativistic potential

$$V_{\rm NR} = \frac{2m^2}{\sqrt{\langle\vec{p}^2\rangle+m^2}} - \sqrt{\langle\vec{p}^2\rangle+m^2} + V = 2\hat{m} - \frac{\langle\vec{p}^2\rangle}{\hat{m}} + V \quad . \tag{25}$$

The effective mass $\hat{m}$ as well as the constant in the potential $V_{\rm NR}$ depend on the average momentum $\langle\vec{p}^2\rangle$ and will thus vary from level to level. The expectation value of the kinetic energy is related by the nonrelativistic virial theorem (16) to $\langle r\frac{d}{dr}V(r)\rangle$. The latter is a constant only for the exceptional case of a logarithmic potential $\propto \ln(r/r_0)$ [11]. Only in this case $\langle\vec{p}^2\rangle$ is independent of the level of excitation.

Secondly, there exists a certain kind of duality between an ultrarelativistic Hamiltonian with harmonic-oscillator potential and a nonrelativistic Hamiltonian with linear potential [12]: The ultra-relativistic Hamiltonian $H_{\rm UR} = 2\sqrt{\vec{p}^2} + \kappa r^2$ is converted into the nonrelativistic Hamiltonian $H_{\rm NR} = \vec{p}^2/m + ar$ by means of the duality transformation $|\vec{p}| \to ar/2, r \to 2|\vec{p}|/a$, provided the parameters κ, a, and m are related by $\kappa = a^2/(4m)$. The eigenfunctions in the corresponding wave equations, $H_{\rm UR}\phi(\vec{x}) = E\phi(\vec{x})$ and $H_{\rm NR}\psi(\vec{y}) = E\psi(\vec{y})$, respectively, are then connected by the Fourier transformation $\phi(\vec{x}) = \int d^3y\, \exp(i\frac{a}{2}\vec{x}\cdot\vec{y})\,\psi(\vec{y})$.

3 Relativistic Description of Fermion–Antifermion Bound States

3.1 The Effective Hamiltonian

Our aim is to describe bound states of a fermion–antifermion pair by an effective Hamiltonian [13] which incorporates the relativistically correct kinetic term $H_0(\vec{p}) = \sqrt{\vec{p}^2+m^2}$ as well as the complete relativistic corrections to the static potential. This potential should, of course, be derived from the underlying quantum field theory. In order to do this, we take advantage of the fact that in scattering theory the potential acting between two particles is nothing else but the Fourier transform of the first Born approximation T_{fi}^B to the transition amplitude for the elastic scattering of the involved particles [14,2]:

$$V(\vec{x}) \sim \int d^3k\, e^{-i\vec{k}\cdot\vec{x}}\, T_{fi}^B \quad , \tag{26}$$

where $\vec{k} \equiv \vec{p} - \vec{q}$ denotes the relevant momentum transfer, $\vec{p}$ and $\vec{q}$ being the relative momenta in initial and final state, respectively.

The proposed procedure consists therefore of two main steps [13]:

1. Compute the effective interaction potential $V(\vec{x})$ from the underlying quantum field theory via the scattering amplitude for elastic two-particle scattering.
2. Construct the effective Hamiltonian

$$H = H_0(\vec{p}) + H_0(-\vec{p}) + V(\vec{x}) \tag{27}$$

in order to determine the energy eigenvalues and respective state vectors of the bound state under consideration from the corresponding multi-particle Schrödinger equation.

The energy in the rest system of the bound state is, of course, nothing else but the mass of the composite particle. Obviously, the proposed method may be regarded as the relativistic generalization of the description of bound states by nonrelativistic potential models.

3.2 The Scattering Amplitude

In order to calculate the scattering amplitude required for the derivation of the effective interaction potential, it is sufficient to consider the elastic scattering of the involved fermion–antifermion pair in their center-of-momentum system:

$$f(\vec{p}) + \bar{f}(-\vec{p}) \longrightarrow f(\vec{q}) + \bar{f}(-\vec{q}) \quad . \tag{28}$$

Expressed in terms of the usual Dirac spinors $u(\vec{p})$ and $v(\vec{p})$, the general form of the scattering amplitude corresponding to this process reads

$$T \sim \bar{u}(\vec{q})\,\Gamma_1\,u(\vec{p})\;\bar{v}(-\vec{p})\,\Gamma_2\,v(-\vec{q})\;K \tag{29}$$

where Γ_i, $i = 1, 2$, represents some Dirac matrix, $\Gamma_i = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$. K is the interaction integral kernel, which has to be determined from the underlying quantum field theory. The Fourier transform of just this kernel yields the static interaction potential. As far as the spin structure of the above scattering amplitude is concerned, we shall only be interested in vector $\Gamma_1 \otimes \Gamma_2 = \gamma_\mu \otimes \gamma^\mu$ —corresponding to a (maybe only effective) exchange of a vector boson—and scalar $\Gamma_1 \otimes \Gamma_2 = 1 \otimes 1$ —corresponding to a (maybe only effective) exchange of a scalar boson—, which appear to be the dominant ones for the quark–antiquark interaction originating from quantum chromodynamics. (For a very recent review on the phenomenological aspects of the forces acting within bound states of quarks see, e.g., Ref. [2].)

In order to obtain the dependence of the scattering amplitude on the momenta $\vec{p}$ and $\vec{q}$, we insert an explicit representation of the Dirac spinors, for instance, the Dirac representation

$$\begin{aligned} u(\vec{p}) &\sim \begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{S_i} \end{pmatrix} \chi \quad , \\ v(\vec{p}) &\sim \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{S_i} \\ 1 \end{pmatrix} \chi^c \quad , \qquad \chi^c \equiv -i\sigma_2\chi^* \quad , \end{aligned} \tag{30}$$

where $\vec{\sigma}$ are the three Pauli matrices, χ is the two-component spinor corresponding to a given spin polarization, and we introduced for the denominators in Dirac spinors the shorthand notation $S \equiv \sqrt{\vec{p}^{\,2} + m^2} + m$.

Up to an overall normalization factor, the general form of the scattering amplitudes is then given for vectorial spin structure by [13,15]

$$\begin{aligned} T_V \sim \Big\{ 1 \;&+\; \frac{1}{S_1 S_3}(\vec{p}\cdot\vec{q} - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_1) \\ &+\; \frac{1}{S_2 S_4}(\vec{p}\cdot\vec{q} - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_2) \\ &+\; \frac{1}{S_1 S_2}\left[\vec{p}^{\,2}\,(1 - \vec{\sigma}_1\cdot\vec{\sigma}_2) + (\vec{p}\cdot\vec{\sigma}_1)(\vec{p}\cdot\vec{\sigma}_2)\right] \\ &+\; \frac{1}{S_3 S_4}\left[\vec{q}^{\,2}\,(1 - \vec{\sigma}_1\cdot\vec{\sigma}_2) + (\vec{q}\cdot\vec{\sigma}_1)(\vec{q}\cdot\vec{\sigma}_2)\right] \\ &+\; \frac{1}{S_1 S_4}\left[\vec{p}\cdot\vec{q}\,(1 + \vec{\sigma}_1\cdot\vec{\sigma}_2)\right. \\ &\qquad \left. - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_+ - (\vec{q}\cdot\vec{\sigma}_1)(\vec{p}\cdot\vec{\sigma}_2)\right] \\ &+\; \frac{1}{S_2 S_3}\left[\vec{p}\cdot\vec{q}\,(1 + \vec{\sigma}_1\cdot\vec{\sigma}_2)\right. \\ &\qquad \left. - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_+ - (\vec{p}\cdot\vec{\sigma}_1)(\vec{q}\cdot\vec{\sigma}_2)\right] \\ &+\; \frac{1}{S_1 S_2 S_3 S_4}\left[(\vec{p}\cdot\vec{q})^2 - i(\vec{p}\cdot\vec{q})(\vec{p}\times\vec{q}\cdot\vec{\sigma}_+)\right. \\ &\qquad \left. - (\vec{p}\times\vec{q}\cdot\vec{\sigma}_1)(\vec{p}\times\vec{q}\cdot\vec{\sigma}_2)\right] \Big\}\, K_V \end{aligned} \tag{31}$$

and for scalar spin structure by [13,15]

$$\begin{aligned} T_S \sim \Big\{ 1 \;&-\; \frac{1}{S_1 S_3}(\vec{p}\cdot\vec{q} - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_1) \\ &-\; \frac{1}{S_2 S_4}(\vec{p}\cdot\vec{q} - i\,\vec{p}\times\vec{q}\cdot\vec{\sigma}_2) \\ &+\; \frac{1}{S_1 S_2 S_3 S_4}\left[(\vec{p}\cdot\vec{q})^2 - i(\vec{p}\cdot\vec{q})(\vec{p}\times\vec{q}\cdot\vec{\sigma}_+)\right. \\ &\qquad \left. - (\vec{p}\times\vec{q}\cdot\vec{\sigma}_1)(\vec{p}\times\vec{q}\cdot\vec{\sigma}_2)\right] \Big\}\, K_S \end{aligned} \tag{32}$$

where $\vec{\sigma}_i$ is understood to act on particle i, $i = 1, 2$, and $\vec{\sigma}_+$ is the sum of $\vec{\sigma}_1$ and $\vec{\sigma}_2$, $\vec{\sigma}_+ \equiv \vec{\sigma}_1 + \vec{\sigma}_2$.

In a nonrelativistic expansion of the above scattering amplitudes up to order $1/c^2$ one recovers, of course, the well-known Breit–Fermi interaction.

3.3 The Massless Case

The case of massless—and thus ultra-relativistically moving—bound-state constituents, i.e., $m_i = 0$, entails a considerable simplification of the scattering amplitudes, since under these circumstances the T-matrix elements only depend on the unit vectors

$$\hat{p} \equiv \frac{\vec{p}}{\sqrt{\vec{p}^2}} \quad , \qquad \hat{q} \equiv \frac{\vec{q}}{\sqrt{\vec{q}^2}} \tag{33}$$

as well as on their difference

$$\vec{k} \equiv \hat{p} - \hat{q} \quad . \tag{34}$$

Explicitly, the scattering amplitudes read for vectorial spin structure [13]

$$\begin{aligned} T_V \;\sim\; & [(3 + \hat{p}\cdot\hat{q})(1 + \hat{p}\cdot\hat{q} - i\,\hat{p}\times\hat{q}\cdot\vec{\sigma}_+) \\ & - \vec{k}^2\,\vec{\sigma}_1\cdot\vec{\sigma}_2 + (\vec{k}\cdot\vec{\sigma}_1)(\vec{k}\cdot\vec{\sigma}_2) \\ & - (\hat{p}\times\hat{q}\cdot\vec{\sigma}_1)(\hat{p}\times\hat{q}\cdot\vec{\sigma}_2)\,]\,K_V \end{aligned} \tag{35}$$

and for scalar spin structure [13]

$$\begin{aligned} T_S \;\sim\; & -\Big[\frac{1}{4}\,\vec{k}^2(\vec{k}^2 + 2\,i\,\hat{p}\times\hat{q}\cdot\vec{\sigma}_+) \\ & - (\hat{p}\times\hat{q}\cdot\vec{\sigma}_1)(\hat{p}\times\hat{q}\cdot\vec{\sigma}_2)\Big]\,K_S \quad . \end{aligned} \tag{36}$$

In contrast to the general case given in Subsection 3.2, this simplified form of the scattering amplitudes allows for an analytical treatment of the bound states under consideration. Accordingly, for the applications of the developed formalism in Section 4, we shall assume the constituents to be massless particles.

3.4 The Variational Method

In order to get an estimate for the energy eigenvalue E we employ a simple variational technique, where we compute the expectation value of the Hamiltonan H with respect to some suitably chosen trial states $|\psi(\lambda)\rangle$ depending on a variational parameter λ,

$$E(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle \quad , \tag{37}$$

and minimize the resulting expression $E(\lambda)$ with respect to λ,

$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda_{\min}} = 0 \quad \rightsquigarrow \quad \lambda_{\min} \quad . \tag{38}$$

The minimum $E(\lambda_{\min})$ found in this way provides, of course, only an upper bound to the proper energy eigenvalue E,

$$E \leq E(\lambda_{\min}) \quad . \tag{39}$$

For reasonable potentials, however, it has been shown that it yields a fairly good approximation to the exact energy.

As our trial functions we shall use Gaussian wave functions,

$$\begin{aligned} \psi_{\ell m}(\vec{x}) &\sim r^\ell\,\exp\left(-\frac{\lambda^2 r^2}{2}\right)\,\mathcal{Y}_{\ell m} \quad , \\ \tilde{\psi}_{\ell m}(\vec{p}) &\sim p^\ell\,\exp\left(-\frac{p^2}{2\lambda^2}\right)\,\mathcal{Y}_{\ell m} \quad , \end{aligned} \tag{40}$$

as well as Hydrogen-like wave functions,

$$\begin{aligned} \psi_{\ell m}(\vec{x}) &\sim r^\ell\,\exp(-\lambda r)\,\mathcal{Y}_{\ell m} \quad , \\ \tilde{\psi}_{\ell m}(\vec{p}) &\sim \frac{p^\ell}{(p^2+\lambda^2)^{\ell+2}}\,\mathcal{Y}_{\ell m} \quad . \end{aligned} \tag{41}$$

Here, $\mathcal{Y}_{\ell m}(\theta,\phi)$ are the spherical harmonics for angular momentum ℓ and projection m.

4 Applications

Let us try to use the formalism developed in the preceding section for the description of mesons as bound states of a (constituent) quark–antiquark pair. It should be clear from the previous discussion that the form of the relevant interaction potential is fixed by the integral kernels K_V, K_S, which have to be determined from the underlying theory.

As is well known, for the strong interaction the potential consists of a short-range and a long-range part.

The short-range part is of perturbative origin—where, since we are only interested in the Born approximation, we only have to deal with lowest order perturbation theory. In gauge theories this contribution is generated by one-gauge-boson exchange and is consequently of vectorial spin structure, with corresponding interaction kernel

$$K_V = -\frac{\kappa}{(p-q)^2} \tag{42}$$

The parameter κ is given, for instance, in quantum electrodynamics from one-photon exchange by $\kappa = Q_f\, Q_{\bar{f}}\, e^2$, and in quantum chromodynamics from one-gluon exchange between colour-singlet states by $\kappa = \frac{4}{3}\, g_s^2$.

The long-range part is of nonperturbative origin and has to rise to infinity for large inter-quark distances r in order to be able to describe confinement: $V_{\rm np}(r) \longrightarrow \infty$ for $r \longrightarrow \infty$. From lattice gauge theories there are hints that this rise is a linear one and that this contribution is of scalar spin structure:

$$V_{\rm np}(r) = a\, r \quad . \tag{43}$$

Under the above assumptions the expectation value of the Hamiltonian (27) is given by [13]

$$\begin{aligned} E_{\vec{0}} \;&=\; \int d^3p\, |\tilde{\psi}(\vec{p})|^2\, [H_{0,1}(\vec{p}) + H_{0,2}(-\vec{p})] \\ &-\; (2\pi)^3 \int d^3p\, d^3q\, \tilde{\psi}^*(\vec{q})\, T^B_{fi}\, \tilde{\psi}(\vec{p}) \\ &+\; \int d^3x\, |\psi(\vec{x})|^2\, V_{\rm np}(\vec{x}) \quad . \end{aligned} \tag{44}$$

In the following we will investigate the question whether or not the above model is able to reproduce some simple features of the meson spectrum.

4.1 Singlet–Triplet Mass Differences

Empirically, the differences of the squared masses of corresponding spin-singlet and spin-triplet partners which contain at least one light quark (u, d, s) are constant.

Table 1: Differences of the squared masses of spin-singlet and spin-triplet partners [16]

Spin triplet	Spin singlet	$M^2_{S=1} - M^2_{S=0}$ [GeV^2]
ρ	π	0.57
K^*	K	0.55
D^*	D	0.55
D_s^*	D_s	0.58
B^*	B	0.56

From Table 1 one finds for the average of all pairs of mesons:

$$M^2_{S=1} - M^2_{S=0} \simeq (0.56 \pm 0.01)\ \text{GeV}^2 \quad . \tag{45}$$

The Fourier transform of $\vec{p}\times\vec{q}$ is the relative orbital angular momentum of the two bound-state constituents. Accordingly, all terms which involve this expression do not contribute to ground states (with $\ell = 0$).

Consequently, the bound-state energy (44) is given by [13]

$$\begin{aligned} E_{\vec{0}} \;&=\; 2 \int d^3p\, \left|\tilde{\psi}(p)\right|^2\, p \\ &-\; \frac{1}{(2\pi)^3}\frac{\kappa}{4} \left| \int_0^\infty dp\, p\, \tilde{\psi}(p) \right|^2 \left[\Omega^2 - \frac{2}{3}(4\pi)^2 \langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle\right] \\ &+\; a \int d^3x\, |\psi(r)|^2\, r \quad , \end{aligned} \tag{46}$$

with

$$\Omega^2 \equiv \int d\Omega_p\, d\Omega_q\, \frac{1}{\vec{k}^2}\, (3 + \hat{p}\cdot\hat{q})(1 + \hat{p}\cdot\hat{q}) \quad . \tag{47}$$

The spin expectation value $\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle$ depends on the total spin S of the two-fermion state:

$$\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle = \begin{cases} -3 & \text{for spin singlets, } S = 0 \\ +1 & \text{for spin triplets, } S = 1 \end{cases} \quad . \tag{48}$$

Applying the variational technique of Subsection 3.4 to the above expression for the energy yields for the mass-squared difference we are looking for [13]

$$M^2_{S=1} - M^2_{S=0} \simeq \begin{cases} \dfrac{32}{3\pi^3}\,\kappa a & \text{for Gaussian trial functions} \\ \dfrac{16}{\pi^3}\,\kappa a & \text{for Hydrogen-like trial functions} \end{cases} . \quad (49)$$

These expressions have to be compared with the result [2] obtained in the nonrelativistic case on the grounds of the instantaneous-limit approximation to the interaction kernel (42),

$$M^2_{S=1} - M^2_{S=0} \simeq \frac{2}{3\pi}\,\kappa a \quad . \quad (50)$$

Obviously, all predictions for the mass-squared differences are independent of the mass of the particles which constitute the bound state. However, in the nonrelativistic case this mass independence follows from the neglect of terms of higher order in the inverse masses of the components [17] and the assumption that light constituents will be mainly affected by the linear part of the potential. In contrast to that, in the ultrarelativistic case this mass independence is enforced by the assumption of vanishing masses of the bound-state constituents. Since in this case there is no other dimensional parameter than the slope a of the linear potential, any quantity of dimension mass-squared has to be proportional to this slope.

4.2 Linear Regge Trajectories

It is a well-known experimental fact that hadrons populate linear Regge trajectories, that is, the square of the mass of a state with orbital angular momentum ℓ is proportional to ℓ:

$$M^2(\ell) = \beta\,\ell + \text{const} \quad , \quad (51)$$

with the same slope $\beta \simeq 1.2$ GeV^2 for all trajectories. This feature of the hadron spectrum is nicely illustrated in Fig. 2.

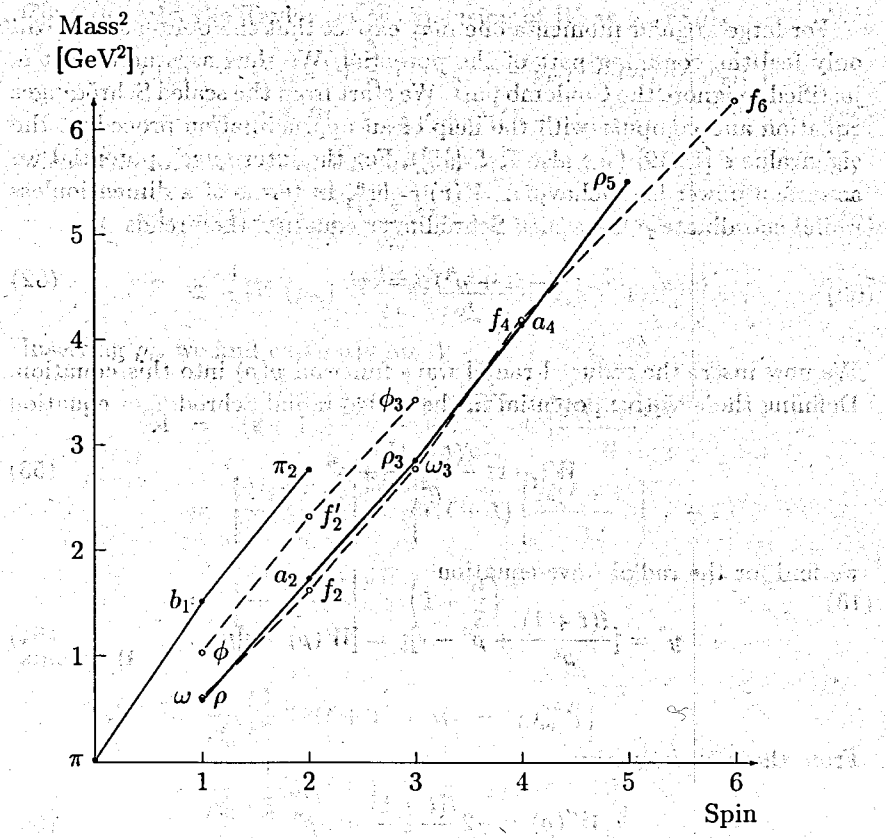

Figure 2: Lowest-lying experimental Regge trajectories for non-strange mesons [16]

4.3 Nonrelativistic Approach to Regge Trajectories

We now ask ourselves whether or not it is possible to find in the framework of nonrelativistic potential models a potential which reproduces the linear behaviour of Regge trajectories mentioned in the preceding subsection. As we will see below, the answer is yes.

For large angular momenta one may expect that the bound states will only feel the confining part of the potential. We thus assume that it is justified to ignore the Coulomb part. We start from the scaled Schrödinger equation and compute with the help of an approximation procedure the eigenvalue ϵ [18,19] (see also Ref. [10]). For the inter-quark potential we assume a power-law behaviour, $V(r) = br^n$. In terms of a dimensionless radial coordinate ρ the scaled Schrödinger equation then reads

$$(-\Delta + \rho^n)\psi = \epsilon\psi \quad . \tag{52}$$

We now insert the reduced radial wave function $y(\rho)$ into this equation. Defining the effective potential in the scaled radial Schrödinger equation

$$W(\rho) \equiv \frac{\ell(\ell+1)}{\rho^2} + \rho^n \quad , \tag{53}$$

we find for the radial wave equation

$$y'' = [\frac{\ell(\ell+1)}{\rho^2} + \rho^n - \epsilon]y = [W(\rho) - \epsilon]y \quad . \tag{54}$$

From the first derivative

$$W'(\rho) = -2\frac{\ell(\ell+1)}{\rho^3} + n\rho^{n-1} \, . \tag{55}$$

we determine the minimum of W,

$$W'(\rho_m) = 0 \quad , \qquad \rho_m = \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{1}{2+n}} \quad , \tag{56}$$

and approximate W near this minimum ρ_m by a parabola. The second derivative of W is

$$W''(\rho) = 6\frac{\ell(\ell+1)}{\rho^4} + n(n-1)\rho^{n-2} \quad . \tag{57}$$

Consequently, the Taylor series expansion of W at ρ_m reads

$$\begin{aligned} W(\rho) &= W(\rho_m) + \frac{1}{2}W''(\rho_m)(\rho-\rho_m)^2 + O((\rho-\rho_m)^3) \\ &= A + B(\rho-\rho_m)^2 + O((\rho-\rho_m)^3) \quad , \end{aligned} \tag{58}$$

where the constants A and B are defined according to

$$A \equiv W(\rho_m) = \frac{\ell(\ell+1)}{\rho_m^2} + \rho_m^n \tag{59}$$

$$B \equiv \frac{1}{2}W''(\rho_m) = \frac{1}{2}\left[6\frac{\ell(\ell+1)}{\rho_m^4} + n(n-1)\rho_m^{n-2}\right] \quad . \tag{60}$$

Inserting ρ_m we find explicitly for A

$$\begin{aligned} A &= \ell(\ell+1)\left[\frac{2\ell(\ell+1)}{n}\right]^{-\frac{2}{n+2}} + \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \left\{\ell(\ell+1)\left[\frac{2\ell(\ell+1)}{n}\right]^{-1} + 1\right\} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \left(1+\frac{n}{2}\right) \end{aligned} \tag{61}$$

and for B

$$\begin{aligned} \sqrt{B} &= \frac{1}{\rho_m^2}\left\{\frac{1}{2}[6\ell(\ell+1) + n(n-1)\rho_m^{n+2}]\right\}^{1/2} \\ &= \frac{1}{\rho_m^2}\left\{\frac{1}{2}[6\ell(\ell+1) + (n-1)2\ell(\ell+1)]\right\}^{1/2} \\ &= \frac{1}{\rho_m^2}\left[2\ell(\ell+1)\left(1+\frac{n}{2}\right)\right]^{1/2} = \frac{1}{\rho_m^2}\frac{2\ell(\ell+1)\left(1+\frac{n}{2}\right)}{[\ell(\ell+1)(n+2)]^{1/2}} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{-\frac{2}{n+2}} \frac{2\ell(\ell+1)\left(1+\frac{n}{2}\right)}{[\ell(\ell+1)(n+2)]^{1/2}} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \left[\frac{2\ell(\ell+1)}{n}\right]^{-1} \frac{2\ell(\ell+1)\left(1+\frac{n}{2}\right)}{[\ell(\ell+1)(n+2)]^{1/2}} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \frac{n(1+\frac{n}{2})}{[\ell(\ell+1)(n+2)]^{1/2}} \\ &= A \cdot \frac{n}{[\ell(\ell+1)(n+2)]^{1/2}} \end{aligned} \tag{62}$$

Abbreviating $\tilde{\epsilon} \equiv \epsilon - A$ our differential equation thus takes the form

$$y'' = [B(\rho - \rho_m)^2 - \tilde{\epsilon}]y \quad . \tag{63}$$

This is obviously a one-dimensional harmonic oscillator. The corresponding eigenvalue is

$$\tilde{\epsilon} = 2\sqrt{B}\left(N + \frac{1}{2}\right) \quad . \tag{64}$$

Hence the eigenvalue of the scaled Schrödinger equation is given by

$$\begin{aligned} \epsilon &= \tilde{\epsilon} + A = A + 2\sqrt{B}\left(N + \frac{1}{2}\right) \\ &= A\left\{1 + \frac{n(2N+1)}{[\ell(\ell+1)(n+2)]^{1/2}}\right\} \\ &= \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}}\left(1+\frac{n}{2}\right)\left\{1 + \frac{n(2N+1)}{[\ell(\ell+1)(n+2)]^{1/2}}\right\} \quad . \end{aligned} \tag{65}$$

Scaling back we obtain for the energy

$$E = \left(\frac{b^2}{(2\mu)^n}\right)^{\frac{1}{2+n}} \cdot \epsilon \quad , \tag{66}$$

and for the mass of the bound state

$$\begin{aligned} M &= m_1 + m_2 \; + \; E \\ &= m_1 + m_2 \; + \; \left(\frac{b^2}{(2\mu)^n}\right)^{\frac{1}{2+n}} \left[\frac{2\ell(\ell+1)}{n}\right]^{\frac{n}{n+2}} \left(1+\frac{n}{2}\right) \\ &\quad \times \left\{1 + \frac{n(2N+1)}{[\ell(\ell+1)(n+2)]^{1/2}}\right\} \quad . \end{aligned} \tag{67}$$

We now determine that value of n which yields linear Regge trajectories. To this end we consider (67) for large ℓ. The leading term is the one containing ℓ^2. Hence

$$M \simeq \left(\frac{b^2}{(2\mu)^n}\right)^{\frac{1}{2+n}} \left(1+\frac{n}{2}\right)\left(\frac{2}{n}\right)^{\frac{n}{n+2}} (\ell)^{\frac{2n}{n+2}} + \text{const} \quad . \tag{68}$$

In order to get $M \propto \sqrt{\ell}$, n has to satisfy $\frac{2n}{n+2} = \frac{1}{2}$ and is thus fixed to $n = \frac{2}{3}$. Consequently, we find for large ℓ

$$M^2 \simeq 16\sqrt{\frac{b^3}{54\mu}} \cdot \ell + \text{const} \quad . \tag{69}$$

In other words, a confining potential of the form $V(r) = br^{2/3}$ leads to the well-known relation

$$M^2 = \beta\ell + \text{const} \quad , \tag{70}$$

where

$$\beta = 16\sqrt{\frac{b^3}{54\mu}} \tag{71}$$

is the so-called Regge slope. Adding a Coulomb-like part to the potential one obtains indeed rather good predictions for the spectrum. We conclude that, when treated nonrelativistically, a linear potential $V(r) = ar$ does not lead to linear Regge trajectories.

4.4 Regge Trajectories in an Ultra-Relativistic Treatment of Fermion–Antifermion Bound States

Let us now look at the meson spectrum from the opposite, that is, the ultra-relativistic, point of view [20]. For simplicity, we only consider spin-singlet mesons, that is, the total spin of the bound-state constituents is equal to zero, $S = 0$. In this case the expectation value of the vectorial scattering amplitude, taken with respect to (S=0) states reduces to [20]

$$\langle T_V \rangle_{S=0} \sim 2\, K_V \quad . \tag{72}$$

Nevertheless, it is not possible to give an analytic expression for the resulting energy spectrum. However, with the help of the asymptotic expansion of the spherical harmonics $\mathcal{Y}_{\ell m}(\theta, \phi)$ for large values of ℓ [21],

$$\begin{aligned} \mathcal{Y}_{\ell m}(\theta,\phi) &\simeq \frac{\sqrt{\ell}}{\pi} \frac{\sqrt{\Gamma(\ell-m+1)\,\Gamma(\ell+m+1)}}{\Gamma(\ell+\frac{3}{2})} \\ &\quad \times \frac{\cos[(\ell+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{m\pi}{2}]}{\sqrt{\sin\theta}}\, e^{im\phi} \qquad \text{for} \quad \ell \longrightarrow \infty \quad , \end{aligned} \tag{73}$$

it can be shown that in the limit $\ell \longrightarrow \infty$ the perturbative contribution to the energy vanishes proportional to ℓ^{-2} for Gaussian trial functions or proportional to $\ell^{-5/2}$ for hydrogen-like trial functions [20]. The reasoning for this is as follows. Both our trial functions are of the form $\tilde{\psi}_{\ell m}(\vec{p}) = f(p)\,\mathcal{Y}_{\ell m}(\theta, \phi)$, differing only by the function $f(p)$ of the radial variable.

The radial integrations in the perturbative contribution to the energy give for Gaussian trial functions

$$\left[\int_0^\infty dp\, p\, f(p)\right]^2 = 2^{\ell+1}\,\frac{\Gamma(\frac{\ell}{2}+1)^2}{\Gamma(\ell+\frac{3}{2})}\,\lambda \qquad (74)$$

and for Hydrogen-like trial functions

$$\left[\int_0^\infty dp\, p\, f(p)\right]^2 = \frac{2^{4\ell+6}}{\pi}\,\frac{\Gamma(\frac{\ell}{2}+1)^4}{\Gamma(2\ell+3)}\,\lambda \quad . \qquad (75)$$

The angular integrations cannot be performed analytically. With the above asymptotic expansion, however, one may estimate the behaviour of the angular integral for large ℓ. It is easy to convince oneself by partial integration that an integral of the form

$$\int_0^\pi dx \cos\left[\left(\ell+\frac{1}{2}\right)x + a\right] f(x)$$

is of the order ℓ^{-1}:

$$\begin{aligned}&\int_0^\pi dx \cos\left[\left(\ell+\frac{1}{2}\right)x + a\right] f(x)\\ &= \frac{1}{\ell+\frac{1}{2}}\{(-1)^\ell \cos a\, f(\pi) - \sin a\, f(0)\\ &\qquad - \int_0^\pi dx \sin\left[\left(\ell+\frac{1}{2}\right)x + a\right]\frac{d}{dx}f(x)\} \quad , \end{aligned} \qquad (76)$$

provided the first derivative of the otherwise arbitrary, ℓ-independent function $f(x)$ exists. We thus find that the angular integral behaves like

$$\ell\,\frac{\Gamma(\ell-m+1)\,\Gamma(\ell+m+1)}{\Gamma(\ell+\frac{3}{2})^2} \times O\left(\ell^{-2}\right) \quad .$$

Consequently, with the help of Stirling's formula

$$\Gamma(x) \simeq \sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}} \qquad \text{for} \quad x \longrightarrow \infty \quad , \qquad (77)$$

the announced result follows:

$$\lim_{\ell\longrightarrow\infty} \int d^3p\, d^3q\, \tilde{\psi}^*(\vec{q})\, T^B_{fi}\, \tilde{\psi}(\vec{p}) = 0 \quad . \qquad (78)$$

Upon vanishing of the perturbative contribution, the bound-state energy E_ℓ corresponding to large orbital angular momentum ℓ is given by

$$E_\ell = 2\int d^3p\, |\tilde{\psi}_{\ell m}(\vec{p})|^2\, p + a\int d^3x\, |\psi_{\ell m}(\vec{x})|^2\, r \quad . \qquad (79)$$

Applying, as before, the variational technique of Subsection 3.4 yields for the large-ℓ dependence of the energy [20] (for both types of trial functions)

$$E_\ell \simeq 2\sqrt{2\,a}\,\frac{\Gamma(\ell+2)}{\Gamma(\ell+\frac{3}{2})} \quad . \qquad (80)$$

Recalling again Stirling's formula (77) in order to express the above gamma functions for large ℓ, we obtain from the ratio

$$\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)} \simeq \sqrt{x} \qquad (81)$$

indeed linear Regge trajectories, with slope $\beta = 8a$ [20]:

$$E_\ell^2 \simeq 8a\ell \qquad \text{for} \quad \ell \longrightarrow \infty \quad . \qquad (82)$$

Herein, the way in which the energy E_ℓ depends on a is no great surprise. Because of the lack of any dimensional parameter other than the slope a of the linear potential, any quantity of dimension mass-squared has to be proportional to a. From (82) the numerical value of a is about $a = 0.15\ \text{GeV}^2$.

The result (82) is the same as the one obtained either in the WKB approximation to a relativistic potential model based on the Klein–Gordon equation [22], or as a lower bound to the asymptotic ground-state energy in a mathematically rigorous discussion for a purely linear potential [23], or within the path-integral formalism when assuming the asymptotic large-area law for the Wilson loop [24] (see also Ref. [2]).

In summary, we investigated bound states of fermion–antifermion pairs by our effective-Hamiltonian method [13], which describes the interaction of the bound-state constituents by an effective potential but incorporates relativistic kinematics. Treating the constituents ultra-relativistically, we have shown that the behaviour of the bound-state masses for large angular momenta is exclusively determined by the non-perturbative contribution to the interaction potential [20]. For a linear rise of this part of the potential with increasing inter-quark distance one obtains an (asymptotically) linear dependence of the squared masses on the corresponding

orbital angular momentum [20]. This result is a consequence of the relativistic kinematics incorporated in this approach. In contrast to that, in the nonrelativistic approximation linear Regge trajectories require a confining potential rising like the inter-quark distance to the power $\frac{2}{3}$, as has been demonstrated in Subsection 4.3.

4.5 Nonrelativistic Versus Ultra-Relativistic Description of Regge Trajectories

In Subsection 4.3 we learned that within a nonrelativistic consideration one is able to obtain linear Regge trajectories for a potential which behaves like $V(r) \sim r^{2/3}$. We found that the nonrelativistic Hamiltonian

$$H_{\rm NR} = m_1 + m_2 + \frac{\vec{p}^{\,2}}{2\,\mu} + b\,r^{2/3} \quad , \tag{83}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the two-particle system, leads to linear Regge trajectories with slope

$$\beta_{\rm NR} = \sqrt{\frac{128\,b^3}{27\,\mu}} \quad . \tag{84}$$

In contrast to that, we showed in Subsection 4.4 that in the ultra-relativistic case a linear confining potential gives linear Regge trajectories with slope

$$\beta_{\rm UR} = 8\,a \quad . \tag{85}$$

We shall now demonstrate that, despite of the apparently different forms of the involved Hamiltonians, the resulting energy eigenvalues are almost identical, that is, not only in their slopes but also in their absolute values [25].

To this end, let us compare the binding energies $\varepsilon \equiv E - m_1 - m_2$ for nonrelativistic and ultrarelativistic limit, $\varepsilon_{\rm NR}$ and $\varepsilon_{\rm UR}$, respectively. We demand equality of the Regge slopes, $\beta_{\rm NR} = \beta_{\rm UR}$, since, if reasonable at all, they should describe one and the same physics. With the help of our variational procedure (with Gaussian trial functions) we then find for the ratio of these binding energies [25]

$$\frac{\varepsilon_{\rm NR}}{\varepsilon_{\rm UR}} = \frac{\left[\Gamma(\ell+\frac{5}{2})\,\Gamma(\ell+\frac{11}{6})^3\right]^{1/4}}{\Gamma(\ell+2)} \longrightarrow 1 \quad \text{for} \quad \ell \to \infty \quad . \tag{86}$$

As indicated, for $\ell \to \infty$ this ratio approaches unity.

From its derivative with respect to ℓ, the behaviour of this ratio for finite ℓ is controlled by the expression

$$\psi(\ell+\tfrac{5}{2}) + 3\,\psi(\ell+\tfrac{11}{6}) - 4\,\psi(\ell+2) \quad , \tag{87}$$

where $\psi(x) \equiv \Gamma'(x)/\Gamma(x)$ denotes the logarithmic derivative of the gamma function, the so-called "digamma function" [21]. Because of the concavity of $\psi(x)$,

$$\psi''(x) = -2\sum_{n=0}^{\infty}\frac{1}{(x+n)^3} < 0 \quad , \qquad \text{for} \quad x \neq 0, -1, -2, \ldots \quad , \tag{88}$$

this expression is always negative. Any function $f(x)$ with $f''(x) < 0$ satisfies the concavity condition

$$\rho\,f(x) + (1-\rho)\,f(y) \leq f(\rho x + (1-\rho)y) \quad . \tag{89}$$

In our case $x = \ell + \frac{5}{2}$, $y = \ell + \frac{11}{6}$, and $\rho = \frac{1}{4}$. The ratio $\varepsilon_{\rm NR}/\varepsilon_{\rm UR}$ is therefore a strictly monotonic decreasing function with increasing ℓ and hence bounded from above by its value for $\ell = 0$, which is about 1.025, and, of course, bounded from below by unity:

$$1 \leq \frac{\varepsilon_{\rm NR}}{\varepsilon_{\rm UR}} \leq \frac{\varepsilon_{\rm NR}}{\varepsilon_{\rm UR}}(\ell=0) \simeq 1.025 \tag{90}$$

Summarizing, we compared the spectra of energy eigenvalues predicted by two different ways of describing hadrons as bound states of quarks [2]: the two approaches might be regarded as opposite extremes as far as the extent of incorporation of relativistic kinematics is concerned; the respective confining inter-quark potential, however, is determined by the requirement that both models should yield linear Regge trajectories in the limit of large angular momenta. By use of a simple variational technique, we found that the binding energies obtained within nonrelativistic and ultra-relativistic treatment agree with an error of less than three percent [25]. These findings have also been confirmed by explicit numerical computation of the mesonic mass spectrum.

5 The Bethe–Salpeter Formalism

As a final topic we would like to clarify the relation between our effective-Hamiltonian approach to bound states and the Bethe–Salpeter formalism. We shall show below that the eigenvalue equation involving our effective Hamiltonian corresponds to a well-defined approximation of the Bethe–Salpeter equation.

Within the framework of the Bethe–Salpeter formalism a bound state is represented by the Bethe–Salpeter amplitude, which (in momentum space) is defined as the Fourier transform of the time-ordered product of the respective field operators of the particles constituting the bound state, taken between the state vector of the bound state and the vacuum, after factorizing off the motion of the center-of-momentum. Accordingly, ignoring all normalization factors, the BS amplitude for fermion–antifermion bound states reads

$$\Psi(p) \sim e^{iPX} \int d^4x\, e^{ipx}\, \langle 0|T(\psi(x_1)\bar{\psi}(x_2))|P\rangle \quad , \tag{91}$$

where X denotes the center-of-momentum coordinate, $x = x_1 - x_2$ the relative coordinate, $P = p_1 + p_2$ the total momentum, and p the relative momentum of the two bound-state constituents.

The BS amplitude $\Psi(p)$ satisfies the Bethe–Salpeter equation [26] (in momentum space)

$$\boxed{(\not{p}_1 - m_1)\, \Psi(p)\, (\not{p}_2 + m_2) \sim \int d^4q\, K(p,q)\, \Psi(q)} \quad , \tag{92}$$

in which the interaction between the particles forming the bound state enters via the Bethe–Salpeter kernel $K(p,q)$, which is defined (only perturbatively!) as the sum of all BS-irreducible Feynman graphs for two-particle into two-particle scattering. (In the above form of the BS equation the inverse propagators on the left-hand side have been approximated by their free counterparts.)

In principle, the BS equation represents the appropriate tool for the description of bound states within quantum field theory. In practice, however, there are two fundamental drawbacks. On the one hand, the BS kernel cannot be computed beyond the tight limits of perturbation theory. On the other hand, even with the BS kernel at one's disposal, it is—except for a few simple cases—not possible to find the general solution of the BS equation.

Assuming for the BS kernel the static approximation

$$K(p,q) = K(\vec{p},\vec{q}) \quad , \tag{93}$$

which corresponds to the assumption of an instantaneous interaction between the bound-state constituents, and defining the equal-time wave function

$$\Phi(\vec{p}) \equiv \int dp_0\, \Psi(\vec{p},p_0) \quad , \tag{94}$$

leads to the Salpeter equation [27]

$$\Phi(\vec{p}) \sim \int d^3q \left[\frac{\Lambda_1^+\, \gamma_0\, K(\vec{p},\vec{q})\, \Phi(\vec{q})\, \gamma_0\, \Lambda_2^{+c}}{E - H_0(\vec{p}_1) - H_0(\vec{p}_2)} - \frac{\Lambda_1^-\, \gamma_0\, K(\vec{p},\vec{q})\, \Phi(\vec{q})\, \gamma_0\, \Lambda_2^{-c}}{E + H_0(\vec{p}_1) + H_0(\vec{p}_2)} \right] \tag{95}$$

where $\Lambda_i^{\pm}$ are the energy projection operators corresponding to positive or negative energy of the particle i, $i = 1, 2$.

Neglection of the second term on the right-hand side of the Salpeter equation—which corresponds to particle–antiparticle annihilation and subsequent creation and thus has no clear interpretation in terms of a potential—on the grounds of the reasonable assumption that the denominator in the first term is much smaller than that in the second term,

$$E - H_0(\vec{p}_1) - H_0(\vec{p}_2) \ll E + H_0(\vec{p}_1) + H_0(\vec{p}_2) \quad , \tag{96}$$

leads to the so-called reduced Salpeter equation

$$\boxed{[E - H_0(\vec{p}_1) - H_0(\vec{p}_2)]\, \Phi(\vec{p}) \sim \int d^3q\, \Lambda_1^+\, \gamma_0\, K(\vec{p},\vec{q})\, \Phi(\vec{q})\, \gamma_0\, \Lambda_2^{+c}} \quad . \tag{97}$$

The BS amplitude $\Phi(\vec{p})$ is a (4×4) matrix. Very similarly as any Dirac spinor may be decomposed into a "large" and a "small" component, the BS amplitude $\Phi(\vec{p})$ consists of "large–large", "large–small", and "small–small" components. In the latter case, however, these components are related by the energy projection operators $\Lambda_i^{\pm}$. Using these relations in order to express everything in terms of the "large-large" component $\tilde{\psi}(\vec{p})$ one ends up with an equation of motion,

$$\boxed{[E - H_0(\vec{p}_1) - H_0(\vec{p}_2)]\, \tilde{\psi}(\vec{p}) \sim \int d^3q\, T_{fi}^B\, \tilde{\psi}(\vec{q})} \quad , \tag{98}$$

which is formally identical to the Schrödinger equation with our effective Hamiltonian (27).

Acknowledgement

We would like to thank W. Kallies for undertaking any efforts to make our visit at the Joint Institute for Nuclear Research in Dubna, where this paper has been prepared, an extraordinarily pleasant stay and Uta Brus for providing a particularly stimulating atmosphere.

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Received by Publishing Department
on November 15, 1991.