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NEW ALGORITHMS OF SPECTRAL MANAGEMENT

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INTRODUCTION

The methods of the quantum inverse problem provide algorithms of arbitrary modifications of the spectral parameters [1]. In particular, for the infinite square well we can shift the chosen energy level E_λ or change its reduced widths γ_λ without varying other spectral parameters. The corresponding perturbation $\Delta v(x)$ (Bargmann-type potentials [1]) of the original potential and new wave functions can be written in an explicit form.

It is shown in the present paper that with the periodical chains of finite range Bargmann-type potentials we can create gaps in the continuous spectrum at a desired place. Due to the possibility to approximate the arbitrary potential form with such an exactly solvable model potentials [1], we get a powerful tool of spectral engineering.

BARGMANN-TYPE POTENTIALS AND EIGENFUNCTIONS

The following perturbation of the flat bottom of the infinite square well potential $(0<x<a)$

$$V(x) = \delta\gamma_\nu^2 \cos(k_\nu x)\, \frac{1}{k_\nu} \sin k_\nu x\; p^{-1}(x) + (\delta\gamma_\nu^2)^2 \sin^4(k_\nu x)\; p^{-2}(x) \qquad (1)$$

where

$$p(x) = 1 + \delta\gamma_\nu^2 \frac{1}{k_\nu^2} \int_x^a \sin^2(k_\nu t)\, dt \qquad (2)$$

is the Bargmann-type potential well with the same spectrum as for the rectangular well ($\mathring{V}$), but with the perturbed reduced

width γ_ν of a single eigenstate Ψ_ν (where $\gamma_\nu^2 = \mathring{\gamma}_\nu + \delta\gamma_\nu^2$). The γ_ν plays also the part of the normalizing constant for the corresponding regular solution ϕ_ν: $\Psi(x) = \gamma_\nu \phi_\nu$. These solutions can be written in a simple closed form

$$\phi_\nu(x) = \frac{1}{k_\nu} \sin(k_\nu x)\ p^{-1}(x) \qquad (3)$$

and for an arbitrary energy E ($k^2 = E$):

$$\phi(E,x) = \frac{1}{k}\sin(kx) - \delta\gamma_\nu^2 \frac{1}{k_\nu^2 k}\sin(k_\nu x) \int_x^a \sin(k_\nu t)\sin(kt)\ dt \qquad (4)$$

So, the increase of the ground state reduced width γ_1^2 causes the asymmetry of the absolute values of the derivatives at the ends of the interval [0,a] for ϕ_ν and the conservation of this symmetry for all other states, as it is shown in Fig.1.

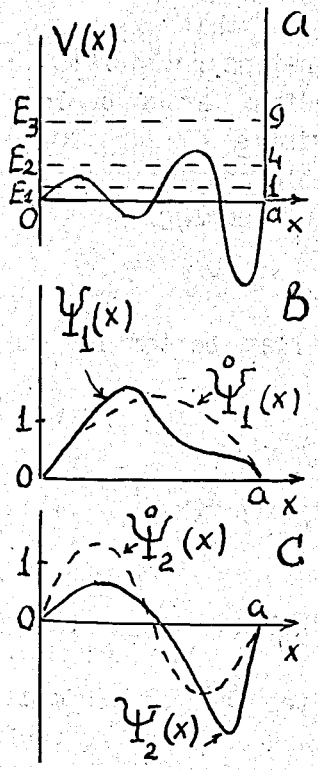

Fig.1 The deformation of the bottom of the infinite square well (a) and of the ground (b) and the first exited (c) states wave functions with variation of the reduced width γ_2(normalizing constant for the first excited state Ψ_2) which is equal to the derivative $\Psi_2'(x=a)$ at the well boundary. All other bound states are deformed without changing the derivatives at the ends of the interval [0,a] as it is shown for the ground state [2]. The periodic continuation of the potential V(x) (as is shown in Fig.3) creates the forbidden zone in the neighborhood of E_2.

A simple explanation of the form of the potential V(x) and the corresponding normalized first and second eigenfunctions

were given in [1]. Now we shall repeat periodically (Fig.2) the perturbed potential shown in Fig.1a and match the corresponding eigenfunctions with perturbed γ_2(exponential growth in the forbidden zone) and unperturbed $\gamma_{i\neq 2}$ (periodic solutions in the allowed zone) reduced widths.

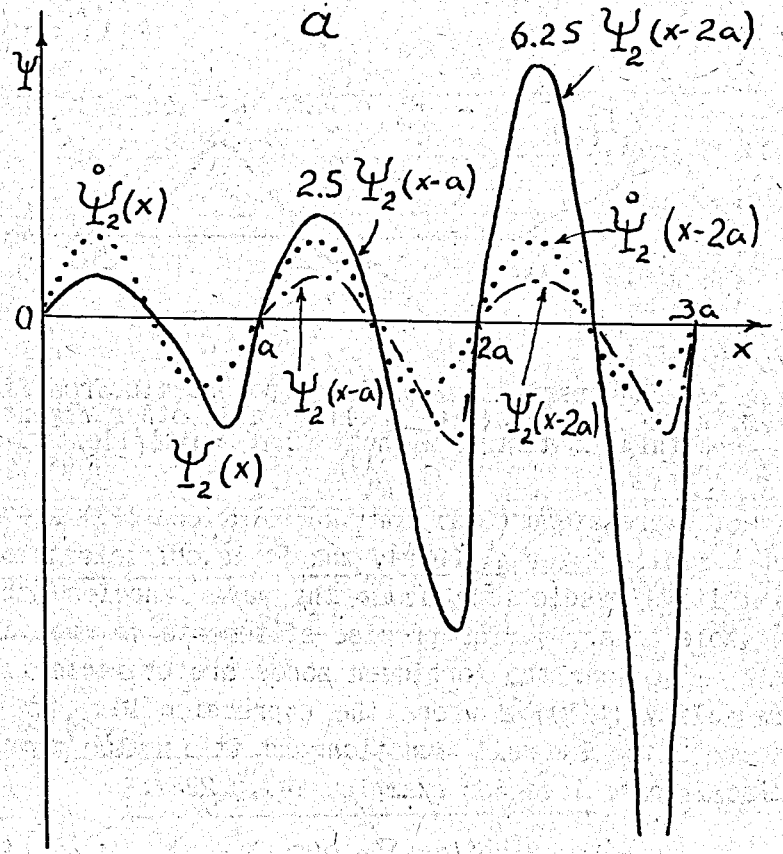

Fig.2 (a) Wave function Ψ_2(solid line), constructed out of the solution $\Psi_2(x)$ on the interval [0,a], which corresponds to the periodically repeated potential V(x) from Fig.1a to the whole axis x (see Fig.2b). The unperturbed $\mathring{\Psi}_2$ function of the first excited state of the rectangular square well is shown by the dotted line (a). Dash-dotted line (a) with broken derivatives at x=na is the chain of repeated perturbed solutions $\Psi_2(0<x<a)$. It has to be multiplied by 2.5 on the interval [a,2a] to make the derivative continuous at x=a and again by 2.5 on [2a,3a] to provide matching at x=2a. This exponential swinging means that E_2 belongs to the forbidden zone.

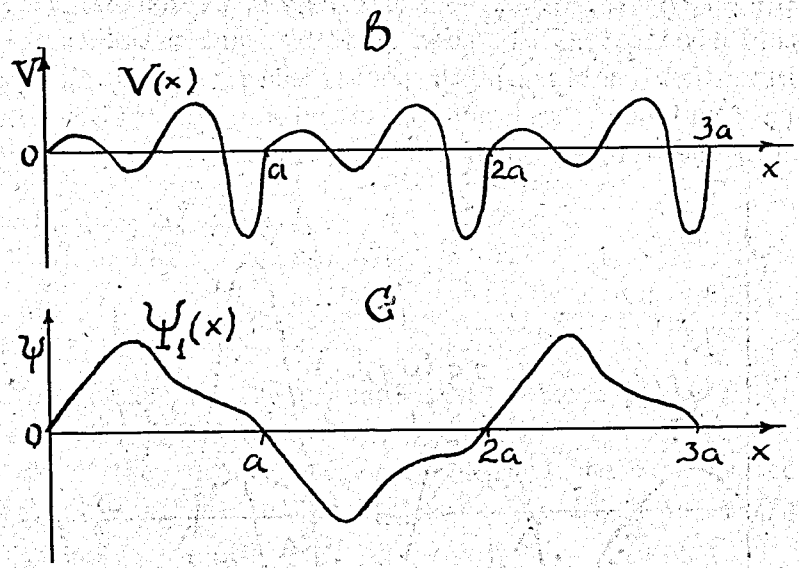

Fig.2b. The periodical chain of potentials from Fig.1a.
Fig.2c.The ground state Ψ like all other eigenstates can be smoothly continued without that multiplication (the allowed zone).

Exact expressions (3,4) for the solutions $\phi(x)$ are valid only on a finite interval $[0,a]$, and it is sufficient to make some intuitive predictions about the wave function behavior on the whole axis. But for precise statements we need a more reliable criterion: the forbidden zones are characterized by the inequality $|D(k)|>2$ where the expression $D(k)$ is up to the factor 2 the Lyapunov function and it is also named the Hill discriminate (see for example, [4],p.234):

$$D(k) = \phi'(x=a) + \psi(x=a) \qquad (5)$$

where special solutions ϕ and ψ satisfy the boundary condition $\phi(0)=\psi'(0)=0$; $\phi'(0)=\psi(0)=1$.

The quality of our predictions is demonstrated in Fig.3. A significant (rather big) lacuna (a gap in the continuous spectrum) appeared just around the energy value of the level E_ν for which the corresponding reduced width γ_ν was changed. It is possible to manage the widths of the gap by choosing the proper value of variation of the normalization factors $d\gamma_\nu^2$ There are some small gaps (almost invisible in our case;

but they increase with growing of $\delta\gamma_\nu^2$) also above other levels. So, the exactly solvable periodic potentials considered here have no finite number of gaps (lacunas) but they enrich the arsenal of tools to approximate arbitrary desired one-dimensional periodic quantum systems.

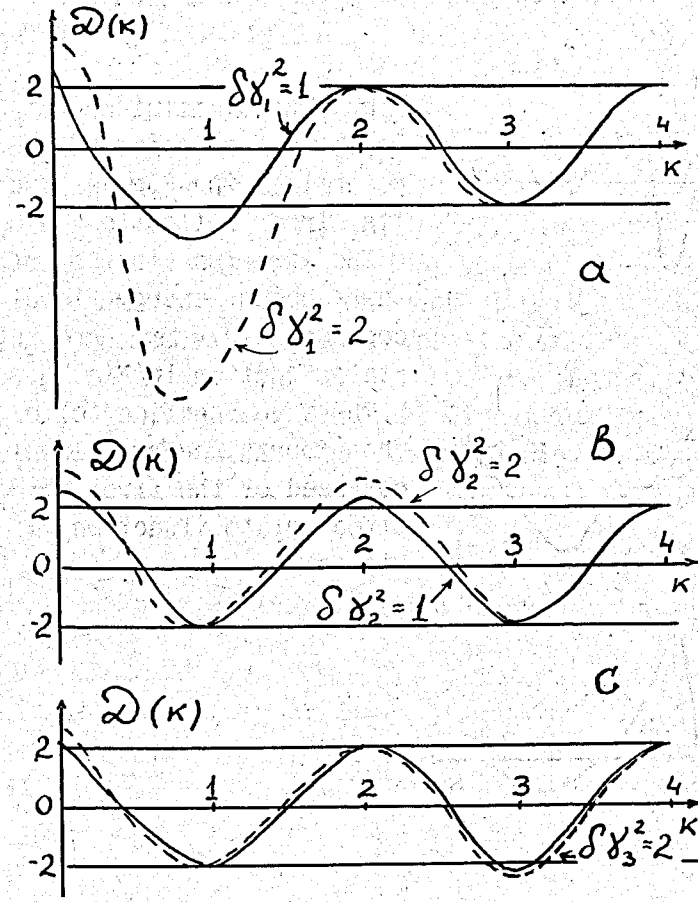

Fig.3. The function D(k) that determines the zone structure. The intervals where $|D(k)|>2$ belong to the forbidden zones (lacunas or gaps in the continuous spectrum). Significant gaps are in the vicinity of the bound states with the perturbed γ_i: (a) i=1; (b) i=2;(c) i=3.

We can construct the chain of potential pairs consisting of the finite-range Bargmann-type potentials (fig.1a) and their reflections with respect to the middle point of the interaction interval (see Fig.4a). All bound states of corresponding infinite potential wells have (anti-)symmetrical derivatives at the well boundaries and can be (anti-)periodically continued along the whole x-axis. So, none of the le-

vels is disposed inside the forbidden zone. But the energy levels of the double well (symmetrized one) are more dense (nearer to the bottom), their reduced widths γ^2 are four times smaller and wave functions are more sensitive to potential perturbations and "unvisible" gaps (of a simple potential chain) can become visible (for symmetrized chain).

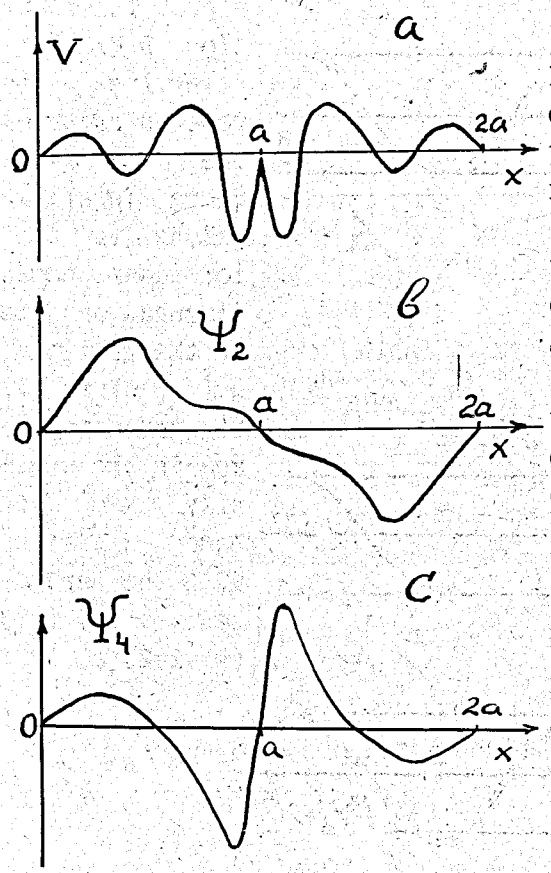

Fig.4 The symmetrical potential (a) composed of the potential in Fig.1a and its mirror reflection.

The wave function (b) of the second energy level composed of the ground state function in Fig.1b.

The wave function (c) of the fourth energy level composed of the first excited state function in Fig.1c

CONCLUSION

The same installation of gaps into the continuous spectrum for the finite-difference analog of the Schroedinger equation is now under consideration. There is only an allowed zone of finite width and the number of gaps is also bounded. For the multichannel generalization of the suggested algorithm, suitable Bargmann-type potential matrices can be

constructed as in [1].

It would also be interesting to investigate the influence on the zone structure of potential perturbations corresponding also to shifts of levels of a rectangular well.

Some exactly solvable three-dimensional models can be constructed out of the one-dimensional periodic potentials of t type shown in Fig.2b.

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