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ELECTROMAGNETIC PROPERTIES OF A TOROIDAL SOLENOID

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1. Introduction

The toroidal solenoid (TS) is an unique object exhibiting a number of interesting properties. For example, the magnetic field H may vanish either inside or outside the solenoid depending on the current distribution on the solenoid surface /1-3/. The small value of magnetic flux leakages from the solenoid has favoured its application in controlled thermonuclear physics /4/. As an accumulator of electromagnetic energy, it is extensively used in an electromagnetic launcher technology (see, e.g., special issues of IEEE Transactions on Plasma Science /5/ and on Magnetics /6/ devoted to this subject). TS is an ideal device for both experimental /7/ and theoretical /8/ investigations of the Aharonov - Bohm effect. According to refs./9,10/ the current flowing in the winding of TS is characterized by the new kind of multipole moments (MM). We mean so called toroidal (or anapole) multipole moments (TMM) They are now an object of extensive theoretical and experimental studies (see, e.g., refs./11-14/). Last (but not least) TS with an electric charge attached to it presents an example of a true three-dimensional anyon /15/. This gives a chance to discover the 3-dimensional Fractional Quantum Hall Effect and 3-dimensional anyonic high-temperature superconductivity. Usually, the electromagnetic field (EMF) of TS is obtained either through numerical integration of the Poisson and Helmholtz Eqs./16/ or through their physical simulation /17/. For the static case, closed expressions for the vector potential (VP) of TS were obtained in ref. /18/ and their properties were discussed in ref. /3/. In the same ref., the EMF of TS with the time-dependent current was considered. Regretfully, only half a page was devoted to the most important case of the periodically varying current. Numerous discussions with radioengineers and

1. Introduction

The majority of realistic problems do not allow exact solutions. The usual way of handling these problems is to resort to an iterative procedure or perturbation theory. The resulting sequence of iterates $\{f_k : k = 0,1,2,\ldots\}$ is very often poorly convergent or even divergent. If one could calculate many $(k \geq 5)$ first terms of an iterative procedure, then one would be able to improve the convergence or find an effective limit of a divergent sequence by invoking resummation techniques such as Padé approximation, Borel transformation, conformal mapping, continued - fraction representation and so on[1].

For example, when solving the Schrödinger equation with anharmonic potentials by iterating the Brillouin - Winger perturbation formula, one meets with the fact that the convergence rate decreases markedly as whether the coupling constant, anharmonicity power or the energy - level number increases[2]. In such a case, the accuracy can be improved by using the hypervirial theorem and Padé approximations[3, 4].

However, to calculate many iterates is often technically impossible. The standard situation is when one is able to find only a few first iterative terms. In this case, the usual resummation techniques fail.

The technical difficulties arising during an iteration process can be explained as follows. Generally, an iterative operator $\hat{I}_k$ transforming the term f_k into f_{k+1} depends on the iteration number k,

$$f_{k+1} = \hat{I}_k f_k.$$

If the iterative operator is not simple enough, then the calculation of a term

$$f_k = \hat{I}_{k-1} f_{k-1} = \hat{I}_{k-1} \hat{I}_{k-2} f_{k-2} = \ldots = \hat{I}_{k-1} \hat{I}_{k-2} \ldots \hat{I}_0 f_0$$

becomes a rather complicated, if in principle possible, task.

This difficulty could be overcome if we would be able to reconstruct the sequence $\{f_k\}$ to $\{f'_k\}$ for which the iterative operator would be independent of the iteration number, i.e.

$$f'_{k+1} = \hat{I} f'_k.$$

If, in addition, the iterative operator is simple and contracting, then one could easily find any iterate

$$f'_k = \hat{I}^k f'_0.$$

Such iterated maps can be treated as dynamical systems[5]. The motion in discrete time is called a cascade. The iterative transformation being a contracting one implies that the sequence $\{f'_k\}$ converges to an attractor representing the sought solution.

In this report we show that an arbitrary sequence of iterates can be approximated by a cascade whose attractor is the sought limit of this sequence. The advantages of this results are obvious: we need to know only a few initial terms in order to find an effective limit of a sequence, and the convergence of the latter can be checked by considering the stability of motion for the cascade. The approach described

below has been developed[6-8] at first by basing on the renormalization - group ideas, although the analogy with dynamical systems has been also emphasized[8, 9]. Here we demonstrate that the dynamical - theory language makes the interpretation of the method much simpler and permits some important generalizations.

2. Iterative Cascade

Suppose we are interested in a function $f(g)$ with the variable $g \in \mathcal{R}$, which satisfies a very complicated equation to be solved using an iterative procedure. The convergence of the latter is known to depend on the choice of an initial approximation $f_0(g)$. The procedure can be contracting for some values of $f_0(g)$ and divergent for others. All possible initial approximations providing the uniform convergence form an attraction region $\mathcal{L}$. In this way, a sequence of iterates interpreted as a cascade moves to an attractor provided that $f_0(g) \in \mathcal{L}$. The convergence is faster for those sequences whose zero approximations are closer to the attractor.

In order that a sequence of iterates would be in the vicinity of an attractor, this sequence should be governed in some way. To this end, by choosing an initial approximation, introduce into it a set of trial parameters z, so that $f_0(g) = f_0(g, z)$. Then, all further iterates also become dependent on these parameters: $f_k(g) \to f_k(g, z)$. Define a set of functions $z_k(g)$ whose role is to govern the behaviour of the sequence

$\{f_k(g)\}$ formed of the terms

$$f_k(g) \equiv f_k(g, z_k(g)); \quad k = 0, 1, 2, \ldots \tag{1}$$

so as to keep these terms close to an attractor. Because of their role, the functions $z_k(g)$ are called[6-9] the governing functions, and the set

$$\mathcal{G} \equiv \{z_k(g) : k = 0, 1, 2, \ldots; \ g \in \mathcal{R}\} \tag{2}$$

can be named the government. By definition, the governing functions guarantee that all terms

$$f_0(g, z_0(g)) \cong f_1(g, z_1(g)) \cong \ldots = f_*(g, z_*(g))$$

are close to an attractor and move to it as k increases; the attractor representing the sought function $f(g)$,

$$f_*(g) \equiv f_*(g, z_*(g)) \cong f(g). \tag{3}$$

Therefore, the general definition of the governing functions can be given as the relation

$$f_{k+p}(g, z_{k+p}(g)) \cong f_k(g, z_k(g)); \quad k, p \geq 0, \tag{4}$$

which is to be understood in the sense of the Cauchy criterion of the uniform convergence

$$|f_{k+p}(g, z_{k+p}(g)) - f_k(g, z_k(g))| < \epsilon,$$

where $k \geq s(\epsilon)$, $p \geq 0$, $g \in \mathcal{R}$. The relation (4) follows from a particular form of the Cauchy criterion in which $s(\epsilon) = 0$, and which can be called the fastest - convergence criterion[8].

The idea of reconstructing a sequence of approximations in order that this sequence would be convergent has been advanced in Refs.10,11 by introducing additional functions satisfying the fastest - convergence criterion. This is now widely known as renormalized, or modified, perturbation theory. However, an option of the governing functions $z_k(g)$ up to now has been heuristic, thus there has been no firm grounds to prefer one of several known variants[10-15].

Note that the introduction of the governing functions can be combined with the use of integral transformations, like the Borel transformation or the Hubbard transformation

$$\exp(-\varphi^2) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{4} \pm iu\varphi\right) du \ .$$

In the dynamical approach we follow here, different adoptions of the governing functions can be classified and analysed, so that it becomes possible to conclude which of the variants is generally preferable.

The interpretation of the sequence $\{f_k(g)\}$ as a dynamical system assumes the necessity of defining a mapping transmuting a term $f_k(g)$ into $f_{k+1}(g)$. To find such a map, let us introduce some new notation. Define the coupling function $g(f)$ by the equation

$$f_0(g, z_0(g)) = f; \quad g = g(f). \tag{5}$$

Introduce the function

$$x_k(f) \equiv f_k(g(f), z_k(g(f))), \tag{6}$$

whose limiting properties in accordance with (5) and (3) are

$$x_0(f) = f \tag{7}$$

and, respectively,

$$x_*(f) = f_*(g(f)). \tag{8}$$

Defining the inverse function $f^{-1}(g)$ by the equation

$$g(f^{-1}) = g; \quad f^{-1} = f^{-1}(g),$$

we can return from (6) to (1) following the relation

$$x_k(f^{-1}(g)) = f_k(g, z_k(g)) = f_k(g).$$

The fastest - convergence condition (4) in terms of (6) reads

$$x_{k+p}(f) \cong x_k(f). \tag{9}$$

Putting here $k = 0$, we get $x_p(f) = f$ substituting which into the right - hand side of (9) we obtain

$$x_{k+p}(f) = x_k(x_p(f)). \tag{10}$$

This relation is often called the property of functional self - similarity. This is why the method based on (10) has been called the method of self - similar approximations[6–9].

On the other hand, the functional property (10) characterizes, as is known[5], a dynamical system in discrete time, that is a cascade. The

ordered sequence of terms (6) starting at the point (7) is named an orbit, or trajectory,

$$\mathcal{T}_f \equiv \{f, x_1(f), x_2(f), \ldots\}.$$

In this way, to any sequence $\{f_k(g)\}$ satisfying the fastest - convergence criterion (4) it is admissible to put into correspondence a sequence $\{x_k(f)\}$ with the relation (10) characteristic of a cascade.

3. Iterative Flow

An additional information can be extracted if we pass from the cascade to a flow. Introduce a continuous variable

$$t \in \mathcal{R}_+ \equiv [0, +\infty), \tag{11}$$

and make an analytical continuation of (6) to a function $x(t, f)$ such that when t crosses a positive integer, say $t = k$, then $x(t, f)$ coincides with the corresponding value of $x_k(f)$,

$$x(k, f) = x_k(f); \quad k = 0, 1, 2, \ldots \tag{12}$$

The cascade property (10) for the function $x(t, f)$ becomes that for the flow

$$x(t + t', f) = x(t, x(t', f)). \tag{13}$$

The initial condition (7) now is

$$x(0, f) = f. \tag{14}$$

The existence of an attractor expressed in (3) and (8) reads

$$x(s_*, f) = f_*(g(f)), \tag{15}$$

where s_* is a saturation number[8].

The self - similar relation (13) can be presented in the differential form. Differentiating (13) with respect to t and then putting $t \to 0,\ t' \to t$, we get

$$\frac{d}{dt}x(t,f) = v(x(t,f)), \tag{16}$$

where

$$v(t) \equiv \lim_{t\to 0}\frac{d}{dt}x(t,f). \tag{17}$$

The latter function is called the vector field, or velocity. Equation (16) is typical of an autonomous dynamical system, that is of a flow. The trajectory is

$$\mathcal{T}_f \equiv \{x(t,f) : t \in [0,\infty)\}. \tag{18}$$

Thus, we have shown that an iterative sequence $\{f_k(g)\}$ can be represented as a flow with the equation of motion (16) describing the trajectory (18). Therefore, the sequence

$$\mathcal{T}_{rep}(g) \equiv \{f_k(g) : k = 0, 1, 2 \ldots\}$$

can be called the representation of trajectory (18). We have managed to obtain the equation of motion (16) by introducing the governing functions providing the property of self - similarity (13). It is possible to say that the self - similar symmetry (13) is imposed by the governing functions.

This fact principally distinguishes our approach from continuous analogs of concrete iterative methods[16, 17] as well as from the renormalization - group method[18] of quantum - field theory, based on symmetry properties of particular equations of motion.

Integrating Eq.(16) from a k -th approximation to the saturation point s_*, we have

$$\int_{x(k,f)}^{x(s_*,f)} \frac{dx}{v(x)} = s_* - k. \tag{19}$$

The substitution $f \to f^{-1}(g)$ for the lower and upper limits gives

$$x(k, f^{-1}(g)) = f_k(g), \quad x(s_*, f^{-1}(g)) = f_*(g).$$

As a result, it follows from Eq.(19) that

$$\int_{f_k(g)}^{f_*(g)} \frac{df}{v(f)} = s_* - k. \tag{20}$$

This equation defines the sought self - similar approximation $f_*(g)$.

4. Vector Field

To integrate (20) we need to know the explicit form of the vector field $v(f)$. In reality, all the information we have is related to the discrete representation. Therefore, we are forced to return to it wishing to write an expression for the vector field. The discrete representation for the latter can be written[6-9] as

$$v_{sk}(f) = \frac{\Delta_{sk}(f)}{s-k} \tag{21}$$

by using the finite difference

$$\Delta_{sk}(f) = f_s(g, z_k) - f_k(g, z_k) + (z_s - z_k)\frac{\partial}{\partial z_k} f_k(g, z_k), \quad (22)$$

in which

$$g = g(f), \quad z_k = z_k(g(f)), \quad k < s.$$

Then, integral (20) is to be replaced by

$$\int_{f_k(g)}^{f^*_{sk}(g)} \frac{df}{v_{sk}(f)} = s_* - k, \quad (23)$$

the self - similar approximation $f^*_{sk}(g)$ being dependent on the chosen velocity (21).

Define the relative fixed - point distance

$$\delta_{sk} \equiv \frac{s_* - k}{s - k}. \quad (24)$$

With notation (24) the integral (23) takes the form

$$\int_{f_k(g)}^{f^*_{sk}(g)} \frac{df}{\Delta_{sk}(f)} = \delta_{sk}. \quad (25)$$

A more elegant expression can be given for (25) by introducing the function

$$y_{sk}(f) \equiv \{\Delta_{sk}(f)\delta_{sk}\}^{-1}. \quad (26)$$

Then, (25) can be written as the normalization law

$$\int_{f_k(g)}^{f^*_{sk}(g)} y_{sk}(f)df = 1. \quad (27)$$

If we assume that the attractors of the considered cascades and flows are fixed points but not limiting cycles or chaotic and strange attractors, then we can readily derive the corresponding stability conditions[19]. To converge to a fixed point, the self - similar mapping (10) has to be contracting, which implies that the corresponding mapping multipliers must be smaller than unity, these multipliers being defined by

$$M^p_{sk}(g) \equiv \lim_{f \to f^*_{sk}(g)} \left| \frac{d}{df} f_p(g(f), z_p(g(f))) \right|. \quad (28)$$

In addition, the equation of motion (16) can be analyzed with respect to the asymptotic Lyapunov stability[19], which requires that the Lyapunov exponent

$$\Lambda_{sk}(g) \equiv \lim_{f \to f^*_{sk}(g)} \frac{d}{df} v_{sk}(f) \quad (29)$$

has to be negative. Thus, the sufficient conditions for the fixed point $f^*_{sk}(g)$ to be stable are

$$M^p_{sk}(g) < 1, \quad \Lambda_{sk}(g) < 0. \quad (30)$$

These two conditions control the choice of governing functions and of the vector field.

5. Ergodic Sequence

It may happen that the dynamical system described by Eq.(16) may have an attractor which is not a fixed point (stable node or stable focus) but which, e.g., is a stable limit cycle, stable torus, quasiattractor, chaotic attractor or strange attractor. Then, the stability conditions (30) are not valid. How is it possible then to define a correct self - similar approximation of the sought function?

When an attractor is not a fixed point, then the analysis of the Lyapunov stability should be replaced by that of the Poisson stability[20], although for a limit cycle and torus the Lyapunov analysis can be applied. A function $x(t,f)$, real and continuous, given for $t \in \mathcal{R}_+$ is called stable á la Poisson, or Poisson stable, if for each $\epsilon > 0$ and any $t \in \mathcal{R}_+$ one can define an infinite sequence $\{t_p = t_p(\epsilon, t) : \ p = 0,1,2,\ldots\}$ for which $t_p \rightarrow \infty$ as $p \rightarrow \infty$, such that

$$|x(t+t_p,f) - x(t,f)| < \epsilon; \quad p = 0,1,2\ldots. \tag{31}$$

A periodic motion corresponding to a limit cycle is, as is obvious, Poisson stable. Then, one may put $t_p = pT$, where T is a period. A quasiperiodic motion corresponds to the motion on a torus. Almost a periodic motion is related to quasiattractor. Both the latter motions are Poisson stable[20]. The motion on a chaotic attractor is mixing. A strange attractor is particular kind of the chaotic attractor with a dimensionality lower than a manifold into which it is embedded. The mixing motion is also Poisson stable.

All kinds of attractors are metrically transitive. Therefore, we can define the ergodic average

$$x_{erg}(f) \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t,f)dt, \tag{32}$$

which must be independent of an initial point,

$$x_{erg}(f) = x_{erg}. \tag{33}$$

The discrete analog of (32) is

$$f_{erg}(g) \equiv \lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} f^*_{sk}(g), \tag{34}$$

where $s = s(k) > k$ and the initial term corresponding to $k = 0$ is omitted in accordance with (33). The ergodic average (34) is the limit

$$\lim_{p\to\infty} f^*_p(g) = f_{erg}(g)$$

of the sequence $\{f^*_p(g) : \ p = 1,2\ldots\}$ composed of the quasiergodic terms

$$f^*_p(g) \equiv \frac{1}{p} \sum_{k=1}^{p} f^*_{sk}(g); \quad s = s(k). \tag{35}$$

Therefore, we may call the sequence $\{f^*_p(g)\}$ the ergodic sequence.

The definition of the ergodic sequence gives us a practical tool for constructing higher orders of self - similar approximations.

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($\Omega = kr - \omega t$, $\Lambda = \pi R^2 g$). The argument of the Bessel functions in (2.7) is $kd\sin\theta$. The nonvanishing field strengths are

$$H_\varphi = \frac{\Lambda}{2r^3}\left[(k^2r^2-1)\sin\Omega + kr\cos\Omega\right]\cdot J_1 + \frac{\Lambda kd}{4r^3}\sin\Omega\cdot\sin\theta\cdot J_0, \tag{2.8}$$

$$E_\theta = \frac{\Lambda k}{2r^2}\cos\Omega\cdot(J_1 - kd\sin\theta\cdot J_2) - \frac{\Lambda k}{4r^3}\sin\Omega\left[d\sin\theta\cdot(J_0-3J_2)-2kr^2J_1\right],$$

$$E_r = -\frac{\Lambda kd\cos\theta}{2r^3}(\sin\Omega - kr\cos\Omega)\,J_0 + \frac{\Lambda kd^2\sin\theta\cos\theta}{2r^4}(3\cos\Omega + 2kr\sin\Omega)J_1$$

In the wave zone ($kr \gg 1$)

$$E_\theta = H_\varphi = \frac{\Lambda k^2\sin\Omega}{2r}\cdot J_1,\quad E_r = \frac{\Lambda k^2 d}{2r^2}\cos\theta\cdot\cos\Omega\; J_0 \tag{2.9}$$

The radial component of the Poynting vector is

$$S_r = \frac{1}{4\pi c}E_\theta\cdot H_\varphi = \frac{1}{4\pi c}\left(\frac{\Lambda k^2\sin\Omega}{2r}\right)^2\left[J_1(kd\sin\theta)\right]^2 \tag{2.10}$$

The integral energy flow (averaged over the period) is

$$\frac{1}{T}\int_0^T dt\int S_r\, dS = \frac{1}{4c}\left(\frac{\Lambda k^2}{2}\right)^2\int_0^\pi\left[J_1(kd\sin\theta)\right]^2\sin\theta\, d\theta \tag{2.11}$$

Being related to the square of the total current ($= NI$) this gives the so-called radiation resistance /23/

$$\frac{\pi^2k^4R^4}{4c^5}\int_0^\pi (J_1)^2\sin\theta\, d\theta \tag{2.12}$$

The integral occurring here can be taken in a closed form for small and large values of kd

$$\int_0^\pi (J_1)^2\sin\theta\, d\theta = \begin{cases}(kd)^2/3 & \text{for } kd \ll 1\\ (kd)^{-1} & \text{for } kd \gg 1\end{cases}$$

Experimental investigations of TS with alternate current were performed almost half a century ago. Their description may be found in an excellent book /24/, published regretfully only in Russian.

§3. The EMF multipole expansion for the toroidal solenoid

3.1. The spherical function expansion. In what follows we shall use the current given by

$$\vec{j} = \vec{j}_0\cdot\exp(-i\omega t) \tag{3.1}$$

(where j_0 as before is given by Eq.(2.4)). As all components of the VP and EMF strengths contain the same factor $\exp(-i\omega t)$, it will be omitted in all intermediatory calculations. In the final expressions this factor should be restored and the real part from them should be taken. We introduce now the VP components having the definite projection of orbital angular momentum

$$A_0 = A_z,\quad A_{\pm 1} = \mp\frac{1}{\sqrt{2}}(A_x \pm iA_y)$$

Their development over states with definite orbital momentum has the form

$$A_0 = -\frac{1}{2}i\pi gkR\sum V_e^0 F_e^0,$$
$$A_1 = -\frac{1}{2\sqrt{2}}i\pi gkR\exp(i\varphi)\sum V_e^1\cdot F_e^1,\quad A_{-1} = -(A_1)^* \tag{3.2}$$

Here $V_e^m = h_e(kr)\cdot P_e^m(\cos\theta)$, h_e is the spherical Hankel function ($h_e(x) = H^{(1)}_{e+1/2}(x)/\sqrt{x}$), P_e^m is the adjoint Legendre function. The coefficients F are determined by the current distribution

$$F_e^0 = \int d\psi\cos\psi\, g_e\cdot P_e,\quad F_e^1 = \int d\psi\sin\psi\, g_e\cdot P_e^1 \tag{3.3}$$

In this Eq. g_e is the spherical Bessel function ($g_e(x) = J_{e+\frac{1}{2}}(x)/\sqrt{x}$) of the argument $k\rho$ ($\rho = (d^2+R^2+2dR\cos\psi)^{1/2}$); the Legendre polynomials in (3.3) have $R\sin\psi/\rho$ as an argument. The cylindrical components of VP are easily obtained from (3.2)

$$A_z = A_0,\quad A_\rho = \frac{1}{\sqrt{2}}\, i\pi k g R \sum V_\ell^1 F_\ell^1,\quad A_\varphi = 0$$

It follows at once from (3.3) that only those coefficients are different from zero which correspond to even values of ℓ. As an example we present F_ℓ^0 and F_ℓ^1 for $\ell = 0,2$ (for $R \ll d$):

$$F_0^0 = \pi\sqrt{2}\left[g_{-1} J_1 - \frac{1}{2}\epsilon g_0 (J_0 - J_2)\right],$$

$$F_2^0 = -\frac{\pi}{2}\sqrt{\frac{5}{2}}\left[g_{-3} J_1 - \epsilon (J_0 - J_2)(3g_2 - 3g_{-2} - g_0)\right], \tag{3.4}$$

$$F_2^1 = \frac{15}{2}\pi\epsilon\left[g_2 (J_0 + J_2) - \frac{\epsilon}{\delta}(J_1 + J_3)\left(6\frac{\cos\delta}{\delta^2} - \cos\delta + \frac{9}{2}\frac{\sin\delta}{\delta}\right)\right.$$

$$\left. + \frac{3\epsilon^2}{2\delta}(J_0 - J_4)\left(5\frac{\sin\delta}{\delta^2} - \frac{1}{2}\sin\delta - 3\frac{\cos\delta}{\delta}\right)\right]$$

$$(\epsilon = R/d)$$

In this Eq. spherical Bessel functions and those with integer indices depend on the arguments $\delta = k d$ and kR, resp. When $k \to 0$ the following asymptotic behaviour of F_n and F_n^1 is valid (for $R \ll d$):

$$F_{2n} \approx \frac{\pi \cdot (-1)^n}{2^{4n+1}} \cdot \binom{2n}{n}\sqrt{4n+1}\,\frac{\epsilon\,\delta^{2n}}{\Gamma(2n+3/2)} \cdot \left(2n - \frac{n+1}{4n+3}\delta^2\right),$$

$$F_{2n}^1 \approx -\frac{\pi(-1)^n}{2^{4n+1/2}}\,\frac{\epsilon\,\delta^{2n}}{\Gamma(2n+\frac{3}{2})}\left[\frac{(4n+1)(2n+1)}{n}\right]^{1/2}\frac{(2n)!}{n!\,(n-1)!}\left(1 - \frac{\delta^2}{2}\frac{1}{4n+3}\right) \tag{3.5}$$

Eqs. (3.2) describing VP of TS $(\rho - d)^2 + z^2 = R^2$ are valid outside the sphere of the radius $r = d + R$. For $r < d - R$ one should use Eqs. (3.2) in which the role of g_ℓ and h_ℓ is interchanged. In a static case, Eqs. (3.2) are transformed into

$$A_0 = -gR\sum_{\ell=2}^{\infty}\frac{1}{r^{\ell+1}}\,\frac{1}{2\ell+1}\,P_\ell(\cos\theta)\, f_\ell^0,$$

$$A_1 = -\frac{1}{\sqrt{2}}\, gR\exp(i\varphi)\sum_{\ell=2}^{\infty}\frac{1}{r^{\ell+1}}\,\frac{1}{2\ell+1}\,P_\ell^1(\cos\theta)\, f_\ell^1 \tag{3.6}$$

Here $f_\ell^0 = \int d\psi \cos\psi\, \rho^\ell\, P_\ell\left(\frac{R\sin\psi}{\rho}\right),$

$$f_\ell^1 = \int d\psi \sin\psi\, \rho^\ell\, P_\ell^1\left(\frac{R\sin\psi}{\rho}\right) \tag{3.7}$$

$$(\rho^2 = d^2 + R^2 + 2dR\cos\psi)$$

For the thin solenoid ($R \ll d$) $f_\ell^{0,1}$ are obtained in the closed form

$$A_0 = \pi g R^2 \sum_{n=0}^{\infty}\frac{2n+1}{2^{2n+1/2}}\,\frac{(-1)^n}{\sqrt{4n+5}}\cdot\binom{2n}{n}\frac{d^{2n+1}}{r^{2n+3}}\cdot P_{2n+2}(\cos\theta),$$

$$A_1 = -\pi g R^2 \exp(i\varphi)\sum_{n=0}^{\infty}\frac{2n+1}{2^{2n+3/2}}\sqrt{\frac{2n+3}{(n+1)(4n+5)}}\,(-1)^n\cdot\binom{2n}{n}\frac{d^{2n+1}}{r^{2n+3}}\, P_{2n+2}^1(\cos\theta) \tag{3.8}$$

For the sake of completeness we consider here the case when the current in the solenoid winding exponentially grows or falls ($\vec{j} = \vec{j}_0 \exp(\pm\omega t)$). The nonvanishing cylindrical components of VP are

$$A_\rho = \frac{gkR}{\sqrt{r}}\exp(\pm\omega t)\sum K_{\ell+\frac{1}{2}}(kr)\cdot P_\ell^1(\cos\theta)\int\frac{1}{\sqrt{\rho}}\, I_{\ell+\frac{1}{2}}(k\rho)\, P_\ell^1\left(\frac{R\sin\psi}{\rho}\right)\sin\psi\, d\psi,$$

$$A_z = -\frac{gkR}{\sqrt{r}}\exp(\pm\omega t)\sum K_{\ell+\frac{1}{2}}(kr)\cdot P_\ell(\cos\theta)\int\frac{1}{\sqrt{\rho}}\, I_{\ell+\frac{1}{2}}(k\rho)\cdot P_\ell\left(\frac{R\sin\psi}{\rho}\right)\cos\psi\, d\psi$$

Here K_ν and I_ν are modified Bessel functions. Under the integral sign $\rho = (R^2 + d^2 + 2dR\cos\psi)^{1/2}$. These Eqs. are valid for $r > d + R$. For $r < d - R$ the role of K and I should be interchanged.

3.2. The vector harmonics expansion. Sometimes it is more convenient to represent VP and field strengths by means of so called vector spherical harmonics. In what follows, we shall adhere notation used in book /21/. This expansion of VP looks as follows:

$$\vec{A} = \frac{2\pi^2 i k}{c}\sum_{\ell m \tau}\vec{B}_\ell^m(\tau, \vec{n})\cdot a_\ell^m(\tau) \tag{3.9}$$

The indices ℓ and m mean here the total angular momentum and its projection onto the z axis (see Appendix 2). Index τ defines the kind of the multipole radiation. The values of $\tau = E, M$ and L correspond to the electric, magnetic and longitudinal multipoles (EM, MM and LM for short). The vectors $\vec{B}_\ell^m$ are given by

$$\vec{B}_\ell^m(L) = \frac{1}{k}\vec{\nabla} h_\ell Y_\ell^m = \sqrt{\frac{\ell+1}{2\ell+1}}\, h_{\ell+1}\vec{Y}^m_{\ell,\ell+1} + \sqrt{\frac{\ell}{2\ell+1}}\, h_{\ell-1}\vec{Y}^m_{\ell,\ell-1},$$

$$\vec{B}_\ell^m(M) = h_\ell \frac{1}{\sqrt{\ell(\ell+1)}}\vec{L}\, Y_\ell^m = -h_\ell \vec{Y}^m_{\ell\ell}, \tag{3.10}$$

$$\vec{B}_\ell^m(E) = -\frac{1}{k}\frac{1}{\sqrt{\ell(\ell+1)}}\vec{\nabla}\times[(\vec{r}\times\vec{\nabla}) h_\ell Y_\ell^m] =$$

$$= -\sqrt{\frac{\ell}{2\ell+1}}\, h_{\ell+1}\vec{Y}^m_{\ell,\ell+1} + \sqrt{\frac{\ell+1}{2\ell+1}}\, h_{\ell-1}\vec{Y}^m_{\ell,\ell-1}$$

Here $\vec{L}$ is the orbital angular momentum $(\vec{L} = -i\,\vec{r}\times\vec{\nabla})$; the vector functions $\vec{Y}^m_{\ell\lambda}$ (see Appendix 2) as well as the vectors $\vec{B}^m_\ell(\tau)$ are eigenfunctions of the total angular momentum square and its z projection. The coefficients occurring in (3.9) are determined by the current density

$$a_\ell^m(\tau) = \int \vec{A}_\ell^{m*}(\tau, \vec{r}\,)\, \vec{j}_0(\vec{r}\,)\, dV$$

The vector quantities $\vec{A}_\ell^m$ are defined by Eqs. similar to (3.10) in which the Hankel spherical functions h_ℓ are replaced by the Bessel ones g_ℓ. The coefficients $a_\ell^m(\tau)$ are referred to as field amplitudes /19-21/ or form factors /10/. The values $\tau = E, M$ and L correspond to electrical, magnetic and longitudinal form factors (EFF, MFF and LFF, resp.). For compleneteness we give here the explicit expressions for $a_\ell^m(\tau)$

$$a_\ell^m(M) = -J^m_{\ell\ell},$$

$$a_\ell^m(L) = \sqrt{\frac{\ell+1}{2\ell+1}}\, J^m_{\ell,\ell+1} + \sqrt{\frac{\ell}{2\ell+1}}\, J^m_{\ell,\ell-1}, \tag{3.11}$$

$$a_\ell^m(E) = -\sqrt{\frac{\ell}{2\ell+1}}\, J^m_{\ell,\ell+1} + \sqrt{\frac{\ell+1}{2\ell+1}}\, J^m_{\ell,\ell-1}$$

$$\left(J^m_{\ell\lambda} = \int g_\lambda(kr)\, \vec{Y}^m_{\ell\lambda}\, \vec{j}_0\, dV \right)$$

VP presented in the form (3.9) with $a_\ell(\tau)$ given by (3.11) satisfies gauge condition /20/

$$\operatorname{div}\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0$$

Here φ is the scalar potential

$$\varphi = \exp(-i\omega t)\int \frac{\exp(ik|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\, \rho_0(\vec{r}\,')\, dV'$$

(It is suggested therefore that charge density ρ harmonically changes with time $\rho = \rho_0 \cdot \exp(-i\omega t)$). In the absence of charge density $(\rho_0 = 0)$ VP meets gauge condition $\operatorname{div}\vec{A} = 0$. In this case LFF are equal to zero. To check this we apply to (3.9) the div operator. It follows at once from (3.10) that $\operatorname{div}\vec{B}_\ell^m(M) = \operatorname{div}\vec{B}_\ell^m(E) = 0$ and $\operatorname{div}\vec{B}_\ell^m(L) = -k h_\ell Y_\ell^m$. Thus,

$$\operatorname{div}\vec{A} = -\frac{2\pi^2 ik^2}{c}\sum h_\ell Y_\ell^m a_\ell^m(L) \tag{3.12}$$

As the particular terms of this sum are linear independent, so $a_\ell^m(L) = 0$ if $\operatorname{div}\vec{A} = 0$. This may be also proved in a more straightforward way by performing integration in $a_\ell^m(L)$ (see Appendix 3). From the Eq. $a_\ell^m(L) = 0$ it follows at once (see Eq.(3.11)) that

$$J^m_{\ell,\ell-1} = -\sqrt{\frac{\ell+1}{\ell}}\, J^m_{\ell,\ell+1} \tag{3.13}$$

Substituting this into $a_\ell^m(E)$ one gets

$$a_\ell^m(E) = -\sqrt{\frac{2\ell+1}{\ell}}\, J^m_{\ell,\ell+1} \tag{3.14}$$

It turns out that for the poloidal current (2.4) $a_\ell^m(M)=0$ and $a_\ell^m(E)=\delta_{m,0}\,a_\ell(E)$. Integrating in (3.14) wrt $\tilde{R}$ and φ one arrives at

$$a_\ell(E) = -\frac{gR}{2\sqrt{2\pi}}\,\frac{1}{\sqrt{\ell}}\sqrt{\frac{2\ell+1}{2\ell+3}}\left(\sqrt{\ell+1}\,F^0_{\ell+1}-\sqrt{\ell+2}\,F^1_{\ell+1}\right) \tag{3.15}$$

($F_\ell^{0,1}$ were defined above (see Eq.(3.3)). For $k\to 0$ we find the following asymptotic behaviour of $a_\ell(E)$ (see Appendix 4):

$$a_\ell(E) = -\frac{1}{\sqrt{\pi}}\,\frac{k^{\ell+1}Rg}{\Gamma(\ell+\frac{3}{2})}\,\frac{1}{2^{\ell+2}}\,\frac{1}{\sqrt{\ell(\ell+1)}}\sqrt{\frac{2\ell+1}{2\ell+3}}\,f^0_{\ell+1} \tag{3.16}$$

(f_ℓ^0 are given by Eq.(3.7)).

Substituting this into (3.9) we get in the static limit

$$\vec{A} = Rg\sqrt{2\pi}\sum_{\ell=1}^{\infty}\frac{1}{\sqrt{(2\ell+1)(2\ell+3)}}\,\frac{1}{r^{\ell+2}}\,f^0_{\ell+1}\,\vec{Y}^0_{\ell,\ell+1} \tag{3.17}$$

It is easy to check that particular components of (3.17) coincide with (3.6). From (3.9) (bearing in mind the omitted factor $\exp(-i\omega t)$) one obtains for the EMF strengths

$$\vec{H} = -ik\sum a_\ell(E)\,\vec{B}_\ell(M), \qquad \vec{E} = ik\sum a_\ell(E)\,\vec{B}_\ell(E) \tag{3.18}$$

Or in the spherical components

$$H_\varphi = \frac{1}{\sqrt{2\pi}}\,k\sum h_\ell P^1_\ell\, a_\ell(E),$$

$$E_\theta = -\frac{ik}{\sqrt{2\pi}}\sum\frac{1}{2\ell+1}\left[(\ell+1)h_{\ell-1}-\ell h_{\ell+1}\right]P^1_\ell\, a_\ell(E), \tag{3.19}$$

$$E_r = \frac{ik}{\sqrt{2\pi}}\sum\frac{\sqrt{\ell(\ell+1)}}{2\ell+1}\left(h_{\ell+1}+h_{\ell-1}\right)P_\ell\, a_\ell(E)$$

The Poynting vector averaged over the time period and integrated over the sphere of sufficiently large radius is

$$\frac{1}{2}\,\frac{1}{4\pi c}\int E_\theta H^*_\varphi\, dS = \frac{1}{4\pi^2 c}\sum a_\ell^2(E) \tag{3.20}$$

The factor 1/2 in the LHS of (3.21) takes into account the difference in time dependences of Eqs.(2.3) and (3.1).

3.3. The toroidal form factors (TFF) and toroidal multipole moments (TMM). In refs. /10/ were introduced so called TFF and TMM. It is interesting to know how they are looking for the particular case under the consideration (that is in the absence of charge density and for the current density defined by Eq.(3.1)). But at first we consider a general case when both the charge and current densities differ from zero. The VP is still given by Eq.(3.9) but $a_\ell^m(L)\neq 0$ now. Exactly (see Appendix 3)

$$q_\ell^m = \frac{i}{c}\,a_\ell^m(L) \tag{3.21}$$

(for the charge density harmonically changing with time). The VP now meets the Lorentz condition $\mathrm{div}\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0$. Some pecularities concerning Eq.(3.21) should be mentioned here. We wright out explicitly quantities entering there

$$q_\ell^m = \int g_\ell\, Y_\ell^{m*}\rho_0\, dV, \tag{3.22}$$

$$a_\ell^m(L) = \sqrt{\frac{\ell+1}{2\ell+1}}\,J^m_{\ell,\ell+1}+\sqrt{\frac{\ell}{2\ell+1}}\,J^m_{\ell,\ell-1} \quad \left(J^m_{\ell n}=\int g_n\,\vec{Y}^{m*}_{\ell n}\cdot\vec{j}_0\, dV\right)$$

By developing Bessel functions entering into both sides of Eq.(3.21) we observe that this equation cannot be satisfied (for the nonvanishing charge density) in any order of k. The reason is that ρ_0 and $\vec{j}_0$ entering into Eq.(3.22) are not independent. The continuity equation $\mathrm{div}\,\vec{j}_0 = i\omega\rho_0$ confirms this. Thus we consider

Eq.(3.21) as the definition of q_ℓ^m. Its dependence of K is carried out by the Bessel functions in the RHS of Eq.(3.21). For the vanishing charge density we may develop the RHS of Eq.(3.21) in powers of K. The disappearance of coefficient at $K^{\ell-1+2n}$ leads to

$$\sqrt{\ell}\,(\ell+n+\tfrac{1}{2})\int r^{\ell-1+2n}\,\vec{Y}^{m*}_{\ell,\ell-1}\,\vec{j}_0\,dV = n\sqrt{\ell+1}\int r^{\ell-1+2n}\,\vec{Y}^{m*}_{\ell,\ell+1}\,\vec{j}_0\,dV$$

For $n=1$ we obtain useful identity

$$\sqrt{\ell}\,(\ell+\tfrac{3}{2})\int r^{\ell+1}\,\vec{Y}^{m*}_{\ell,\ell-1}\,\vec{j}_0\,dV = \sqrt{\ell+1}\int r^{\ell+1}\,\vec{Y}^{m}_{\ell,\ell+1}\,\vec{j}_0\,dV \qquad (3.23)$$

We present now Eq.(3.9) in a slightly extended form

$$\vec{A} = \frac{2\pi^2 iK}{c}\Big[-a_\ell^m(M)\,h_\ell\,\vec{Y}^m_{\ell\ell} + a_\ell^m(L)\Big(\sqrt{\tfrac{\ell+1}{2\ell+1}}\,h_{\ell+1}\vec{Y}^m_{\ell,\ell+1} + \sqrt{\tfrac{\ell}{2\ell+1}}\,h_{\ell-1}\vec{Y}^m_{\ell,\ell-1}\Big) + a_\ell^m(E)\Big(-\sqrt{\tfrac{\ell}{2\ell+1}}\,h_{\ell+1}\vec{Y}^m_{\ell,\ell+1} + \sqrt{\tfrac{\ell+1}{2\ell+1}}\,h_{\ell-1}\vec{Y}^m_{\ell,\ell-1}\Big)\Big] \qquad (3.24)$$

It is easy to check that the terms relating to EFF and LFF (2-nd and 3-d lines in Eq.(3.24)) taken separately diverge in the long-wavelength limit $K\to 0$. In fact (see Eqs.(3.11)), $a_\ell^m(E)$ and $a_\ell^m(L)$ fall like $K^{\ell-1}$ as $K\to 0$ while $B_\ell^m(E)$ and $B_\ell^m(L)$ (see Eqs.(3.10)) grow like $K^{-\ell-2}$ in the same limit. With the account of the overall K factor in Eq.(3.24) this gives K^{-2} divergence either for electrical or longitudinal multipole terms. This drawback is lacking for the vanishing charge density. In this case (see Appendix 3) $a_\ell^m(L)=0$. This condition changes the behaviour of $a_\ell^m(E)$ for small values of K (see Eqs.(3.14)-(3.16) and Appendix 4). As $a_\ell^m(E)$ falls now like $K^{\ell+1}$ for $K\to 0$, the terms of Eq.(3.9) (or (3.24)) corresponding to the particular multipoles are well-behaved. Turning again to the general case we observe that there are no divergences in Eqs.(3.2) from which Eq.(3.9) (or (3.24)) easily follows. Substituting $a_\ell^m(E)$ and $a_\ell^m(L)$ into Eq.(3.24) we arrive at the equation in which the divergent terms entering into EFF and LFF compensate each each other:

$$\vec{A} = \frac{2\pi^2 iK}{c}\sum\Big[h_\ell\vec{Y}^m_{\ell\ell}\cdot J^m_{\ell\ell} + h_{\ell+1}\vec{Y}^m_{\ell,\ell+1}J^m_{\ell,\ell+1} + h_{\ell-1}\vec{Y}^m_{\ell,\ell-1}J^m_{\ell,\ell-1}\Big]$$

The other way /10/ to deal with singularities in Eq.(3.9) (or (3.24)) is to separate explicitly their contribution to EFF. Combining Eqs.(3.11) and (3.21) one gets

$$a_\ell^m(E) = -ic\sqrt{\frac{\ell+1}{\ell}}\,q_\ell^m - \sqrt{\frac{2\ell+1}{\ell}}\,J^m_{\ell,\ell+1} \qquad (3.25)$$

Now we present this Eq. in the form

$$a_\ell^m(E) = a_\ell^m(E,T) - ic\sqrt{\frac{\ell+1}{\ell}}\,q_\ell^m(0) \qquad (3.26)$$

Here

$$q_\ell^m(0) = \lim_{K\to 0} q_\ell^m(K) = \frac{i}{c}\,\frac{K^{\ell-1}}{2^{\ell-1/2}}\,\frac{1}{\Gamma(\ell+\frac{1}{2})}\int r^{\ell-1}\,\vec{Y}^{m*}_{\ell,\ell-1}\,\vec{j}_0\,dV$$

and

$$a_\ell^m(E,T) = -ic\sqrt{\frac{\ell+1}{\ell}}\,[q_\ell^m - q_\ell^m(0)] - \sqrt{\frac{2\ell+1}{\ell}}\,J^m_{\ell,\ell+1} \qquad (3.27)$$

The second term in the RHS of (3.26) being iserted into (3.24) exactly compensates the singularity of LFF. The ratio

$$T_\ell^m(K) = a_\ell^m(E,T)/K^{\ell+1} \qquad (3.28)$$

is referred to as TFF /10/. Using Eqs.(3.21) and (3.22) we develop q_ℓ^m and $J^m_{\ell,\ell+1}$ entering into Eq.(3.27) in powers of K. It follows from this that in the long-wavelength limit $(K\to 0)$ $T_\ell^m(K)$ tends to the finite value

$$T_l^m(0) = -\frac{1}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l+1}{2l+1}}\,\frac{1}{2^{l+3/2}}\left(\sqrt{\frac{l}{l+1}}\,\frac{1}{l+\frac{3}{2}}\int r^{l+1}\vec{Y}^{m*}_{l,l+1}\vec{j}_0\,dV + \int r^{l+1}\vec{Y}^{m*}_{l,l-1}\vec{j}_0\,dV\right) \tag{3.29}$$

This quantity is called TMM /10/. On the other hand putting $q_l^m = 0$ in Eqs.(3.25) and (3.26) we obtain for the case of vanishing charge density

$$a_l^m(E) = a_l^m(E,T) = -\sqrt{\frac{2l+1}{l}}\,J^m_{l,l+1} \tag{3.30}$$

Thus

$$T_l^m(K) = a_l^m(E)/K^{l+1} \tag{3.31}$$

For $K \to 0$ it tends to the finite value

$$T_l^m(0) = -\sqrt{\frac{2l+1}{l}}\,\frac{1}{\Gamma(l+\frac{5}{2})}\,\frac{1}{2^{l+3/2}}\int r^{l+1}\vec{Y}^{m*}_{l\,l+1}\vec{j}_0\,dV \tag{3.32}$$

Using Eqs. (3.15) and (3.16) we may simplify these expressions:

$$T_l^m(K) = -\delta_{0,m}\frac{gR}{2\sqrt{2\pi}}\,\frac{1}{K^{l+1}}\,\frac{1}{\sqrt{l}}\sqrt{\frac{2l+1}{2l+3}}\left(\sqrt{l+1}\,F^0_{l+1} - \sqrt{l+2}\,F^1_{l+1}\right),$$

$$T_l^m(0) = -\delta_{m,0}\frac{1}{\sqrt{\pi}}\,\frac{Rg}{\Gamma(l+\frac{3}{2})}\,\frac{1}{2^{l+2}}\,\frac{1}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{2l+3}}\,f^0_{l+1}$$

Eqs. (3.29) and (3.32) were obtained in two different ways. In fact, Eq.(3.32) follows directly from Eqs.(3.25)-(3.28) by putting $q_l^m = 0$ in them. On the other hand the derivation of Eq.(3.32) uses the expansion of q_l^m in powers of K as an intermediatory step. Evidently these equations should coincide when the charge density disappears. But the condition for this is just Eq.(3.23). The latter being taken into account proves identity of Eqs.(3.29) and (3.32). Thus, $T_l^m(0)$ (for the vanishing charge density) may be presented in three different albeit equivalent forms. They are given by Eqs. (3.29), (3.32) and by

$$T_l^m(0) = -\sqrt{\frac{2l+1}{l+1}}\,\frac{1}{\Gamma(l+\frac{3}{2})}\,\frac{1}{2^{l+3/2}}\int r^{l+1}\vec{Y}^{m*}_{l,l-1}\vec{j}_0\,dV \tag{3.33}$$

As an illustrative example, consider the toroidal dipole moment $(l=1)$. Its cartesian components are usually presented in the form /10/:

$$T_i = \frac{1}{10}\int [x_i(\vec{r}\vec{J}) - 2r^2 J_i]\,dV \tag{3.34}$$

We prove now that this equation reduces (up to nonessential factor) to Eq.(3.33) for $l=1$. For this we use the identity (for $\rho_0 = 0$)

$$\mathrm{div}(x_i x_k x_l \vec{J}) = x_k x_l J_i + x_i x_l J_k + x_i x_k J_l$$

Integrating this equation over the 3-dimensional volume and then contracting wrt k, l indices we get

$$\int [r^2 J_i + 2x_i(\vec{r}\vec{J})]\,dV = 0$$

Substituting this into Eq.(3.34) we obtain

$$T_i = -\frac{1}{4}\int r^2 J_i\,dV \tag{3.35}$$

This expression is completely equivalent (for $l=1$) to Eq.(3.33), and as a consequence, to EMM given by Eq.(3.30). We conclude: for the poloidal current (2.4) and the vanishing charge density the toroidal multipole moments and form factors are proportional to the electrical ones.

§4. <u>The interaction of toroidal solenoid with an external electromagnetic field</u>

The interaction of the current $\vec{j}$ with the external magnetic field $\vec{H}_{ext}$ is given by

$$U = -\frac{1}{c}\int \vec{j}\vec{A}\,dV \tag{4.1}$$

Here $\vec{A}$ is the VP of the magnetic field $\vec{H}_{ext}$ $(= \mathrm{rot}\,\vec{A})$

Let the distance between the magnetic field source and the constant current $\vec j_0$ flowing in the TS winding be much larger than solenoid's dimensions. Then, in the neighbourhood of TS the VP may be presented in the form

$$A_i(\vec r)=A_i(\vec a)+\frac{\partial A_i(\vec a)}{\partial x_k}x_k+\frac{1}{2}\frac{\partial^2 A_i(\vec a)}{\partial x_k\partial x_l}x_k x_l \tag{4.2}$$

(It is suggested therefore that $\vec A$ varies rather slowly in the solenoid's vicinity). Here $\vec a$ is some fixed point near the solenoid and $\vec r$ defines the position of a particular current element wrt this point. Inserting expansion (4.2) into Eq.(4.1) we obtain (for the poloidal current (2.4))

$$U=-\vec\mu_1\vec H_{ext}-\frac{1}{3}\vec\mu_2\cdot\mathrm{rot}\,\vec H_{ext} \tag{4.3}$$

Here $\vec\mu_1$ is the dipole magnetic moment: $\vec\mu_1=\int\vec M\,dV$

$$\vec\mu_2=\int\vec r\times\vec M\,dV \tag{4.4}$$

and $\vec M$ is the magnetic moment density $\vec M=\frac{1}{2c}(\vec r\times\vec j)$. For the poloidal current (2.4) the nonvanishing components of $\vec M$ are:

$$M_x=-M\sin\varphi,\quad M_y=M\cos\varphi,$$
$$M=\frac{g}{8\pi}\delta(\tilde R-R)\frac{R+d\cos\psi}{d+R\cos\psi} \tag{4.5}$$

This means that only the φ component of $\vec M$ differs from zero $(M_\varphi=M)$. It follows from Eqs.(4.5) that $\vec\mu_1=0$, i.e. the magnetic dipole moment equals zero for TS. For the poloidal current (2.4) the single nonvanishing component of $\vec\mu_2$ is

$$\mu_{2z}=\int(xM_y-yM_x)\,dV=\frac{3}{4}\pi g d R^2 \tag{4.6}$$

For the arbitrary orientation of TS' symmetry axis all three components of $\vec\mu_2$ differ from zero. Writing out the triple vector product in Eq.(4.4) we get

$$\mu_{2i}=\frac{1}{2c}\int[x_i(\vec r\vec j)-r^2 j_i]\,dV=-\frac{3}{4c}\int r^2 j_i\,dV \tag{4.7}$$

Comparing Eq.(4.6) with Eqs. (3.29), (3.32), (3.33) and (3.35) we recover the coincidence of μ_{2i} with either EMM or TMM. Since $\vec\mu_1$ disappears for TS the interaction (4.1) may be presented in the form

$$U=-\frac{1}{3}\vec\mu_2\,\mathrm{rot}\,\vec H_{ext}(\vec r=\vec a) \tag{4.8}$$

Using the Maxwell equation $\mathrm{rot}\,\vec H_{ext}=\frac{1}{c}\frac{\partial\vec E_{ext}}{\partial t}+\frac{4\pi}{c}\vec j_{ext}$ and taking into account that expansion (4.2) is valid at sufficiently large distances from the external field source (where $\vec j_{ext}=0$) one may rewrite Eq.(4.8) as

$$U=-\frac{1}{2c}\dot{\vec E}\cdot\vec\mu_2 \tag{4.9}$$

It follows from this that TS interacts with the external EMF if the electric field has a nonvanishing and changing with time component along the symmetry axis of the solenoid. This assertion grounds essentially on the fact that dimensions of the solenoid are small enough (wrt distance from the EMF source). The interaction with static magnetic field is possible if this condition fails. To see this we introduce instead of the current $\vec j_0$ the magnetization $\vec{\mathcal M}$:

$$c\cdot\mathrm{rot}\,\vec{\mathcal M}=\vec j_0 \tag{4.10}$$

The magnetization $\vec{\mathcal M}$ corresponding to $\vec j_0$ is given by

$$\vec{\mathcal M}=\mathcal M\cdot\vec n_\varphi,\quad \mathcal M=\frac{g}{4\pi}\frac{\theta(R-\tilde R)}{d+\tilde R\cos\psi} \tag{4.11}$$

As the current and magnetization formalisms are entirely equivalent /20,25/, one may forget about solenoid's current and treat solenoid as a magnetized ring with magnetization defined by Eq.(4.11). Its physical realization /7/ is a hard ferromagnetic ring having magnetization (4.11) that is independent of applied fields. Now we substitute Eq.(4.10) into (4.1):

$$\mathcal{U} = -\int \vec{A}\, \mathrm{rot}\, \vec{M}\, dV$$

Integrating this equation by parts one gets

$$\mathcal{U} = \int \vec{H}_{ext} \cdot \vec{M}\, dV \qquad (4.12)$$

It follows from this that TS interacts with the external magnetic field the φ component of which has non-zero overlapping with TS'magnetization. As an example, consider the toroidal magnetized ring and the linear current. It turns out that interaction energy (4.12) differs from zero only if the linear current passes through the torus hole.

Consider two TS with constant currents in their windings. Do these solenoids interact? (This question was posed by J.A.Smorodinsky /26/). Their interaction is given by

$$-\frac{1}{2c^2}\int \frac{\vec{j}_1(\vec{r}_1)\cdot\vec{j}_2(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|}\, dV_1 dV_2 = \frac{1}{2c}\int A_2(\vec{r}_1)\, \vec{j}_1(\vec{r}_1)\, dV_1$$

Using the relation $\vec{j}_1 = c\, \mathrm{rot}\, \vec{M}_1$ and integrating by parts one gets

$$\frac{1}{2}\int \vec{H}_2(\vec{r})\cdot\vec{M}_1(\vec{r})\, dV$$

From this it follows that there is no interaction between the non-overlapping solenoids (as the magnetic strengths and magnetizations are confined inside the solenoids).

One may wonder what is the profit to present VP of TS in three different ways (Eqs.(2.7), (3.2) and (3.9))? There are certain reasons for it. As Eq.(2.7) contains all multipoles in a closed form, it is easier to operate with it in practice. Since each particular term of the sum occurring in (3.2) tends to a finite value in the long-wavelength limit $K\to 0$ (see Eqs.(3.6) and (3.7)), one should expect the same for the development of (3.9). We have seen in §3.3 that in a

general case when both the charge and current densities are present, the separate contributions of EFF and LFF diverge as $K\to 0$. Only their sum remains to be finite. Thus, expansion (3.2) may serve as a guiding point.

5. The model of toroidal solenoid current

Consider torus (2.1). In what follows we shall extensively use the toroidal coordinates

$$x = \frac{a\,\mathrm{sh}\mu\cos\varphi}{\mathrm{ch}\mu-\cos\theta},\quad y = \frac{a\,\mathrm{sh}\mu\sin\varphi}{\mathrm{ch}\mu-\cos\theta},\quad z = \frac{a\sin\theta}{\mathrm{ch}\mu-\cos\theta} \qquad (5.1)$$

$$(0<\mu<\infty,\ -\pi<\theta<\pi,\ 0<\varphi<2\pi)$$

For μ fixed the points $P(x,y,z)$ fill the surface of the torus with the parameters $d = a\,\mathrm{cth}\mu$ and $R = a/\mathrm{sh}\mu$. Let $\mu=\mu_0$ correspond to the torus T_0. Then, for $\mu>\mu_0$ ($\mu<\mu_0$) the point $P(x,y,z)$ (where x,y,z are given by (5.1)) lies inside (outside) T_0. The infinitesimal volume element being expressed in toridal coordinates is $dV = \frac{a^3\,\mathrm{sh}\mu\, d\mu\, d\theta\, d\varphi}{(\mathrm{ch}\mu-\cos\theta)^3}$. The current density $\vec{j}_0$ is given by:

$$\vec{j}_0 = -\frac{gc}{4\pi a^2}\frac{\delta(\mu-\mu_0)}{\mathrm{sh}\mu_0}(\mathrm{ch}\mu_0-\cos\theta)^2\,\vec{n}_\theta \qquad (5.2)$$

Here $\vec{n}_\theta$ is the same as $\vec{n}_\psi$ (see §2) but being expressed in toroidal coordinates:

$$\vec{n}_\theta = [(\vec{n}_x\cos\varphi+\vec{n}_y\sin\varphi)\,\mathrm{sh}\mu\sin\theta+\vec{n}_z(1-\mathrm{ch}\mu\cos\theta)](\mathrm{ch}\mu-\cos\theta)^{-1}$$

For the constant current $\vec{H}=\vec{n}_\varphi g/\rho$ inside the solenoid and $\vec{H}=0$ outside it. The VP has two nonvanishing cylindrical components (A_ρ and A_z). At large distances they fall as r^{-3}

$$A_z \approx \frac{\pi g a^3}{8}\frac{\mathrm{ch}\mu_0}{\mathrm{sh}^3\mu_0}\frac{1+3\cos 2\theta_s}{r^3},\quad A_\rho \approx \frac{3\pi g a^3}{8}\frac{\mathrm{ch}\mu_0}{\mathrm{sh}^3\mu_0}\frac{\sin 2\theta_s}{r^3}$$

(θ_s is the spherical polar angle)

Their explicit expressions are given in ref./18/. Now we try to simulate the solenoid current as a relative motion of one of charged layers, out of which the surface charge distribution is composed. We require the following conditions to be fulfilled: 1) the total charge should be equal zero; 2) the electrostatic potential should vanish both inside and outside the solenoid; 3) the rotation of the elements composing the particular charged layer in the $\varphi=$ const plane (each element rotates in that $\varphi=$ const plane in which it lies) should reproduce the current distribution (5.2) and as a consequence the VP of the toroidal solenoid.

We seek the charge distribution in the form

$$\sigma = \sigma_1 \cdot \delta(\mu-\mu_1) + \sigma_2 \cdot \delta(\mu-\mu_2) + \sigma_0\, \delta(\mu-\mu_0) \tag{5.3}$$

For the definiteness we choose $\mu_1 > \mu_2 > \mu_0$. The charge distribution consists of three charged toroidal shells encompassing each other (it is impossible to meet all conditions 1)-3) using the charge distribution consisting of two layers). The layers corresponding to $\mu=\mu_0$ and $\mu=\mu_1$ are external and internal ones (fig. 1). Let each element of the external layer ($\mu=\mu_0$) rotate in the $\varphi=$ const plane with angular velocity ω. We choose σ_0 so as to reproduce the current density (5.2). This gives

$$\sigma_0 \omega R = -\frac{gc}{4\pi a^2} \frac{(\mathrm{ch}\mu_0 - \cos\theta)^2}{\mathrm{sh}\mu_0} \tag{5.4}$$

Here $R = a/\mathrm{sh}\mu_0$ is the radius of the external shell. Or in a slightly different form

$$\sigma_0 = f\,\frac{(\mathrm{ch}\mu_0 - \cos\theta)^2}{\mathrm{sh}\mu_0} \qquad \left(f = -gc\,\frac{\mathrm{sh}\mu_0}{4\pi a^2 \omega}\right) \tag{5.5}$$

We give only the final answer. The electrostatic potential vanishing both inside ($\infty > \mu > \mu_1$) and outside ($0 < \mu < \mu_0$) TS equals

$$\Phi = 8\sqrt{2} f a^2 (\mathrm{ch}\mu - \cos\theta)^{1/2} \sum \frac{1}{1+\delta_{n0}} Q_{n-\frac12}(0)\cdot \cos n\theta \cdot \left[P_{n-\frac12}(0)\cdot Q_{n-\frac12} - P_{n-\frac12}\cdot Q_{n-\frac12}(0)\right]$$

between "0" and "2" charged shells ($\mu_0 < \mu < \mu_2$) and

$$\Phi = 8\sqrt{2} f a^2 (\mathrm{ch}\mu - \cos\theta)^{1/2} \sum \frac{1}{1+\delta_{n0}} Q_{n-\frac12}(0)\cdot \frac{\tau_n(0,2)}{\tau_n(1,2)}\cdot \cos n\theta \left[P_{n-\frac12}(1)\cdot Q_{n-\frac12} - P_{n-\frac12}\, Q_{n-\frac12}(1)\right]$$

between "1" and "2" charged shells ($\mu_2 < \mu < \mu_1$). Here we put $P_\nu^\lambda(i) \equiv P_\nu^\lambda(\mathrm{ch}\mu_i)$, $Q_\nu^\lambda(i) \equiv Q_\nu^\lambda(\mathrm{ch}\mu_i)$ $(i=0,1,2)$; $\tau_n(i,j) = P_{n-\frac12}(i)\, Q_{n-\frac12}(j) - P_{n-\frac12}(j)\cdot Q_{n-\frac12}(i)$. From now we do not indicate the argument of the Legendre functions if it equals $\mathrm{ch}\mu$.

The discontinuity of $\frac{d\Phi}{d\mu}$ at $\mu=\mu_1$ and $\mu=\mu_2$ fixes σ_1 and σ_2

$$\sigma_1 = -\frac{2\sqrt{2}}{\pi} f\, \frac{(\mathrm{ch}\mu_1 - \cos\theta)^{5/2}}{\mathrm{sh}\mu_1} \sum \frac{\cos n\theta}{1+\delta_{n0}}\cdot Q_{n-\frac12}(0)\, \frac{\tau_n(0,2)}{\tau_n(1,2)},$$

$$\sigma_2 = -\frac{2\sqrt{2}}{\pi} f\, \frac{(\mathrm{ch}\mu_2 - \cos\theta)^{5/2}}{\mathrm{sh}\mu_2} \sum \frac{\cos n\theta}{1+\delta_{n0}}\cdot Q_{n-\frac12}(0)\, \frac{\tau_n(1,0)}{\tau_n(1,2)}$$

The constant f determines the value of the charge on each of the layers 0, 1 and 2:

$$e_0 = \int \sigma_0\, \delta(\mu-\mu_0)\, dV = 4\pi^2 f a^3/\mathrm{sh}\mu_0,$$

$$e_1 = \int \sigma_1\, \delta(\mu-\mu_1)\, dV = -16 a^3 f \sum \frac{1}{1+\delta_{n0}} Q_{n-\frac12}(0)\cdot Q_{n-\frac12}(1)\, \frac{\tau_n(0,2)}{\tau_n(1,1)},$$

$$e_2 = \int \sigma_2\, \delta(\mu-\mu_2)\, dV = -16 a^3 f \sum \frac{1}{1+\delta_{n0}} Q_{n-\frac12}(0)\cdot Q_{n-\frac12}(2)\, \frac{\tau_n(1,0)}{\tau_n(1,2)}$$

It is easy to check that

$$e_0 + e_1 + e_2 = 0$$

i.e. the treated charged distribution is electrically neutral. The uniform rotation of the elements composing "0" shell in the const plane with angular velocity ω imitates the surface current (5.2). As a result, the VP $\vec{A}_\omega$ satisfying the Poisson Eq.

$$\Delta \vec{A}_\omega = -\frac{4\pi}{c}\vec{j}_0$$

is generated. The constant g entering into the definition of $\vec{j}_0$ (see Eq.(5.2)) may be expressed through ϱ_0

$$g = -\omega\varrho_0/\pi c$$

This means that for $\vec{A}_\omega$ we may use explicit expressions of the VP obtained in /18/ with g defined by the last equation. Thus obtained electrostatic potential φ' and surface densities σ_i' meet conditions 1)-3) mentioned above.

§6. <u>The rotating toroidal solenoid</u>

6.1. General considerations. The magnetic field of a toroidal solenoid at rest differs from zero only inside it. The magnetic field appears outside the solenoid when it moves through the medium $(\epsilon\mu \neq 1)$ with a constant velocity /22/. This fact seems strange, as using the Lorentz transformation one may always pass to a coordinate system where a solenoid is at rest. However, in this system the medium is in motion and this fact leads to the situation which is not equivalent to the initial one (when the solenoid was at rest relative to the medium). A similar situation arises when one considers the uniform rotation of a toroidal solenoid around its symmetry axis z. It seems first that the magnetic field remains to be enclosed inside the solenoid. In fact let a solenoid be first at rest in the laboratory system S. Now we pss to the coordinate system S′ rotating around the solenoid with angular velocity Ω. Using the Lorentz transformation one may evaluate in this noninertial system the electromagnetic strengths $\vec{E}$

and $\vec{H}$ (note that the accelerated motion of S′ relative to S does not invalidate the use of the Lorentz transformation (see, e.g., /27/). The field $\vec{H}$ is enclosed inside the solenoid in S, so the same is valid in S'. Now consider two physical situations: 1) the solenoid is at rest, the observer rotates and 2) the observer is at rest, the solenoid rotates. As the relative motion of the solenoid and observer is in both the cases the same, so one may erroneously conclude that magnetic field strengths vanish outside the solenoid for the second situation too. The drawback of these considerations is that the identity of relative motion does not guarantee the equivalence of physical situations. In fact an observer is situated in the noninertial reference frame in the first case and in the inertial one in the second case. For a rotating charged sphere this paradox was investigated by L.Schiff /28/. The modern treatment of these questions may be found e.g. in refs. /29/. As equivalence is lost, one should make concrete calculations to evaluate the electromagnetic field of the rotating solenoid. One precaution is needed. In what follows we consider the rotation of TS with current distribution constructed in a previous section. The charge density of this solenoid differs from zero (it consists of 3 charged shells). This means that EMF generated by the rotation of this solenoid does not coincide with EMF of the magnetized ring with magnetization defined by Eq.(4.11) (for which charge density equals zero).

6.2. EMF calculations of the rotating toroidal solenoid. Let the solenoid rotate as a whole around the symmetry axis z with angular velocity Ω (fig. 2). Then in addition to the poloidal current (5.2) there appears the current

$$\vec{j} = j_\varphi \cdot \vec{n}_\varphi, \quad j_\varphi = \sigma \cdot v_\varphi = \Omega\rho\sigma \tag{6.1}$$

flowing in the latitude direction. Here σ is given by Eq. (5.3), $\rho=\frac{a\,sh\mu}{ch\mu-\cos\theta}$ is the distance between a particular, element of the charge shell and the z axis. The current (6.1) generates VP satisfying the Poisson Eq.: $\Delta\vec{A}_{\Omega}=-\frac{4\pi}{c}\vec{j}_{\varphi}$. It turns out that has the single nonvanishing component (A_{φ}^{Ω}). It equals

$$A_{\varphi}^{\Omega}=(ch\mu-\cos\theta)^{1/2}\sum\frac{1}{1+\delta_{n0}}A_n(\mu)\cdot\cos n\theta \qquad (6.2)$$

The functions $A_n(\mu)$ are defined in Appendix 5. At large distances A_{φ}^{Ω} falls as r^{-2}

$$A_{\varphi}^{\Omega}\sim d\,\Omega\sin\theta_s/r^2 \qquad (6.3)$$

Here θ_s is the polar angle. The constant d is also given in Appendix 5. The non-vanishing components of magnetic field decrease as r^{-3}

$$H_r\approx\frac{2d\,\Omega\cos\theta_s}{r^3},\quad H_\theta\approx\frac{d\,\Omega\sin\theta_s}{r^3}$$

The total VP of the rotating TS is

$$\vec{A}=\vec{A}_{\omega}+\vec{A}_{\Omega}$$

where $\vec{A}_{\omega}$ is VP of the resting TS /18/ and $\vec{A}_{\Omega}=A_{\varphi}^{\Omega}\cdot\vec{n}_{\varphi}$.

Let the solenoid rotate as a whole around its symmetry axis with the velocity linearly growing with time: $v=\Omega\rho t$. Then the total VP turns out to be equal to

$$\vec{A}=\vec{A}_{\omega}+(tA_{\varphi}^{\Omega}-\frac{1}{c}\beta)\vec{n}_{\varphi} \qquad (6.4)$$

Here $\vec{A}_{\omega}$ and A_{φ}^{Ω} were defined above. The constant β is given in Appendix 5. As a result, the constant electric field $(E_{\varphi}=-\frac{1}{c}A_{\varphi}^{\Omega})$ and the linearly increasing magnetic field $\vec{H}=t\cdot rot(A_{\varphi}^{\Omega}\vec{n}_{\varphi})$ arise outside the solenoid. At large distances one has

$$E_{\varphi}\approx-\frac{1}{c}\frac{d\,\Omega\sin\theta_s}{r^2},\quad H_r\approx\frac{2d\,\Omega t\cos\theta_s}{r^3},\quad H_\theta\approx\frac{d\,\Omega t\sin\theta_s}{r^3}$$

The radial component of the Poynting vector is directed off the solenoid

$$S_r=-\frac{E_{\varphi}H_{\theta}}{4\pi c}=t\,\frac{d^2\Omega^2\sin^2\theta_s}{4\pi c^2 r^5}$$

Now we surround the toroidal solenoid by the impenetrable (for the observer, not for the EMF) sphere S (fig. 3). Is it possible to establish the existence of the current in the solenoid's winding? As we have seen the magnetic field comes out of the solenoid (and the sphere) if the whole construction (the solenoid plus sphere) undergoes the rotation. This may be verified experimentally.

The following considerations /26/ show that a charged particles should exhibit classical scattering on an impenetrable toroidal solenoid. Due to the recoil effects the solenoid gets finite acceleration. As a result, the nonvanishing electromagnetic field strengths appear outside the solenoid, thus distorting (due to the Lorentz force) the particle trajectory. There are two reasons for the recoil effects. The first one is trivial. It is due to the collision of the incident particles with the surface of an impenetrable torus (or impenetrable sphere surrounding it). The second reason is rather subtle /7,26/. It is associated with the fact that inside the solenoid both electric and magnetic field strengths differ from zero. The electric one is that of the incident charged particle while the magnetic strength is that of the solenoid's internal magnetic field. As a result, the momenta $\int\vec{E}(\vec{r},\vec{r}')\times\vec{H}(\vec{r}')\,dV'$ and torque $\int\vec{r}'\times(\vec{E}\times\vec{H})\,dV$ arise which tend to shift and rotate the solenoid. This in turn leads to the appearance of magnetic field outside TS and to the scattering of charged particles on this field.

7. Conclusion

We briefly summarize the main results obtained.

1) The electromagnetic radiation field of the toroidal solenoid with alternate current in its winding is obtained. The properties of this field are investigated.

2) The multipole expansion of this field is obtained. Its relation to the toroidal multipole moments is established.

3) It is shown how the toroidal solenoid interacts with an external electromagnetic field.

4) The explicit realization of the current flowing in a solenoid's winding is derived.

5) It is shown that rotation of the toroidal solenoid leads to the appearance of the electromagnetic field outside that solenoid.

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Appendix 1

There are two spherical components of VP which are different from zero

$$A_r = \frac{R d g \cos\theta}{4\pi} \iint \frac{d\psi\, d\varphi}{q\, R_{01}} \cos\psi \cos(kR_{01} - \omega t),$$

$$A_\theta = \frac{R g r}{4\pi} \iint \frac{d\psi\, d\varphi}{q\, R_{01}} \cos\varphi \cos\psi \cos(kR_{01} - \omega t) \qquad \text{(A1.1)}$$

$$(R_{01}^2 = p^2 - 2Rq\cos\psi,\quad p^2 = r^2 + d^2 - 2d\rho\cos\varphi,\quad q^2 = (\rho\cos\varphi - d)^2 + z^2$$

For the thin solenoid $(R \ll d)$ these Eqs. are simplified

$$A_r = \frac{R d g \cos\theta}{4} \left(\cos\omega t \int F\, d\varphi + \sin\omega t \int G\, d\varphi\right), \qquad \text{(A1.2)}$$

$$A_\theta = \frac{R g r}{4} \left(\cos\omega t \int F \cos\varphi\, d\varphi + \sin\omega t \int G \cos\varphi\, d\varphi\right) - \operatorname{tg}\theta \cdot A_r$$

Here $F = \frac{R}{p^3}\cos kp\,(J_0 - J_2) + \frac{2}{pq}\sin kp \cdot J_1,$

$$G = \frac{R}{p^3}\sin kp \cdot (J_0 - J_2) - \frac{2}{pq}\cos kp \cdot J_1 \qquad \text{(A1.3)}$$

The Bessel functions entering into (A1.3) depend on $x = kRq/p$. If $kR \ll 1$, then

$$A_r = \frac{1}{4} R^2 d g \cos\theta \left(\cos\omega t \int d\varphi \cdot f + \sin\omega t \int d\varphi \cdot g\right), \qquad \text{(A1.4)}$$

$$A_\theta = \frac{1}{4} R^2 g r \cdot \left(\cos\omega t \int d\varphi \cdot \cos\varphi \cdot f + \sin\omega t \int d\varphi \cdot \cos\varphi \cdot g\right) - \operatorname{tg}\theta \cdot A_r$$

$$\left(f = \frac{\cos kp}{p^3} + \frac{k}{p^2}\sin kp,\quad g = \frac{\sin kp}{p^3} - \frac{k}{p^2}\cos kp \right)$$

From this one easily obtains Eq.(2.7) if additional conditions $d \ll r$ and $kd^2 \ll r$ are imposed.

Appendix 2

The total angular momentum is defined /21/ as a geometrical sum of orbital and spin momenta $\vec{J} = \vec{L} + \vec{S}$. Here $\vec{L} = -i\,\vec{r} \times \vec{\nabla}$ and $\vec{S}$ is the vector matrix with the components

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},\quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The vector harmonics $\vec{Y}^m_{e\lambda}$ are vectorially coupled quantities of the spherical harmonics and unity spherical vectors

$$\vec{Y}^m_{e\lambda} = \sum_\mu C(1, \lambda, e; -\mu, m+\mu)\, Y^{m+\mu}_\lambda\, \vec{n}_{-\mu}$$

$\vec{Y}^{m}_{e\lambda}$ are eigenfunctions of $\vec{J}^2, \vec{L}^2$ and J_0

$$J_0 \vec{Y}^{m}_{e\lambda} = m \vec{Y}^{m}_{e\lambda}, \quad \vec{J}^2 \vec{Y}^{m}_{e\lambda} = \ell(\ell+1) \vec{Y}^{m}_{e\lambda}, \quad \vec{L}^2 \vec{Y}^{m}_{e\lambda} = \lambda(\lambda+1) \vec{Y}^{m}_{e\lambda}$$

The vector harmonics form an orthonormal set

$$\int \vec{Y}^{m*}_{e\lambda} \vec{Y}^{m'}_{e'\lambda'} d\Omega = \delta_{ee'} \delta_{\lambda\lambda'} \delta_{mm'}$$

Appendix 3

We start with the definition of $a^m_e(L)$

$$a^m_e(L) = \int \vec{A}^{m*}_e(L,\vec{r}) \cdot \vec{j}_0 \, dV = \frac{1}{K} \int \vec{\nabla}(g_e Y^{m*}_e) \, \vec{j}_0 \, dV$$

Integrate this Eq. by parts

$$a^m_e(L) = \frac{1}{K} \int div(g_e Y^{m*}_e \vec{j}_0) \, dV - \frac{1}{K} \int g_e Y^{m*}_e \, div \, \vec{j}_0 \cdot dV \tag{A3.1}$$

The first term in (A3.1) disappears if the current density occupies the finite portion of space. Bearing in mind that $div \vec{j} = -\frac{\partial \rho}{\partial t}$ we find for the charge density periodically changing with time $(\rho = \rho_0 \exp(-i\omega t))$

$$div \, \vec{j}_0 = i\omega \rho_0, \quad a^m_e(L) = -ic \, q^M_e \tag{A3.2}$$

where $q^m_e = \int g_e Y^{m*}_e \rho_0 \, dV$ (ρ_0 is the charge density).

Substitution of (A3.2) into (3.12) leads to

$$div \vec{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0 \tag{A3.3}$$

Here φ is the scalar potential

$$\varphi = 2\pi^2 i K \exp(-i\omega t) \sum h_e Y^m_e q^m_e$$

From (A3.1) it follows that $a^m_e(L) = 0$ for the vanishing charge density.

Appendix 4

From the fact that for $K \to 0$

$$F^{0,1}_e \approx \frac{K^{\ell}}{\Gamma(\ell+\frac{3}{2})} \frac{1}{2^{\ell+1/2}} \cdot f^{0,1}_{\ell} \tag{A4.1}$$

it follows that in this limit Eq.(3.15) reduces to

$$a_e(E) = -\frac{gR}{2\sqrt{2\pi}} \frac{1}{\sqrt{\ell}} \sqrt{\frac{2\ell+1}{2\ell+3}} \frac{K^{\ell+1}}{\Gamma(\ell+5/2)} \frac{1}{2^{\ell+3/2}} \left(\sqrt{\ell+1} \, f^0_{\ell+1} - \sqrt{\ell+2} \, f^1_{\ell+1}\right) \tag{A4.2}$$

The disappearance of LFF leads to the following relation between F_ℓ

$$\sqrt{\ell} \, F^0_{\ell-1} + \sqrt{\ell-1} \, F^1_{\ell-1} = \left(\frac{\ell+1}{\ell} \frac{2\ell-1}{2\ell+3}\right)^{1/2} \left(\sqrt{\ell+1} \, F^0_{\ell+1} - \sqrt{\ell+2} \, F^1_{\ell+1}\right) \tag{A4.3}$$

From (A4.1) it seems that RHS and LHS of (A4.3) behave differently for small K. Substitute (A4.1) into (A4.3) and divide both its sides by $K^{\ell-1}$

$$\sqrt{\frac{\ell}{2\ell-1}} \left(\sqrt{\ell} \, f^0_{\ell-1} + \sqrt{\ell-1} \, f^1_{\ell-1}\right) = \frac{K^2}{2\ell+3} \frac{1}{2\ell+1} \sqrt{\frac{\ell+1}{2\ell+3}} \left(\sqrt{\ell+1} \, f^0_{\ell+1} - \sqrt{\ell+2} \, f^1_{\ell+1}\right) \tag{A4.4}$$

The LHS of this Eq. does not depend on K, while the RHS tend to zero as K^2. This means that LHS of (A4.4) should vanish identically

$$\sqrt{\ell} \, f^0_{\ell-1} + \sqrt{\ell-1} \, f^1_{\ell-1} = 0 \quad \text{or} \quad f^1_{\ell-1} = -\sqrt{\frac{\ell}{\ell-1}} \, f^0_{\ell-1}$$

Substituting this into (A4.2) we arrive at Eq.(3.16):

$$a_\ell(E) \approx -\frac{1}{\sqrt{\pi}} \frac{K^{\ell+1}}{\Gamma(\ell+\frac{3}{2})} \frac{Rg}{2^{\ell+2}} \frac{1}{\sqrt{\ell(\ell+1)}} \sqrt{\frac{2\ell+1}{2\ell+3}} \, f^0_{\ell+1}$$

Appendix 5

Functions $A_n(\mu)$ entering into Eq.(6.2) are defined as follows. They are equal

$$A_n = d_n\left[C_{0n}\, Q^1_{n-\frac{1}{2}}(0) + C_{1n}\, Q^1_{n-\frac{1}{2}}(1) + C_{2n}\, Q^1_{n-\frac{1}{2}}(2)\right] P^1_{n-\frac{1}{2}}$$

outside the solenoid $(0<\mu<\mu_0)$; (A5.1)

$$A_n = d_n\left[C_{0n}\, P^1_{n-\frac{1}{2}}(0) + C_{1n}\, P^1_{n-\frac{1}{2}}(1) + C_{2n}\, P^1_{n-\frac{1}{2}}(2)\right] Q^1_{n-\frac{1}{2}}$$

inside it $(\mu_1<\mu<\infty)$; Further

$$A_n = d_n\left\{\left[C_{1n}\, Q^1_{n-\frac{1}{2}}(1) + C_{2n}\, Q^1_{n-\frac{1}{2}}(2)\right] P^1_{n-\frac{1}{2}} + C_{0n}\, P^1_{n-\frac{1}{2}}(0)\cdot Q^1_{n-\frac{1}{2}}\right\}$$

between shells "0" and "2" and

$$A_n = d_n\cdot\left\{\left[C_{2n}\, P^1_{n-\frac{1}{2}}(2) + C_{0n}\, P^1_{n-\frac{1}{2}}(0)\right] Q^1_{n-\frac{1}{2}} + C_{1n}\, Q^1_{n-\frac{1}{2}}(1)\cdot P^1_{n-\frac{1}{2}}\right\},$$

between shells "2" and "1".

Here $d_n = \dfrac{8\sqrt{2}\, f a^3 \Omega}{c\,(n^2-1/4)}$. The coefficients C depend only on the charge distribution parameters:

$$C_{0n} = 2\cdot Q^1_{n-\frac{1}{2}}(0) = -\sum \frac{1}{1+\delta_{m0}}\, Q_{m-\frac{1}{2}}(0)\cdot f_{mn}(0),$$

$$C_{1n} = \sum \frac{1}{1+\delta_{m0}}\, Q_{m-\frac{1}{2}}(0)\, \frac{\tau_m(0,2)}{\tau_m(1,2)}\, f_{mn}(1)$$

$$C_{2n} = \sum \frac{1}{1+\delta_{m0}}\, Q_{m-\frac{1}{2}}(0)\, \frac{\tau_m(1,0)}{\tau_m(1,2)}\, f_{mn}(2)$$

$$\left(f_{mn}(i) = \exp\left[-\mu_i\,(m+n)\right] - \exp\left[-\mu_i\, |m-n|\right],\quad i = 0,1,2 \right)$$

The constant α occurring in Eq.(6.3) equals

$$\alpha = \frac{16\sqrt{2}\, f a^3}{c}\sum \frac{1}{1+\delta_{n0}}\left[C_{0n}\, Q^1_{n-\frac{1}{2}}(0) + C_{1n}\, Q^1_{n-\frac{1}{2}}(1) + C_{2n}\, Q^1_{n-\frac{1}{2}}(2)\right] \quad \text{(A5.1)}$$

Eqs. (A5.1) and (A5.2) are simplified for small thickness of the charge distribution $(\mu_1 = \mu_0 + \Delta_1,\ \mu_2 = \mu_0 + \Delta_2,\ \Delta_1 \ll \mu_0,\ \Delta_2 \ll \mu_0)$

In this case

$$A_n(\mu) = d_n\cdot \operatorname{cth}\mu_0\, (\Delta_1+\Delta_2)\cdot \left[Q^1_{n-\frac{1}{2}}(0)\right]^2\cdot P^1_{n-\frac{1}{2}}$$

outside the solenoid and

$$A_n = d_n\cdot \operatorname{cth}\mu_0\, (\Delta_1+\Delta_2)\cdot P^1_{n-\frac{1}{2}}(0)\cdot Q^1_{n-\frac{1}{2}}(0)\cdot Q^1_{n-1/2}$$

inside it. Further

$$\alpha = \frac{e_0}{2\sqrt{2}\, e}\, \operatorname{cth}\mu_0\cdot(\Delta_1+\Delta_2)\left(1+\frac{6}{x_0}+\frac{6}{x_0^2}\right),\quad x_0 = \exp(2\mu_0)-1$$

Finally, constant β appearing in Eq.(6.4) is:

$$\beta = 32\, f a^4 \Omega\left[\frac{1}{\operatorname{sh}\mu_1}\sum \frac{1}{1+\delta_{n0}}\, Q^1_{n-\frac{1}{2}}(1)\cdot Q_{n-\frac{1}{2}}(0)\,\frac{\tau_n(0,2)}{\tau_n(1,2)} + \right.$$

$$+\frac{1}{\operatorname{sh}\mu_2}\sum \frac{1}{1+\delta_{n0}}\, Q^1_{n-\frac{1}{2}}(2)\, Q_{n-\frac{1}{2}}(0)\,\frac{\tau_n(1,0)}{\tau_n(1,2)} -$$

$$\left. -\frac{1}{\operatorname{sh}\mu_0}\sum \frac{1}{1+\delta_{n0}}\, Q^1_{n-\frac{1}{2}}(0)\, Q_{n-\frac{1}{2}}(0)\right].$$

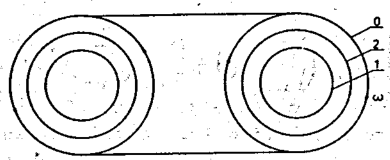

Fig. 1.

The model of the TS current. The charge distribution consists of three charged shells (0,1 and 2). The electrostatic potential differs from zero only between shells 0 and 2. The rotation of the external shell with the angular velocity ω simulates the TS current.

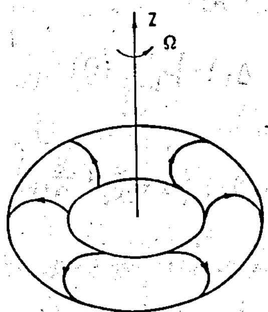

Fig. 2.

The rotation of TS around its symmety axis leads to the appearance of the magnetic field outside TS.

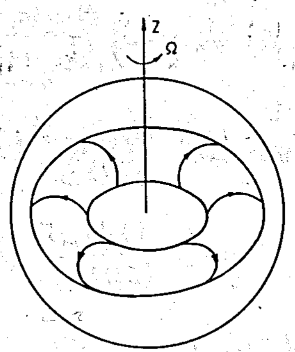

Fig. 3.

The TS is imbedded into the impenetrable sphere. The electric field of the incoming charged particles, being combined with the magnetic field inside TS, leads to the nonzero torque which tends to rotate TS. As a result the magnetic field arises outside TS which in turn affects on the incoming charged particles.

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