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NEW ADIABATIC REPRESENTATION FOR THREE-BODY SCATTERING PROBLEM

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1. Introduction

In this paper we develop a new adiabatic approach reported in [1-3] to the solution of the three-body scattering problem for an appropriate class of pair short-range potentials. It is based on the global adiabatic representation of the three-body wave function Ψ, consisting of the sum of Faddeev's components F_α ($\alpha=1,2,3$) in the configuration space of the c.m. relative motion $\{\underset{\sim}{X} = \underset{\sim}{x}_\alpha + \underset{\sim}{y}_\alpha\} \in \mathbb{R}^6 \setminus \{0\}$ locally isomorphic to the manifold M : $\{X, \hat{X}\} \in \mathbb{R}^1_+ \times \hat{M}$ defined as $\mathbb{R}^1_+ \times S^1 \times S^2 \times S^2$ or $\mathbb{R}^1_+ \times S^3 \times S^2$. In the present approach the correct boundary conditions corresponding to all possible scattering processes, including both the break-up and rearrangement ones, automatically follow from Faddeev's integral equations. Thus, it is also a certain generalization of [4], where the adiabatic set-up has been used for a correct formulation of the scattering theory in the three-body charged system below the threshold of break-up.

The adiabatic description of quantum systems leads to a modern geometrical treatment of the scattering problem. The use of the hyperradius X ($X^2 = x_\alpha^2 + y_\alpha^2$) as an invariant adiabatic variable in M and decomposition of the total Hamiltonian into the slow and fast parts $H = H^s(X) + H^f(X; \hat{X})$ enable us to introduce the global adiabatic basis $\Phi = \Phi(X; \hat{X})$ in the framework of a Hilbert fibre bundle $\mathcal{H}(B, \mathcal{F}_X, \pi)$. Here $B = \mathbb{R}^1_+ \ni X$ is the base, universal for all Faddeev's components F_α, $\mathcal{F}_X$ is the typical fibre isomorphic to $L_2(\hat{M}, d\hat{M})$ with appropriate measure $d\hat{M}$ and inner scalar products $\langle \cdot | \cdot \rangle$. Each fibre is spanned over the orthogonal and complete set $\{\varphi_i\}$ of the real analytic eigenfunctions of the fast Hamiltonian H^f, while $\pi: \mathcal{H} \to B$ is the projection $\mathcal{H}$ into the corresponding point $X \in B$. Moreover, there exist in $\mathcal{H}$ a natural connection A: $A_{ij}(X) = -i\langle\varphi_i|\partial_X|\varphi_j\rangle$ and a parallel transport operator $\mathcal{U}(X, \mathring{X})$ related with it and acting from $\mathcal{F}_{\mathring{X}}$ to $\mathcal{F}_X$. When the corresponding to A curvature $\int A$ vanishes and the projection π is a trivial one, the basis set $\{\varphi_i\}$ is globally defined [2].

Based on the above global adiabatic representation and correct boundary conditions we have consistently formulated the three-body scattering problem by reducing it to the multi-channel one for the"slow" radial system of the gauge type and the parametric quasiangular one in the fibre $\mathcal{F}_X$ describing the fast motion. We investigate also properties and give the complete classification of the states $|i\rangle$ of the global adiabatic basis $\{\varphi\}$ by introducing the special block structure of the parallel transport operator $\mathcal{U}$, which defines a direct decomposition of the Hilbert fibres $\mathcal{F}_X = \mathcal{F}_X^+ \oplus \mathcal{F}_X^-$. The introduced decomposition of the complete set $\{\varphi_i\}$ of the basis states in the form $\varphi = \varphi_+ \oplus \varphi_-$ corresponds to different asymptotic behaviours of the eigenvalues $\mathcal{E}_i(X)$ of the fast Hamiltonian H^f: $\mathcal{E}_+(X \to \infty) \geqslant 0$ and $\mathcal{E}_-(X \to \infty) < 0$. The potential curves (or terms) $\mathcal{E}_+(\infty)$ and $\mathcal{E}_-(\infty)$ reproduce all threshold peculiarities of the three-body system including the break-up and cluster ones, respectively. Thus, the basis functions φ are responsible for two different types ($\pm$) of channels: φ_+ correspond to the break-up surface functions and φ_- correspond to the cluster surface functions.

From the physical point of view, the existence of the unique basis φ consisting of φ_+ and φ_- is due to the fact that all static pair interactions V_α of different fragments constituting the three-body system are contained completely in the fast Hamiltonian $H^f = H_0^f + \sum_\alpha V_\alpha$. Owing to the unique basis and the compatibility conditions of the physical asymptotic solutions of the Schrödinger equation with the known boundary conditions for the Faddeev components, we can formulate the three-body scattering problem taking into account all possible processes, including not only the rearrangement ones but also break-up processes. It is worth noting that in the alternative approaches [5-7] - dealing with different sets of the local adiabatic surface functions in the more complicate Schrödinger picture, the question about the orthogonality and completeness relations for these sets remains open. In our approach the above relations are satisfied automatically, since we are working with the trivial projection π of the Hilbert fibre bundle $\mathcal{H}$.

The above assertions give us a more clear knowledge of a more subtle situation with the Coulomb potential in the ordinary adiabatic approaches initiated in [8].Actually therein,as a rule, one deals only with cluster Coulomb surface functions like φ_- and a misty view appears that states like φ_+ are absent. This feeling occurs, because usually one considers only an asymptotics of the surface functions with a fixed number i in the limit of X tending to infinity. While it is obviously that the density of the basis states becomes sufficiently high if the number i goes to infinity with the same velocity as X.

As a result, we immediately get such basis states, which correspond to the break-up surface Coulomb functions like Φ_+ . For an appropriate class of pair short-range potentials the above limit transformation practically coincides with the famous conversion of Jacobi polynomials into Bessel functions [9]. This fact is taken for granted also in our geometrical construction of the consistency conditions for the connection coefficients diag A^2 , which provide the manifestation of geometrical phases of the radial solutions – a la Berry phase [2]. For the Coulomb case the situation remains in principle the same but it needs obviously a more careful consideration.

It should be noted that the parametric problem for basis quasiangular functions is related with the multichannel problem for the radial coefficients of the adiabatic expansion of the three-body function. The unitary bilocal operator $\mathcal{U}(X,\mathring{X})$ acting from $\mathcal{F}_{\mathring{X}}$ to $\mathcal{F}_X$ provides a parallel transport of the moving frame $|\Phi(X,\hat{X})\rangle$ from some fixed point $\mathring{X}$ to an arbitrary point X of the base B and at the same time it is straightforwardly linked with the connection A . The connection operator A represents an effective gauge field in the "slow" system of radial equations and realizes the coupling between channels in contrast with the ordinary coupled channel method. The solutions of the "slow" system and their asymptotics are analysed with the use of the above operators $\mathcal{U}$ and A, the latter being essential for sticking the asymptotics of the physical three-body wave function with those of the Faddeev components. Thus, in the framework of the adiabatic approach using the above geometrical and spectral arguments we have ela-

borated selfconsistent solution of the three-body scattering problem on the basis of a gauge invariant formulation of the multichannel quantum scattering theory within the traditional statement [10] and the nonstandard parametric problem in the fibre $\mathcal{F}_X$. The Jost matrices and regular, physical, Jost matrix solutions are introduced for the "slow" system of radial equations in which there appears a covariant derivative $(1\otimes\partial_X+A(X))^2$. The latter corresponds to the matrix of the potentials dependent on the velocity [11]. We have derived also expressions for S-matrix scattering in the global adiabatic basis and for invariant partial and total amplitudes compatible with the realistic amplitudes for the three-body system.

The paper is organized as follows. In section 2 we have established general relationships between adiabatic representation for the Faddeev components and Schrödinger three-body wave functions. The geometrical interpretation of the three-body scattering problem is given in the new global adiabatic approach. In Sections 3-5 the above general statement of the problem is developed in detail for the wide class of short-range interparticle potentials. Section 3a-e is devoted to the study of properties of the global adiabatic basis $\{\Phi_i\}$ composed of the local basis Faddeev components $\{F_{\alpha i}\}$. The classification of the total set of basis states consisting of two types, the break-up Φ_+ and cluster Φ_- surface functions is given. In Section 3a we investigated a multichannel system of radial equations of the gauge type obtained via the above unique adiabatic basis. In Section 4 we have introduced and studied the Jost matrix and regular, physical, Jost matrix solutions for the "slow" system of radial equations in which there appears a covariant derivative. In Section 5 we have derived relationships between the invariant partial S-matrix, the matrix amplitudes in the global adiabatic basis and realistic physical amplitudes following from the Faddeev integral equations as $X \to \infty$. Thus, in Sections 3-5 we have developed a gauge-invariant treatment of the multichannel quantum scattering theory as applied to the three-body problem. In Section 6 we briefly discuss the possible applications of the present approach to a new formulation of the three-body inverse scattering problem.

2. Statement of the Problem

The differential formulation of the modified Faddeev equations is a suitable tool for a correct investigation of scattering processes in the system of three nonrelativistic particles [12]. In the configuration space of the relative particle motion $\{\underset{\sim}{X} = \underset{\sim}{x}_\alpha + \underset{\sim}{y}_\alpha\} \in \mathbb{R}^6 \setminus \{0\}$ where α =1,2,3 denotes the number of a pair of particles with Jacobi coordinates $\underset{\sim}{x}_\alpha$, the equations for the partial Faddeev components F_α are of the form

$$\{-\Delta_{\underset{\sim}{X}} + \hat{V}_\alpha + \sum_\beta V_\beta^0 - E\} F_\alpha = -\hat{V}_\alpha \sum_{\beta \neq \alpha} F_\beta, \tag{1}$$

where the functions $\hat{V}_\alpha$ and V_α^0 are the short- and long-range parts of the pair potentials $V_\alpha \equiv V_\alpha(\underset{\sim}{x}_\alpha) = \hat{V}_\alpha + V_\alpha^0$, $E = P^2 = k_\alpha^2 + p_\alpha^2$ is the c.m.s. energy, $\underset{\sim}{P} = \{\underset{\sim}{k}_\alpha, \underset{\sim}{p}_\alpha\}$ corresponds to $\underset{\sim}{X} = \{\underset{\sim}{x}_\alpha, \underset{\sim}{y}_\alpha\}$. The correct asymptotic boundary conditions, containing information about all possible scattering processes in the three-particle system, result from the compact integral equations and they allow one to find out a unique solution for the sys-

tem (1). The corresponding Schrodinger wave function, satisfying

$$(H - E)\Psi = 0; \quad H = -\Delta_{\underset{\sim}{X}} + \sum_\alpha V_\alpha \tag{2}$$

is simply expressed as a sum of its Faddeev components:

$$\Psi(\underset{\sim}{X}) = \sum_\alpha F_\alpha(\underset{\sim}{X}_\alpha(\underset{\sim}{X})). \tag{3}$$

i) Let us choose now the coordinate representation in M: $\underset{\sim}{X} \in \mathbb{R}_+^1 \times S_X^5(\hat{X})$ in which $X \in B = \mathbb{R}_+^1$ being the square root of the trace of the inertia tensor $X^2 = x_\alpha^2 + y_\alpha^2$ defines the hyperradius of the sphere $S_X^5(\hat{X}) \sim S^1 \times S^2 \times S^2$ or $S^3 \times S^2$.

ii) We introduce for the components F_α the following local adiabatic expansion:

$$F_\beta(\underset{\sim}{X}_\beta) = \sum_j F_{\beta j}(X; \hat{X}_\beta) X^{-1} \chi_j(X), \tag{4}$$

where the components of the basis $F_{\beta j}(X; \hat{X}_\beta)$ are defined as solutions of the spectral problem for the system (1) on $\hat{M}$ at a fixed value of $X \in B$:

$$\{-X^{-2}\Delta_{\hat{X}_\alpha} + \hat{V}_\alpha + \sum_\beta V_\beta^0 - \mathcal{E}_j(X)\} F_{\alpha j}(X; \hat{X}_\alpha) = -\hat{V}_\alpha \sum_{\beta \neq \alpha} F_{\beta j}(X; \hat{X}_\beta). \tag{5}$$

Here $\Delta_{\hat{X}_\alpha}$ is the angular part of the Laplace operator $\Delta_{\underset{\sim}{X}}$ on M, $\mathcal{E}_j(X)$ is the spectral parameter; j being the set of indices labelling the spectrum $\sigma(H^f(X;\hat{X}))$ of the three-particle Hamiltonian operator

$$H^f(X;\hat{X}) = -X^{-2}\Delta_{\hat{X}} + \sum_\alpha V_\alpha(X; \hat{X}_\alpha) \tag{6}$$

with domain $\mathcal{D}(H^f(X)) \subset \mathcal{F}_X$ and range $\mathcal{R}(H^f(X)) \subset L_2(S_X^5(\hat{X}))$ depending on X parametrically. Solutions of the Schrödinger problem (2) on the sphere $\hat{M} = S_X^5(\hat{X})$ with the Hamiltonian operator (6)

$$\{H^f(X;\hat{X}) - \mathcal{E}_j(X)\}\Phi_j(X;\hat{X}) = 0 \tag{7}$$

coincides up to renormalization with functions of the adiabatic basis, composed from the components $\{F_{\alpha j}\}$ like (3)

$$\Phi_j(X;\hat{X}) = \sum_{\alpha=1}^{3} F_{\alpha j}(X;\hat{X}_\alpha(\hat{X})) \tag{8}$$

thus inheriting the appropriate asymptotic conditions for $\{\Phi_j\}$ at $X \to \infty$. Define the scalar product $\langle\cdot|\cdot\rangle$ in $\mathcal{F}_X = L_2(\hat{M}, d\hat{M}^\gamma)$ with O(5)-invariant measure $d\hat{M}^\gamma \equiv X^{2\gamma} d\Omega_{S^5}$ (γ being the constant 3/2). Then the following orthonormalization and completeness relations hold:

$$\langle i|j\rangle = \int dM^\gamma \Phi_i^+(X;\hat{X})\Phi_j(X;\hat{X}) = \delta_{ij}, \tag{9a}$$

$$X^{2\gamma}\sum_j \Phi_j(X;\hat{X})\Phi_j^+(X;\hat{X}') = \delta(\hat{X} - \hat{X}'). \tag{9b}$$

iii) The use of the above definitions allows us to construct the Hilbert fibre bundle $\mathcal{H}$ with base $B = R_+^1 \ni X$, typical fibre $\mathcal{F}_X = L_2(\hat{M}, d\hat{M}^\gamma)$ where the real-analytic eigenfunctions $\{\Phi_j(\cdot;\hat{X})\}$ of the operator $H^f(\cdot;\hat{X})$ are considered as sections defining a local basis in a given neighbourhood $U \subset B$. The induced connection form A in this bundle (see e.g. [13]), defined as $-i\langle\Phi|\partial_X|\Phi\rangle$, generates the parallel transport $\mathcal{U}$ of the vectors $|\Phi\rangle$ along the base B. In the case when the corresponding to A curvature form $\int A$ vanishes the basis set $\{\Phi_j(X;\hat{X})\}$ is globally defined and therefore the following generalized global adiabatic expansion is valid:

$$\Psi(\underset{\sim}{X}) = \sum_{\beta}^{3}\sum_j F_{\beta j}(X;\hat{X})X^{-1}\chi_j(X) = \sum_j \Phi_j(X;\hat{X})X^{-1}\chi_j(X). \tag{10}$$

This is a new generalization of the known hyperspherical adiabatic expansion like those given in [6-8], owing to the use in (2)-(10)) of the asymptotic boundary conditions following from the compact integral equations and moreover the decomposition (8) for the definition of the new global adiabatic basis $\{\Phi_i\}$ which is constructed in our approach with the help of the local basis components $\{F_{\alpha i}\}$ being the solutions of the eigenvalue problem for the Faddeev equations (5) for any fixed value $X \in B$. A simple example of the numerical solution to the problem for the positronium ion Ps^- has been given recently in [14]. Also therein it is shown that the obtained solution may be used as a good initial approximation for the straightforward numerical solution of the original eigenvalue problem for Faddeev equations (1). Substitution of (10) into (2) results in the system of ordinary differential equations of second order for the coefficients $\chi \equiv \chi(X, \mathcal{P})$:

$$\sum_j [-D_{ij}^2(X) + \{\mathcal{E}_i(X) - \mathcal{E}_i(\infty) + \gamma(\gamma+1)X^{-2} - \mathcal{P}_i^2\}\delta_{ij}]\chi_j = 0$$

$$(\mathcal{P}_i^2 = E - \mathcal{E}_i(\infty), \quad \mathcal{P} = diag\, \mathcal{P}_i). \tag{11}$$

Here $\gamma(\gamma+1)X^{-2}$ is the nonvanishing centrifugal potential, $\gamma = 3/2$, D is the covariant derivative: $D_{ij} = \delta_{ij}\partial_X + A_{ij}$, and the operator A is the above-mentioned connection operator on $\mathcal{H}$:

$$A_{ij} = -i\langle\Phi_i|\partial_X|\Phi_j\rangle. \tag{11a}$$

3. Properties of the Global Adiabatic Representation

Let us consider in detail the scattering problem in a given system of three spinless particles with pair real-analytic potentials $V_\alpha \equiv \hat{V}_\alpha$ satisfying the conditions

$$\int_0^\infty dx_\alpha |V_\alpha(x_\alpha)| < \infty , \qquad \int_0^\infty dx_\alpha |V_\alpha(x_\alpha)| x_\alpha < \infty$$

and respecting the L-representation of the total momentum: $\underset{\sim}{L} = \underset{\sim}{\ell}_\alpha + \underset{\sim}{\lambda}_\alpha$. Where $\underset{\sim}{\ell}_\alpha = -i \underset{\sim}{x}_\alpha \wedge \partial \underset{\sim}{x}_\alpha$ and $\underset{\sim}{\lambda}_\alpha = -i \underset{\sim}{y}_\alpha \wedge \partial \underset{\sim}{y}_\alpha$ are the orbital momenta of the pair and third particle respectively. In what follows we shall omit in our notation the set of exact quantum numbers $L = \{L, M, \zeta\}$ of the momentum, its projection and total parity $\zeta = (-1)^{\ell_\alpha + \lambda_\alpha}$.

a. Transport operator and system of radial equations

For definiteness let us choose the pair potentials $\hat{V}_\alpha$ and Neumann boundary conditions for the problem (7) so that the spectrum $\sigma(H^f)$ of the operators $H^f(X;\hat{X})$ is a purely discrete and analytic for $X \in (0,\infty)$ and consists of two parts $\sigma_+ \cup \sigma_-$ corresponding to different asymptotic behaviour of the terms $\mathcal{E}_i(X)$: $\mathcal{E}_i(X \to \infty) \geq 0$ and $\mathcal{E}_-(X \to \infty) < 0$. Consequently for the real analytic basis functions one has: $\Phi = \Phi_+ \oplus \Phi_-$ that defines a direct-sum decomposition of the Hilbert fibres $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$. The functions Φ_+ correspond to the break-up surface states, Φ_- correspond to the cluster surface states. Using the relations of completeness and orthogonality (9) we define a fixed frame $|e(\hat{X})\rangle \equiv |\Phi(\mathring{X};X)\rangle$ picking some fixed point $X = \mathring{X} \in B$ in the base of the bundle $\mathcal{H}(B,\mathcal{F},\mathcal{T})$ where the projection $\mathcal{T}$ is a trivial one. Now for every pair of points

$\mathring{X}$ and X in B we can introduce the unitary bilocal operator $\mathcal{U}(X,\mathring{X})$ acting from $\mathcal{F}_{\mathring{X}}$ to $\mathcal{F}_X$:

$$|\Phi(X,\hat{X})\rangle = X^{-\nu} \mathring{X}^{\nu} |e(\hat{X})\rangle\, \mathcal{U}(X,\mathring{X}) \tag{12a}$$

establishing a parallel transport of the frame $|\Phi(X,\hat{X})\rangle$ from $\mathring{X}$ to arbitrary point X :

$$\mathcal{U}(X,\mathring{X}) = \mathcal{P} \exp i \int_{\mathring{X}}^{X} A(y)\, dy . \tag{12b}$$

The direct sum structure on the fibres $\mathcal{F}$ induced by a spectral projections $Q_\pm = \sum_j |\Phi_{j\pm}\rangle\langle\Phi_{j\pm}|$ extends naturally on the whole bundle $\mathcal{H}$:

$$\mathcal{T}(\Pi_\pm) = Q_\pm .$$

where $\Pi_\pm = 1/2 \cdot (\sigma_1 \pm i\sigma_2)$, σ_1, σ_2 are Pauli matrices. Accordingly for the transport operator one gets

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{++} & \mathcal{U}_{+-} \\ \mathcal{U}_{-+} & \mathcal{U}_{--} \end{pmatrix} . \tag{12c}$$

Let us rewrite the expansion (10) for the partial wave function Ψ_i explicitly separating the break-up states Ψ_i^+ from the cluster states Ψ_i^-

$$\Psi_i = \Psi_i^+ + \Psi_i^- = \sum_{j+} \Phi_{j+} X^{-1} \chi_{j+i} + \sum_{j} \Phi_{j-} X^{-1} \chi_{j-i} . \tag{10a}$$

Here the radial function $\chi \equiv \{\chi_{ji}(X,\mathcal{P})\}$ is also represented in the block form

$$\chi = \begin{pmatrix} \chi_{++} & \chi_{+-} \\ \chi_{-+} & \chi_{--} \end{pmatrix} .$$

and the diagonal momentum matrix $\mathcal{P}$ is given in the same form

$$\mathcal{P}=\begin{pmatrix}\mathcal{P}_+ & 0\\ 0 & \mathcal{P}_-\end{pmatrix}=\begin{pmatrix}\sqrt{E\otimes 1-\mathcal{E}_+(\infty)} & 0\\ 0 & \sqrt{E\otimes 1-\mathcal{E}_-(\infty)}\end{pmatrix}.$$

Here $\mathcal{P}_+$ is the diagonal matrix of the momentum of three free particles. Substitution of (10a) into the Schrödinger equation (2) leads to a block decomposition of the system of coupled differential equations (11)

$$\langle\Phi|\{-X^{-5}\partial_X X^5\partial_X+H^f(X;\hat{X})-E\}(1+\Pi_++\Pi_-)|\Psi\rangle_{L_2(S^5_X,\,d\Omega^\gamma)}= \tag{11b}$$

$$=\{-D^2(X)+\gamma(\gamma+1)X^{-2}+V(X)-\mathcal{P}^2\}\chi(X,\mathcal{P})=0.$$

Here we adopted the notation $D(X)=1\otimes\partial_X+A(X)$ for the covariant derivative, $A=-\tilde{A}=i(\mathcal{U}^{-1}\partial_X\mathcal{U}+\gamma X^{-1})$ is the connection operator in the bundle $\mathcal{H}$; $\mathcal{U}$ being the unitary operator (12) generating a parallel transport of the basis from the fiber $\mathcal{F}_\infty$ to the fibre $\mathcal{F}_X$; $V(X)=\mathcal{E}(X)-\mathcal{E}(\infty)$ is the effective potential energy operator; 1 being unit operator. Taking into consideration the block structure (12c) of the operator $\mathcal{U}$ we have

$$\begin{pmatrix}-D^2_{++}+V_{++}-\mathcal{P}_+^2 & -\partial_X A_{+-}-A_{+-}\partial_X-A^2_{+-}\\ -\partial_X A_{-+}-A_{-+}\partial_X-A^2_{-+} & -D^2_{--}+V_{--}-\mathcal{P}_-^2\end{pmatrix}\begin{pmatrix}\chi_{++} & \chi_{+-}\\ \chi_{-+} & \chi_{--}\end{pmatrix}=0. \tag{11c}$$

Going back to the properties of the adiabatic basis $\{\Phi_i\}$ we have to note that (at each point $X\in B$ of the base) it can be specified by a set of three exact quantum numbers $L=\{L,M,\xi\}$ and three approximate $i=\{i_1,i_2,i_3\}$ ones

whose meaning is established in the vicinities of the origin $\{X=0\}$ and infinite point $\{X=\infty\}$. The rule of correspondence between both sets of quantum numbers $i(0)$ and $i(\infty)$ (so-called correlation diagram) could be expressed in terms of the transport operator $\mathcal{U}(0,\infty)$.

b. Classification of the basis states in the limit $X\to 0$

In the limit of small X in eq.(7) the effective potential $\sum_\alpha V_\alpha$ can be considered as the perturbation. Then the basis functions $\Phi(X\to 0,\hat{X})\simeq X^{-\gamma}\Phi^{(0)}(0,X)$ are defined by a linear combination

$$\Phi_i^{(0)}=\sum_{\ell\lambda\in K}\sum_{\alpha}b^{i(0)}_{\alpha\,\ell\lambda}Y_K(\hat{X})=\sum_{\ell\lambda\in K}b^{i(0)}_{\ell\lambda}Y_K(\hat{X}) \tag{13a}$$

of free hyperspherical functions $Y_K\in L_2(S^5(\hat{X}))$ at a fixed eigenvalue E^0_K of the square of hypermomentum $\vec{\mathcal{K}}^2$:

$$\vec{\mathcal{K}}^2 Y_K(\hat{X})=E^0_K Y_K(\hat{X}),\quad E^0_K=\mathcal{K}(\mathcal{K}+4).$$

Here $-\vec{\mathcal{K}}^2$ equals the angular part of the Laplace operator on the sphere $S^5(\hat{X})$

$$\Delta_{\hat{X}}=\frac{4}{\sin^2 u}\frac{\partial}{\partial u}\sin^2 u\frac{\partial}{\partial u}-\frac{2\,\ell^2}{1-\cos u}-\frac{2\,\lambda^2}{1+\cos u}.$$

The five angles $\hat{X}=\{u,\hat{x},\hat{y}\}$ are given by the usual spherical angles $\hat{x},\hat{y}$ of the Jacobi vectors $\underset{\sim}{x}$, and $\underset{\sim}{y}$, and the doubled hyperangle $u=2v$, where v is the stereographic projection angle $v=\mathrm{arctg}\,(x/y)$, $0\le v\le\pi/2$ [15].

Note the doubled hyperangle like the above one was introduced in the framework of the three-body hyperspherical parametrization by V.A.Fock [16] and rediscovered by A.Kuppermann [17]. The set of hyperspherical quantum numbers $K=\{\mathcal{K}\ell\lambda\}$ is defined by orbital momentum ℓ of the pair α, orbital momentum λ of the third particle and hypermomentum $\mathcal{K}=2N+\ell+\lambda$. Here N equals the number of nodes (with respect to the variable u) of the free hyperspherical function:

$$Y_K(\hat{X}) = C_K \sin^\ell u/2 \cdot \cos^\lambda u/2 \cdot P_N^{(\ell+1/2,\, \lambda+1/2)}(\cos u)\, \mathcal{Y}^L_{\ell\lambda}(\hat{x},\hat{y}).$$

where C_K is the normalization constant and $\{\mathcal{Y}^L_{\ell\lambda}\}$ is the bispherical basis in $L_2(S^2(\hat{x})\otimes S^2(\hat{y}))$ [18]. The coefficients $b^{i(0)}_{\ell\lambda}$ in the linear combination (13a) are determined simultaneously with the second order correction to the energy

$$\mathcal{E}_{i(0)}(X) \underset{X\to\infty}{=} E^0_K X^{-2} + \mathcal{E}^{(2)}_{i(0)} + \ldots \tag{13b}$$

obtained from the secular equation

$$\sum_{\ell'\lambda'\in K} \{\langle \mathcal{K}\ell\lambda | \sum_\alpha V_\alpha | \mathcal{K}\ell'\lambda'\rangle - \delta_{\ell\ell'}\delta_{\lambda\lambda'}\mathcal{E}^{(2)}_{\mathcal{K}\ell'\lambda'}\}\, b^{i(0)}_{\ell'\lambda'} = 0 .$$

Thus, in the vicinity $U(X\to 0)$ the set i of asymptotic quantum numbers equals $i(0)=\{0,\mathcal{K},\nu=\nu(\ell,\lambda)\}$, where ν denotes roots of the secular equation with increasing $\mathcal{E}^{(2)}_{i(0)}$ at fixed $\mathcal{K}, L$. The degeneracy number ν_{max} equals $[(\mathcal{K}-L)/2+1](L+1)$ and $[\mathcal{K}-(L+1)/2+1]L$ for the basis states with the fixed parity $\sigma=\xi(-1)^L$ equal +1 and -1, respectively [19].

Note that the total inversion operator $\hat{I}$ on $M=R^1_+\times S^1(u)\times S^2(\hat{x})\times S(\hat{y})$ differs from the total parity operator $P_{\hat{x}\hat{y}}=P_{\hat{x}}\cdot P_{\hat{y}}$ on the four-dimensional torus $S^2(\hat{x})\times S^2(\hat{y})$:

$$\hat{I} = \hat{I}_u P_{\hat{x}\hat{y}} \hat{I}_u = \hat{I}_u P_{\hat{x}} P_{\hat{y}} \hat{I}_u , \tag{14}$$

where $\hat{I}_u: u=\pi-u$ is the reflection operator on the sphere $S^2(u,v)$, $v\equiv v_{\hat{x}\hat{y}}=v(\hat{x},\hat{y})$ [5]. The eigenvalues I and ξ of the operators $\hat{I}$ and $P_{\hat{x}\hat{y}}$ are different though formally their action coincides on the free hyperspherical functions Y_K

$$\hat{I}\,Y_K = I\,Y_K , \quad I=(-1)^{\mathcal{K}}=(-1)^{2N+\ell+\lambda}$$

and

$$P_{\hat{x}\hat{y}}\,Y_K = \xi Y_K , \quad \xi=(-1)^{\ell+\lambda}.$$

c. Classification in the limit $X\to\infty$

We obtain the leading terms of the asymptotic expansion of the basis functions $\Phi(X;\hat{X}) \simeq X^{-\delta}\Phi(\infty;\hat{X})$, exploiting the decomposition (8) in the limit $X\to\infty$. Thus, we reduce the eigenvalue problem (5) for the local basis components $\overset{3}{\underset{\alpha=1}{\oplus}} F_{\alpha i}$ as follows

$$\{h(X;\hat{X}) - 1\otimes\mathcal{E}_i(X)\}\{\overset{3}{\underset{\alpha=1}{\oplus}} F_{\alpha i}(X;\hat{X})\} = 0 . \tag{15a}$$

Here $h(X;\hat{X}) = 1_\alpha\otimes h_\alpha(X;X_\alpha)$ is the operator in the left hand side of the equations (5), while $h_\alpha(X;X_\alpha)$ is the pair Hamiltonian on the sphere $S^5_X(\hat{X}_\alpha)$:

$$h_\alpha(X;X_\alpha) = -X^{-2}\Delta_{\hat{X}_\alpha} + V_\alpha(X;u_\alpha). \tag{15b}$$

The spectrum $\sigma_\alpha(h_\alpha(X))$ of the operator h_α is a purely discrete and real-analytic and consists of two parts $\sigma^+_\alpha \cup \sigma^-_\alpha$

with $\mathcal{E}_i^+(X\to\infty)\geq 0$ and $\mathcal{E}^-(X\to\infty)<0$. The local basis components $F_{\alpha i^+}$, $F_{\alpha i^-}$ have different asymptotics on the sphere $S^5(\hat{X}_\alpha)$. The first one corresponds to the α-pair "scattering" states $\varphi_{\alpha\varepsilon^+\ell m}(X\to\infty,\omega_\alpha)$ with the energy $\mathcal{E}_\alpha^+=k_\alpha^2\geq 0$ while the second one is related to the α-pair bound-state $\varphi_{\alpha\varepsilon^-\ell m}(X\to\infty,\omega_\alpha)$ with the energy $\mathcal{E}_\alpha^-=-\varkappa_\alpha^2<0$. For these components $F_{\alpha i^\pm}(X\to\infty,\hat{X}_\alpha)$ the following representation holds:

$$F_{\alpha i^\pm}(X\to\infty,\hat{X}_\alpha)=\sum_m \varphi_{\alpha\varepsilon^\pm\ell m}(X\to\infty,\omega_\alpha)\,\mathcal{Z}^L_{\alpha\ell m\lambda}(\hat{y}_\alpha), \qquad (15c)$$

where the set $i^\pm$ equals $i^\pm=\{\alpha\varepsilon^\pm\ell\lambda\}$ and

$$\mathcal{Z}^L_{\alpha\ell m\lambda}(\hat{y}_\alpha)=\langle Y_{\ell m}(\hat{x}_\alpha)|\,\mathcal{Y}^L_{\ell\lambda}(\hat{x}_\alpha,\hat{y}_\alpha)\rangle\in L_2(S^2(\hat{y}_\alpha)).$$

Here $\{\mathcal{Y}^L_{\ell\lambda}(\hat{x},\hat{y})\}$ is the known bispherical basis for the parametrization of $\hat{M}=S^1\times S^2\times S^2$ [18], or is the rotation bipolar basis for the alternative parametrization of $\hat{M}=S^3\times S^2$ [19]. The coefficients $\varphi_{\alpha\varepsilon^\pm\ell m}(X\to\infty,\omega_\alpha)$, where $\omega_\alpha\in S^1(u_\alpha)\times S^2(\hat{x}_\alpha)$, in the limit $X\to\infty$ correspond to the pair wave functions $\varphi_A(\hat{x}_\alpha)\equiv|A\rangle=|\alpha k\ell m\rangle$ and $|A\rangle=|\alpha n\ell m\rangle$ of the pair Hamiltonian in the Jacobi parametrization $\underset{\sim}{x}_\alpha\in R^1_+(x_\alpha)\times S^2(\hat{x}_\alpha)$

$$(-\Delta_{\underset{\sim}{x}_\alpha}+V_\alpha(x_\alpha)-\mathcal{E}^\pm)\,\varphi_A(\underset{\sim}{x}_\alpha)=0 \qquad (15d)$$

describing both the very scattering states (with energy $\mathcal{E}_A^+$) and the bound states (the energy $\mathcal{E}_A^-$) of the pair α, n being the principal quantum number [11].

Among the "scattering" states $F_{\alpha i^+}$ with positive energy $\mathcal{E}_i^+(X\to\infty)\geq 0$ the zero energy states $F_{\alpha i^0}=F^0_{\alpha i}$ corresponding to zero pair energy $(\mathcal{E}_\alpha^+=0)$ play a special role. These states will represent "truly-break-up" states on the sphere $S^5(\hat{X})$. They correspond to the free hyperspherical functions distorted by the pair potentials $V_\alpha(X,u_\alpha)$ and therefore are characterized like (13a) by the set of quantum numbers $i^0=\{0,\mathcal{K},\nu(\ell,\lambda)\}$. Thus we have three sorts of states $\{|i^+\rangle=\Phi_{i^+},\ |i^0\rangle=\Phi_{i^0}\}$ and $|i^-\rangle=\Phi_{i^-}$ belonging to $L_2(S^5(\hat{X}))$, and respectively three sets of asymptotic quantum numbers $\{i^+=\{\alpha k\ell,\lambda\}$, $i^0=\{0\mathcal{K}\nu(\ell,\lambda)\}\}$ and $i^-=\{\alpha n\ell\lambda\}$. They exhaust the total asymptotic set $i(\infty)$ that is necessary for labelling the spectrum $\sigma(H^f(X))=\bigcup_\alpha\sigma_\alpha(h_\alpha(X))$ of the Hamiltonian $H^f(X)$ in the vicinity of $X\to\infty$. Note that the "truly-break-up" surface components $F^0_{\alpha i^+}$ for the states $|i^+\rangle$ with the energy $\mathcal{E}_\alpha^+>0$ may be introduced too. Those practically coincide with the free hyperspherical functions Y_K with such quantum numbers of the hypermomentum $\mathcal{K}\geq\mathcal{K}^0\sim X$ that the limiting values of the centrifugal energy $\mathcal{E}_i^+=E^0_K X^{-2}\Longrightarrow k_\alpha^2>0$. The above components $F^0_{\alpha i^+}$ correspond to free solutions φ^0_A of the equation (15d) with the fixed energy $\mathcal{E}_\alpha^+=k_\alpha^2>0$. Thus, we obtain explicitly the "rescattering". surface components $\Delta F_{\alpha i^+}=F_{\alpha i^+}-F^0_{\alpha i^+}$ corresponding to ones $\Delta\varphi_A=\varphi_A-\varphi^0_A$ in the above Jacobi parametrization.

d. Consistency conditions for connection form coefficients diag A^2

Now we need to fit the asymptotic boundary conditions for the Faddeev components F_α with the behaviour of the Schrödinger wave function Ψ in its adiabatic representation (10a). This requirement imposes on the perturbative correction coefficients Ξ^+, Ξ^0, Ξ^- in the asymptotic behaviour of the terms $\mathcal{E}_i^\pm(X)$ at $X \to \infty$ (see (17)) the following asymptotic relations:

$$\text{I)}\quad \Xi^+ + \gamma(\gamma+1) - \lim_{X\to\infty} \{X^2 \mathrm{diag}\, A^2_{++}(X)\ |\mathcal{K} \geq \mathcal{K}^0 \sim X\} = E^0_\lambda \qquad (16)$$

$$\text{II)}\quad \Xi^0 + \gamma(\gamma+1) - \lim_{X\to\infty} \{X^2 \mathrm{diag}\, A^2_{00}(X)\ |\ \mathcal{K}^- < \mathcal{K} < \mathcal{K}^0\} = E^0_K$$

$$\text{III)}\quad \Xi^- + \gamma(\gamma+1) - \lim_{X\to\infty} \{X^2 \mathrm{diag}\, A^2_{--}(X)\ |\ 0 \leq \mathcal{K} \leq \mathcal{K}^-\} = E^0_\lambda. \qquad (16)$$

Here the quantities $E^0_\lambda = \lambda(\lambda+1)$ are the eigenvalues of the operator $\underset{\sim}{\lambda}^2$ on the sphere $S^2(\hat{y})$ and $E^0_K = \mathcal{K}(\mathcal{K}+4)$ are eigenvalues of the operator $\vec{\mathcal{K}}^2 = -\Delta_{\hat{X}}$ on the sphere $S^5(\hat{X})$. Moreover the latter defines the asymptotic behaviour (13b) of $\mathcal{E}^+(X)$ and $\mathcal{E}^-(X)$ in the limit $X \to 0$. As a result, the diagonal matrix elements of the connection operator $A(X)$ squared which change the phase of the wave function $\chi(X)$, have a consistent asymptotics with those of the energy terms $\mathcal{E}(X)$ in the limit $X \to \infty$. The asymptotic behaviour of $\mathcal{E}^\pm(X)$ is given by

$$\mathcal{E}^+(X) \underset{X\to\infty}{\longrightarrow} \begin{cases} \varepsilon^+_\alpha + \Xi^+_\alpha X^{-2} \\ \qquad\ \ \Xi^0_\alpha X^{-2} \end{cases}$$

$$\mathcal{E}^-(X) \underset{X\to\infty}{\longrightarrow} (\varepsilon^-_\alpha + \Xi^-_\alpha X^{-2}) \qquad (17)$$

up to order $O(X^{-2})$ provided that conditions (16) are satisfied.

e. Asymptotic states

The introduced classification of the states $|i\rangle$ and the structure $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ allows one to get for the adiabatic expansion (10) in the limit $X \to \infty$ (L fixed) the following "direct-sum" form

$$\{\Psi_i^{as}\} = \{\Psi_i^{+as}\} \oplus \{\Psi_i^{-as}\} = \{\{\Psi_i^{+as}\} \oplus \{\Psi_i^{0as}\}\} \oplus \{\Psi_i^{-as}\}$$

$$\begin{aligned}\Psi_i^{as}(\underset{\sim}{X}, \mathcal{P}) \cong \Big\{ &\sum_{j^+} \Phi_{j^+}(X\to\infty, \hat{X})\, X^{-1} \chi^{as}_{j^+i}(X, \mathcal{P}) + \\ &+ \sum_{j^0} \Phi_{j^0}(\infty, \hat{X})\, X^{(-1-\gamma)} \chi^{as}_{j^0 i}(X, \mathcal{P}) + \\ &+ \sum_{j^-} \Phi_{j^-}(\infty, \hat{X})\, X^{-1} \chi^{as}_{j^- i}(X, \mathcal{P}). \end{aligned} \qquad (18)$$

Here the first multi-index j of the radial function χ corresponds to the channel component, the second one i is labelling the initial state of the system. The consistency conditions (16) and (17) lead to the fact that the asymptotics of the radial solutions χ_{+i} and χ_{-i} of the system (11c) as $X \to \infty$ is determined only by the centrifugal barrier $E^0_\lambda X^{-2}$ (with respect to the relative motion of the third particle and the pair α). In this case the dependence on γ disappears except for χ_{0i}

$$\chi^{as}_{+i} \underset{X\to\infty}{\approx} (-2i)^{-1} \{\exp(-iX\mathcal{P}_+)\delta_{+i} - \exp(iX\mathcal{P}_+)\mathcal{P}_+^{-1/2} \hat{S}^{as}_{+i} \mathcal{P}_i^{1/2}\},$$

$$\chi^{as}_{0i} \underset{X\to\infty}{\approx} (-2i)^{-1} \{\exp(-i(X\mathcal{P}_0 - \tfrac{\pi\gamma}{2}))\delta_{0i} - \exp(i(X\mathcal{P}_0 - \tfrac{\pi\gamma}{2}))\mathcal{P}_0^{-1/2} \hat{S}^{as}_{0i} \mathcal{P}_i^{1/2}\},$$

$$\chi^{as}_{-i} \underset{X\to\infty}{\approx} (-2i)^{-1} \{\exp(-iX\mathcal{P}_-)\delta_{-i} - \exp(iX\mathcal{P}_-)\mathcal{P}_-^{-1/2} \hat{S}^{as}_{-i} \mathcal{P}_i^{1/2}\}. \qquad (19)$$

Thus, the asymptotic decomposition of Ψ allows one to separate the Ψ^+ of total break-up channel with energy $E(P)=P^2=P_d^2+\varepsilon_d\in[0,\infty)$ from the states Ψ^- of the clusterization channel $E=E_A(P_A)=P_A^2+\varepsilon_A^-\in[-\varkappa_A^2,\infty)$. Remind that $\underset{\sim}{P}_A$ is the momentum of the relative motion of the third particle with respect to the pair d. Moreover the extraction of the "truly-break-up" states $\Psi_i^0(\varepsilon_d^+\geq 0)$ from Ψ_i^+ according to the latter assertion of subsection 3c makes possible to single out explicitly the contributions of the known rescattering processes, thus defining the three-particle scattering S-matrix only on the physical states.

4. Radial Solutions

To investigate the structure of the solutions χ we now turn to the ordinary procedure exploring the Jost matrix solutions. First, we introduce the regular solutions χ^{reg} for the system of radial equations (11b) with boundary conditions

$$\lim_{X\to 0}\chi^{reg}(X,\mathcal{P})X^{-(\mathcal{K}+\gamma+1)}=\mathbb{1}.$$

This condition follows from the asymptotic behaviour for $\mathcal{E}_i(X\to 0)$ $(V(X)\approx\mathcal{K}(\mathcal{K}+4)X^{-2})$ the regularity of $A(X)$ and the derivative $(d/dX)\chi(X,\mathcal{P})$ as $X\to 0$. The matrix of regular solutions can be represented in the form of a linear combination of the Jost solutions

$$\chi^{reg}(X,\mathcal{P})=(-2i)^{-1}\{F_-(X,\mathcal{P})\mathcal{P}^{-1}F_+(\mathcal{P})-F_+(X,\mathcal{P})\mathcal{P}^{-1}F_-(\mathcal{P})\} \tag{20}$$

that satisfy the system of equations (11b) and the boundary conditions

$$F_\pm(X,\mathcal{P})\xrightarrow[X\mathcal{P}\to\infty]{}\exp\{\pm i(X\mathcal{P}-\mathbb{1}\pi\gamma/2)\}.$$

Hereafter the signs plus and minus should correspond to outgoing and incoming waves. The Jost matrix function by definition is equal to

$$F_\pm(\mathcal{P})=W_D\{F_\pm,\chi^{reg}\}=\tilde{F}_\pm D\chi^{reg}-\{\widetilde{DF_\pm}\}\chi^{reg}.$$

Here the notation "$\sim$" means transposition; $F_\pm(\mathcal{P})$ is related with the asymptotics $F_\pm(X,\mathcal{P})$ by

$$F_\pm(\mathcal{P})=2(\mathcal{K}+\gamma+1/2)\lim_{X\to 0}X^{\mathcal{K}+\gamma}\tilde{F}_\pm(X,\mathcal{P}).$$

Thus the regular solutions have the asymptotic behaviour

$$\chi^{reg}\xrightarrow[X\mathcal{P}\to\infty]{}(-2i)^{-1}\{e^{-i(X\mathcal{P}-\mathbb{1}\cdot\pi\gamma/2)}\mathcal{P}^{-1}F_+(\mathcal{P})-e^{i(X\mathcal{P}-\mathbb{1}\cdot\pi\gamma/2)}\mathcal{P}^{-1}F_-(\mathcal{P})\}. \tag{20a}$$

The physical solutions are expressed through the regular ones (20) in the usual way

$$\chi^{Ph}=\chi^{reg}F_+^{-1}(\mathcal{P})\mathcal{P}=-(2i)^{-1}\{F_-(X,\mathcal{P})-F_+(X,\mathcal{P})\bar{S}(\mathcal{P})\}\to \tag{21}$$
$$\xrightarrow[X\mathcal{P}\to\infty]{}(-2i)^{-1}\{e^{-i(X\mathcal{P}-\mathbb{1}\cdot\pi\gamma/2)}-e^{i(X\mathcal{P}-\mathbb{1}\cdot\pi\gamma/2)}\bar{S}(\mathcal{P})\}$$

The Jost matrix functions define the S matrix

$$\bar{S}(\mathcal{P})=\mathcal{P}^{-1}F_-(\mathcal{P})[F_+(\mathcal{P})]^{-1}\mathcal{P}. \tag{22}$$

From the Wronskian definition $W_D\{\chi^{reg},\chi^{reg}\}=0$ we have the known relations

$$\tilde{F}_+ \mathcal{P}^{-1} F_- = \tilde{F}_- \mathcal{P}^{-1} F_+ \; ; \quad \mathcal{P}^{-1} F_- F_+^{-1} \mathcal{P} = \tilde{F}_+^{-1} \tilde{F}_- .$$

From here one can introduce the symmetric $\hat{S} \equiv \hat{S}(\mathcal{P})$ matrix

$$\hat{S} = \mathcal{P}^{1/2} \bar{S} \mathcal{P}^{-1/2}, \qquad \hat{S} = \tilde{\hat{S}}. \tag{23}$$

As a result, from the symmetric S-matrix via projection onto the open channels, one gets the unitary scattering operator

$$\hat{S}^+ . \hat{S} = \hat{S} . \hat{S}^+ = 1 . \tag{24}$$

The relevant physical solutions are

$$\chi^{Ph} = -(2i)^{-1} \{ F_-(X,\mathcal{P}) - F_+(X,\mathcal{P}) \mathcal{P}^{-1/2} \hat{S} \mathcal{P}^{1/2} \} \tag{25}$$

with the asymptotics

$$\chi^{Ph} \underset{X\to\infty}{\longrightarrow} -(2i)^{-1} \{ e^{-i(X\mathcal{P} - 1\cdot\pi\vartheta/2)} - e^{i(X\mathcal{P}-1\cdot\pi\vartheta/2)} \mathcal{P}^{-1/2} \hat{S} \mathcal{P}^{1/2} \} \tag{25a}$$

The imposing of boundary conditions (25a) leads to the conservation of the radial current

$$\partial_X J = 0 \; ; \quad J = -i W_D \{ \tilde{\chi}^{Ph}, \chi^{Ph} \} \tag{26}$$

i.e. it guarantees the conservation of the probability flow we need for calculation of the cross sections. It is worth to note here that the arbitrariness of unitary gauge: $\chi \to u\chi \equiv \chi'$ leaves (26) invariant, or in other words $\partial_X J = 0$ is also a gauge invariant equality. Moreover, the gauge freedom allows one to reduce the system (11b) to a standard form

$$\{ -\partial_X^2 + u V u^{-1} + 1 . \vartheta(\vartheta+1) . X^{-2} - \mathcal{P}^2 \} \chi' = 0 \tag{27}$$

and to use the known procedure for proving the orthogonality and completeness relations for the solutions χ'^{Ph}

$$2/\pi \int_0^\infty \tilde{\chi}'^{*Ph}(X,\mathcal{P}) . \chi'^{Ph}(X,\mathcal{P}')\, dX = \delta(\mathcal{P}-\mathcal{P}'),$$

$$2/\pi \int_0^\infty \chi'^{Ph}(X,\mathcal{P}) \tilde{\chi}'^{*Ph}(X',\mathcal{P})\, d\mathcal{P} + \sum_\nu \chi_\nu'^{Ph}(X) \tilde{\chi}_\nu'^{Ph}(X') = 1 \cdot \delta(X-X') \tag{28}$$

Index ν labels for a discrete spectrum $\sigma_d(H)$ of the three particle Hamiltonian H .

For the parametrization of S-matrix it is more convenient to pass to the so-called eigen-phase-shifts representation δ, $\operatorname{Im} \delta = 0$:

$$\hat{S} . B = B e^{2i\delta} , \qquad F_\pm(\mathcal{P}) = \mathcal{P}^{1/2} B e^{\mp i\delta} .$$

where B is an orthogonal matrix $B = \mathcal{P}^{-1/2} \operatorname{Re}(F_\pm(\mathcal{P}) e^{\pm i\delta})$.

The physical wave function Ψ^{Ph} (10) with χ^{Ph} (25), in the eigen-phase-shift representation

$$\Psi^{Ph\,e} = \Psi^{Ph} B = -(2i)^{-1} X^{-1} \Phi(X,\hat{X}) \{ F_-(X,\mathcal{P}) B - F_+(X,\mathcal{P}) B e^{2i\delta} \}$$

$$\underset{X\to\infty}{\longrightarrow} -(2i)^{-1} X^{-1} \Phi(X,\hat{X}) \{ e^{-i(X\mathcal{P} - 1\cdot\pi\vartheta/2)} B - e^{i(X\mathcal{P} - 1\cdot\pi\vartheta/2)} B e^{2i\delta} \}$$

contains incoming waves in all channels and the coefficients B play the role of mixing parameters. Via the Cayley transformation $S = (1 + iK) . (1 - iK)^{-1}$ one can go to the real symmetric K-matrix instead of S-matrix: $KB = B \operatorname{tg} \delta$.

Note that the solution of (11b) with asymptotics (25a) in the form:

$$\chi_{+i}^{Ph} \longrightarrow -(2i)^{-1} \{ \exp(-i(X\mathcal{P}_+ - 1\pi\vartheta/2)) \delta_{+i} - \exp i(X\mathcal{P}_+ - 1\pi\vartheta/2) \mathcal{P}_+^{-1/2} \hat{S}_{+i} \mathcal{P}_i^{1/2} \}$$

$$\chi_{-i}^{Ph} \longrightarrow -(2i)^{-1} \{ \exp(-i(X\mathcal{P}_- - 1\cdot\pi\vartheta/2)) \delta_{-i} - \exp i(X\mathcal{P}_- - 1\cdot\pi\vartheta/2) \mathcal{P}_-^{-1/2} \hat{S}_{-i} \mathcal{P}_i^{1/2} \} \tag{25b}$$

allows one to get the partial unitary $\hat{S}$-matrix (23), (24) for the three particle system in the presence of break-up (+) and rearrangement (-) channels. According to (22) and (21) for both break-up and rearrangement channels the S-matrix is defined uniquely by the amplitude of the incoming wave F_+^{-1} propagating from $|in\rangle$ state to the vicinity of the triple collision point ($X=0$) where the states are mixed and by the amplitude of outgoing wave F_- propagating to the $|out\rangle$ states.

5. Scattering Amplitude

The total wave function Ψ^{ph} describing all the possible three-body scattering processes in $M = \mathbb{R}_+^1 \times \hat{M}$ has the form

$$\Psi^{Ph}(\underset{\sim}{X}, \underset{\sim}{P}_\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{L\xi' a' a} \Phi^L_{\xi' a'}(X, \hat{X}) X^{-1} \chi_{\xi' a' \xi a}(X, \mathcal{P}) \Phi^{*L}_{\xi a}(\infty, -\hat{P}_\xi) \mathcal{P}_\xi^{-1-\nu}, \tag{29}$$

where $\xi = k_\alpha$ for the "scattering" states $\Phi_{k_\alpha a}$ and $\xi = n_\alpha$ for the cluster states $\Phi_{n_\alpha a}$ in accordance with the complete classification of the basis states $\{\Phi^L_{\xi a}\} \in L_2(\hat{M})$ introduced in Section 3. Using the asymptotic behaviour of the physical solutions (25a,b) we may rewrite (29) as $X\mathcal{P}_\xi \to \infty$ tends to infinity in the form

$$\Psi^{Ph}(\underset{\sim}{X}, \underset{\sim}{P}_\xi) \underset{X\mathcal{P}_\xi}{\approx} (2\pi)^{-3/2-\nu} \{\chi(\nu, \underset{\sim}{X}, \underset{\sim}{P}_\xi) + \tag{30}$$

$$+ \sum_{\xi'} X^{-1-\nu} \exp(i(X\mathcal{P}_{\xi'} - \pi\nu/2)) \mathcal{P}_{\xi'}^{-1/2} f_{\xi'\xi}(\hat{X}, \underset{\sim}{P}_\xi) \mathcal{P}_\xi^{1/2}\}.$$

Here we explicitly extract the incoming wave in M

$$(2\pi)^{-3/2-\nu} \chi(\nu, \underset{\sim}{X}, \underset{\sim}{P}_\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{La} \Phi^L_{\xi a}(\infty, \hat{X}) X^{-1-\nu}.$$

$$(-2i)^{-1}\{e^{-i(X\mathcal{P}_\xi - \frac{\pi\nu}{2})} - e^{i(X\mathcal{P}_\xi - \frac{\pi\nu}{2})}\hat{I}\}\Phi^{*L}_{\xi a}(\infty, -\hat{P}_\xi)\mathcal{P}_\xi^{-1-\nu} \tag{31}$$

and accordingly give the relationship between the transition amplitude $f_{\xi'\xi}(\hat{X}, \underset{\sim}{P}_\xi)$ from the state ξ to the state ξ' of the three-body system and partials amplitudes $\hat{f}_{\xi' a' \xi a}(\mathcal{P})$

$$f_{\xi'\xi}(\mathcal{P}) = 4\pi \sum_{La'a} \Phi^L_{\xi' a'}(\infty, \hat{X}) \hat{f}^L_{\xi' a' \xi a}(\mathcal{P}) \Phi^{*L}_{\xi a}(\infty, -\hat{P}_\xi) \mathcal{P}_\xi^{-\nu}. \tag{32}$$

The invariant unitary partial amplitudes

$$\hat{f}^L_{\xi' a' \xi a}(\mathcal{P}) = \{\hat{S}^L(\mathcal{P}) - 1 \otimes \hat{I}\}_{\xi' a' \xi a} (2i\mathcal{P}_\xi)^{-1} \tag{33}$$

are defined by matrix elements (23), (24) of the unitary scattering operator $\hat{S} = S \cdot \hat{I}$, where $\hat{I}$ is the total inversion operator (14) acting on the $L_2(\hat{M})$ and the S-matrix acts as an integral operator on the $L_2(\hat{M})$.

The physical wave function $\Psi^{Ph}(\underset{\sim}{X}, \underset{\sim}{P}_\xi)$ describing the $3 \to 3$ and $3 \to 2$ processes with the three free particles in the initial state $\Phi_{0a}, \xi = k_\alpha = 0$, at the fixed direction $\hat{P}_0$ of the momentum $\underset{\sim}{P}_0$ in M has the form

$$\Psi_0^{Ph}(\underset{\sim}{X}, \underset{\sim}{P}_0) \underset{X\mathcal{P}_0 \to \infty}{\approx} (2\pi)^{-3/2-\nu}\{\chi_0(\nu, \underset{\sim}{X}, \underset{\sim}{P}_0) +$$

$$+ X^{-1-\nu} e^{i(X\mathcal{P}_0 - \pi\nu/2)} \mathcal{P}_0^{-1/2} f_{00}(\hat{X}, \underset{\sim}{P}_0) \mathcal{P}_0^{1/2} + \tag{34}$$

$$+\sum_{k_\beta} X^{-1-\vartheta} e^{i(X\mathcal{P}_{k_\beta}-\pi\vartheta/2)}\mathcal{P}_{k_\beta}^{-1/2} f_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)\mathcal{P}_0^{1/2}+$$

$$+\sum_{B}\varphi_B(X\to\infty,\omega_\beta)X^{-1}e^{i(X\mathcal{P}_{n_\beta}-\pi\vartheta/2)}\mathcal{P}_{n_\beta}^{-1/2} f_{B0}(\hat{y}_B,\underset{\sim}{P}_0)\mathcal{P}_0^{1/2}\}\,.$$

Here the incoming wave (31) corresponds to the plane six-dimensional wave in M

$$(2\pi)^{-3/2-\vartheta}\chi_0(\vartheta,\underset{\sim}{X},\underset{\sim}{P}_0)\underset{X\mathcal{P}_0\to\infty}{\approx}\left(\frac{2}{\pi}\right)^{1/2}X^{-1-\vartheta}(2i)^{-1}\cdot$$

$$\cdot\{e^{-i(X\mathcal{P}_0-\pi\vartheta/2)}\delta(\hat{X}+\hat{P}_0)-e^{i(X\mathcal{P}_0-\pi\vartheta/2)}\delta(\hat{X}-\hat{P}_0)\}\mathcal{P}_0^{-1-\vartheta}$$

taking into account the completeness relation (9b), $f_{B0}(\hat{y}_\beta,\underset{\sim}{P}_0)$ is the $3\to2$ transition amplitude of the outgoing cluster wave in M

$$f_{B0}(\hat{y}_\beta,\underset{\sim}{P}_0)=4\pi\sum_{L\lambda,a}\mathcal{Z}^{L}_{\beta\ell m\lambda}(\hat{y}_\beta)\hat{f}^{L}_{n\ell\lambda,0a}(\mathcal{P})\Phi^{*L}_{0a}(\infty,-\hat{P}_0) \qquad (35)$$

according to the definition (15c). The transition amplitudes $f_{00}(\hat{X},\underset{\sim}{P}_0)$ and $f_{k_\beta 0}(\hat{X},P_0)=f_{+0}$ are used for describing the scattering processes $(3\to3)$ with $\mathcal{E}_\beta=0$ and $\mathcal{E}^+_\beta>0$ respectively

$$f_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)=4\pi\sum_{La a'}\Phi^{L}_{k_\beta a'}(\infty,\hat{X})\hat{f}^{L}_{k_\beta a',0a}(\mathcal{P})\Phi^{*L}_{0a}(\infty,-\underset{\sim}{P}_0)\mathcal{P}_0^{-\vartheta}\,.$$

In the adiabatic scattering picture the information about the three-body interaction is contained not only in eigenphases δ and mixing parameters B or in the unitary scattering operator $\hat{S}=S\cdot\hat{I}$, but also in the asymptotic quasiangular adiabatic basis functions, which form it's matrix elements (32).

One may explicitly pick out the contribution of the known rescattering processes if one will exploite the representation of the "trully-break-up" surface functions in accordance with Section 3c-e. The amplitude $f_{00}(\hat{X},\underset{\sim}{P}_0)$ corresponds the "trully-break-up" scattering. A straightforward approach compatible with the conservation of the radial current (26) consists in a simple extraction of the "truly-break-up" amplitude

$$f^{0}_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)=4\pi\sum_{La a'}\Phi^{(0)L}_{k_\beta a'}(\infty,\hat{X})\hat{f}^{L}_{k_\beta a',a0}(\mathcal{P})\Phi^{*L}_{0a}(\infty,-\hat{P}_0)\mathcal{P}_0^{-\vartheta}$$

from the $3\to3$ transition one $f_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)$ and implicitly definition of the rescattering amplitude

$$\Delta f_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)=f_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)-f^{0}_{k_\beta 0}(\hat{X},\underset{\sim}{P}_0)\,.$$

It is worth noting that the explicit rescattering picture corresponding to above one is obtained as usual via simple iterations of the Faddeev equations taking into account the consistency conditions (16) and the asymptotic radial solution (19).

The physical wave function $\Psi^{Ph}_A(\underset{\sim}{X},\underset{\sim}{P}_A)$ describing the processes $2\to2$ and $2\to3$ in the configuration space M with a cluster α in the initial state $|A\rangle$ is obtained in the limit $X\mathcal{P}_{n_\alpha}\to\infty$ from (29) via averaging on the angles ω_α in the asymptotic state $|A\rangle$

$$\Psi^{Ph}_A(\underset{\sim}{X},\underset{\sim}{P}_A)=\int d\omega_\alpha\Psi^{Ph}(\underset{\sim}{X},\underset{\sim}{P}_{n_\alpha})\mathcal{P}_{n_\alpha}^{-\vartheta}\varphi_A(\infty,\omega_\alpha)\,. \qquad (36)$$

Taking into consideration both (15c), (25a) and (30)-(33) one gets

$$\Psi_A^{Ph}(\underset{\sim}{X}, \underset{\sim}{P}_A) \underset{X\mathcal{P}_{n_\alpha}\to\infty}{\approx} (2\pi)^{-3/2} \{ X_A(\gamma, \underset{\sim}{X}, \underset{\sim}{P}_A) +$$
$$+\sum_B \varphi_B(X\to\infty, w_\beta) X^{-1} e^{i(X\mathcal{P}_{n_\beta} - \pi\gamma/2)} \mathcal{P}_{n_\beta}^{-1/2} f_{BA}(\hat{y}_\beta, \underset{\sim}{P}_A) \mathcal{P}_{n_\alpha}^{1/2}$$
$$+\sum_{k_\beta} X^{-1-\nu} e^{i(X\mathcal{P}_{k_\beta} - \pi\gamma/2)} \mathcal{P}_{k_\beta}^{-1/2} f_{k_\beta A}(\hat{X}, \underset{\sim}{P}_A) \mathcal{P}_{n_\alpha}^{1/2} \}. \tag{37}$$

Here the incoming cluster wave

$$(2\pi)^{-3/2} X_A(\gamma, \underset{\sim}{X}, P_A) \underset{X\mathcal{P}_{n_\alpha}\to\infty}{\approx} \left(\frac{2}{\pi}\right)^{1/2} \varphi_A(X\to\infty, w_\alpha)(-2iX)^{-1}$$
$$\sum_{L\lambda} \mathcal{Y}^L_{\alpha\ell m\lambda}(\hat{y}_\alpha)\{e^{-i(X\mathcal{P}_{n_\alpha} - \pi\gamma/2)} - e^{i(X\mathcal{P}_{n_\alpha} - \pi\gamma/2)}\hat{I}\} \mathcal{Y}^{*L}_{\alpha\ell m\lambda}(-\hat{P}_A)\mathcal{P}_{n_\alpha}^{-1} I_{AA} \tag{38}$$

appears in M after the projection of (31) on $|A\rangle$ state, $f_{BA}(\hat{y}_\beta, \underset{\sim}{P}_A)$ is the $2\to2$ transition amplitude of the outgoing cluster spherical wave

$$f_{BA}(\hat{y}_\beta, \underset{\sim}{P}_A) = 4\pi \sum_{L\lambda_\alpha\lambda_\beta} \mathcal{Y}^L_{\ell_\beta m_\beta\lambda_\beta}(\hat{y}_\beta) f^L_{n_\beta\ell_\beta\lambda_\beta, n_\alpha\ell_\alpha\lambda_\alpha}(\mathcal{P}) \mathcal{Y}^{*L}_{\ell_\alpha m_\alpha\lambda_\alpha}(-\underset{\sim}{P}_A) I_{AA} \tag{39}$$

$f_{k_\beta A}(\hat{X}, \underset{\sim}{P}_A)$ is the $2\to3$ transition amplitude of the outgoing six-dimensional spherical wave

$$f_{k_\beta A}(\hat{X}, \underset{\sim}{P}_A) = 4\pi \sum_{L\lambda_\alpha d'} \Phi^L_{k_\beta d'}(\infty, \hat{X}) \hat{f}^L_{k_\beta d', n_\alpha\ell_\alpha\lambda_\alpha}(\mathcal{P}) \mathcal{Y}^{*L}_{\ell_\alpha m_\alpha\lambda_\alpha}(-\hat{P}_A) I_{AA}. \tag{40}$$

The latter transition amplitudes are defined via the invariant unitary partial ones (33) and the matrix elements of the inversion operator $\hat{I}_{AA}$ (14) acting on the subspace $L_2(S^3(w_\alpha))$ spanned over the set of functions (15c). The incoming cluster wave (38) corresponds to the usual one in the Jacobi mapping $R^6 \setminus \{0\}$

if the quantity γ gets equal to zero. Actually eq.(38) can be rewritten in the suitable form

$$(2\pi)^{-3/2} X_A(\gamma, \underset{\sim}{X}, \underset{\sim}{P}_A) \underset{X\mathcal{P}_{n_\alpha}\to\infty}{\approx} \left(\frac{2}{\pi}\right)^{1/2} \varphi_A(X\to\infty, w_\alpha)(-2iX)^{-1}$$
$$\{e^{-i(X\mathcal{P}_{n_\alpha} - \pi\gamma/2)}\delta(\hat{y}_\alpha + \hat{P}_A) I_{AA} - e^{i(X\mathcal{P}_{n_\alpha} - \pi\gamma/2)}\delta(\hat{y}_\alpha - \hat{P}_A)\}\mathcal{P}_{n_\alpha}^{-1} \tag{38a}$$

taking into consideration the completeness relations

$$\sum_{L\lambda} \mathcal{Y}^L_{\alpha\ell m\lambda}(\hat{y}_\alpha) \mathcal{Y}^{*L}_{\alpha\ell m'\lambda}(\hat{P}_A) = \delta(\hat{y}_\alpha - \hat{P}_A)\delta_{m'm}.$$

The consistency conditions (16) lead to the fact,that the above dependence on γ disappears and the stiking of the solutions χ^{Ph}_{--} (25b) and χ^{as}_{--} (19) in the vicinity of $X\mathcal{P}\to\infty$ can be used to establish a relationship

$$\hat{S}^{as}_{--} = \hat{S}_{--} e^{-i\pi\gamma} = \hat{S}_{--} e^{-\frac{i}{2}\operatorname{Im}\oint_C A(X)dX}. \tag{41}$$

Thus, we have the manifestation of the geometrical phase of the Berry type [20] in the three-body scattering problem because the factor $-i\pi\gamma$ is produced by the nonzero curvature of the non-trivial diagonal part $A(X) = i\langle\varphi|\partial_X|\varphi\rangle$ of the connection $A(X)$. In our case the curvature is defined as one-half of the integral

$$\frac{1}{2}\operatorname{Im}\oint_C \langle\varphi|\partial_X|\varphi\rangle dX = -\frac{1}{2}\operatorname{Im}\oint_C \frac{\gamma}{X} dX = -\pi\gamma$$

along the close circle C around the triple collision point in the complex plane of the slow variable X. From the physical point of view the above difference between $\hat{S}^{as}$ and $\hat{S}$ in the $2\to2$ sector of the three-body S-matrix is connected with the elastic scattering on the additional centrifugal barrier $\gamma(\gamma+1)X^{-2}$ in the vicinity of the triple collision point which is obviously absent in the usual two-body scattering. As a result we have an asymptotic wave function $\Psi_A^{as}(\underset{\sim}{X},\underset{\sim}{P}_A)$ whose behaviour is adjusted to the asymptotic boundary conditions for the Faddeev components $F_{\beta A}(\underset{\sim}{X},\underset{\sim}{P}_A)$

$$\Psi_A^{as}(\underset{\sim}{X},\underset{\sim}{P}_A) \underset{X\mathcal{P}\to\infty}{\approx} (2\pi)^{-3/2}\chi_A(\gamma=0,\underset{\sim}{X},\underset{\sim}{P}_A)+\sum_{\beta}F_{\beta A}^{as}(\underset{\sim}{X},\underset{\sim}{P}_A)$$
$$\approx (2\pi)^{-3/2}\Big\{\chi_A(\gamma=0,\underset{\sim}{X},\underset{\sim}{P}_A)+$$
$$+\sum_B \varphi_B(X\to\infty,\omega_\beta)X^{-1}e^{iX\mathcal{P}_{n_\beta}}\mathcal{P}_{n_\beta}^{-1/2}f_{BA}^{as}(\hat{y}_\beta,\underset{\sim}{P}_A)\mathcal{P}_{n_\alpha}^{1/2}+$$
$$+\sum_{k_\beta}X^{-1-\gamma}e^{i(X\mathcal{P}_{k_\beta}-\pi\gamma/2)}\mathcal{P}_{k_\beta}^{-1/2}f_{k_\beta A}^{as}(\hat{X},\underset{\sim}{P}_A)\mathcal{P}_{n_\alpha}^{1/2}. \qquad (42)$$

Here the transition amplitudes f_{BA}^{as} and $f_{k_\beta A}^{as}$ have to be constructed by means of (39) and (40) respectively via the substitution of $\hat{S}^{as}$ in eqs. (33) instead of $\hat{S}$.

Solving the system of radial equations (11c) with asymptotic conditions (19) for χ^{as} (and (25a) for χ^{Ph}) and exploiting the consistency conditions (16) like (41), one can calculate (say numerically) the matrix elements $\hat{S}_{--}$, $\hat{S}_{+-}$ for the $2\to2$, $2\to3$ processes and $\hat{S}_{++}$, $\hat{S}_{-+}$ for the $3\to3$, $3\to2$ ones and the corresponding transition amplitudes (39), (40) and (32), (35), respectively. The asymptotic solutions $\chi^{as}\approx O(X^{-2})$ necessary for this purpose follow from consitency conditions for the nondiagonal elements $A(X\to\infty)$ [21]. Then the flux conservation law for the wave function (37) and (34), in terms of the generalized Wronskian (26), allows one to obtain the usual exp-

ressions for the differential cross section of $2\to2$, $2\to3$ and $3\to3$, $3\to2$ processes in the three-body system.

6. <u>Inverse Problem</u>

The three-body inverse scattering problem is the reconstruction of the interaction potential from the known scattering amplitude [3,23,24].

Consider now the reconstruction problem of some unknown short-range three-particle potentials $V_{123}(\underset{\sim}{X})$ in M from the scattering data. Let us add to the Faddeev (1) and Schrodinger (2) equations some real regular bounded potentials $V_{123}(\underset{\sim}{X})$ satisfying conditions

$$\int_0^\infty \int X^{2\gamma+\varepsilon}|V_{123}(X,\hat{X})|\,dX\,d\hat{X}<\infty, \quad \varepsilon=0,1. \qquad (43)$$

Suppose the central pair potentials V_α to be known and obey analogous condition (see Section 3). We assume their relevant three-body amplitudes $f(\hat{X},\underset{\sim}{P})$ and $\overset{\circ}{f}(\hat{X},\underset{\sim}{P})$ to be known as well. In our case the inverse problem reduces to the determination of the S-matrix via the amplitude $f(\hat{X},\underset{\sim}{P})$ (32), (33) and finding afterwords the effective matrix potential $\mathcal{V}(X)$, solution χ of the system (11b) and finally the interaction potential $V_{123}(\underset{\sim}{X})$. The functions of the hyperspherical adiabatic basis $\{\varphi_j(X;\hat{X})\}$ are obtained as eigenfunctions of Eq.(7) on $\overset{\circ}{M}$ with the known Hamiltonian $H^f(X;\hat{X})$ (6) without the potential $V_{123}(X,\hat{X})$. Simultaneously we find also eigenvalues $\mathcal{E}_j(X)$ which give the effective potential $\mathcal{V}(X)$. With the known $\{\varphi_j\}$ we get matrix elements of the connection operator $A(X)$ from (11a) and according to (12b) we can find the bilocal transport operator $\mathcal{U}(X) \equiv \mathcal{U}(X, \mathring{X})$. The unitary operator $\mathcal{U}(X)$ allows us to pass from the system (11b) with the potential matrix

$$\mathcal{V}(X) = \mathring{\mathcal{V}}(X) + \langle \Phi(X,\hat{X}) | V_{123}(X, \hat{X}) | \Phi(X, \hat{X}) \rangle,$$
$$\mathring{\mathcal{V}}_{ij}(X) = (\mathcal{E}_i(X) - \mathcal{E}_i(\infty))\,\delta_{ij} \tag{44}$$

to the standard system of coupled equations (27) for the coefficients

$$\chi'(X, \mathcal{P}) = \mathcal{U}(X)\,\chi(X, \mathcal{P}) \tag{45}$$

with the new effective potential including

$$\mathcal{V}'(X) = \mathring{\mathcal{V}}' + \mathcal{V}'_{123} = \mathcal{U}\,\mathcal{V}\,\mathcal{U}^{-1} =$$
$$= \langle e | H(X,\hat{X}) - \mathcal{E}(\infty) | e \rangle + \langle e | V_{123}(X, \hat{X}) | e \rangle \tag{46}$$

in the fixed frame $|e\rangle$ representation, that in our case is conveniently chosen as $\mathring{X} \to \infty$. Thus, we can apply methods of the multichannel inverse scattering theory because completeness relations (28) are valid for the physical (25) and regular (20) solutions of the system of eqs.(27). Basic generalized equations of the multichannel inverse scattering problem corresponding to (27) are the following [22,25]

$$K(X,X') + Q(X,X') + \int_{X(0)}^{\infty(X)} K(X,t)\,Q(t,X')\,dt = 0, \tag{47}$$

$$\mathcal{V}'(X) = \mathring{\mathcal{V}}'(X) + \mathcal{V}'_{123}(X) = \mathring{\mathcal{V}}'(X) \mp 2\frac{d}{dX}K(X,X), \tag{48}$$

$$\chi'(X, \mathcal{P}) = \mathring{\chi}'(X,\mathcal{P}) + \int_{X(0)}^{\infty(X)} K(X,X')\,\mathring{\chi}'(X,\mathcal{P})\,dX'. \tag{49}$$

The integration limits in (47), (49) and the signs in (48) depend on the specific statement of the inverse problem. In particular, the limits of integration from "X" to "∞" (from "0" to X) (47),(49) and the sign "-" (sign "+") in (48) correspond to the Marchenko (Gel'fand - Levitan) method. In the R-matrix version of the inverse problem [22] the limits of integration are from "X" to "a" and sign "+" is in (48). For the known kernel $Q(X,X')$ determined by scattering or spectral data the multichannel system of Eqs. (47) is solved with respect to the kernel $K(X,X')$. Then, $K(X,X')$ defines the three-body potential matrix $\mathcal{V}'_{123}(X)$ (46) from (48) and the relevant wave functions χ'_{123} (49) of the system (27).

In the generalized Marchenko approach [25] the integral kernel $Q(X,X')$ is given by

$$Q(X,X') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathring{F}'(X,\mathcal{P})\left(\hat{\mathring{S}}'(\mathcal{P}) - \hat{S}'(\mathcal{P})\right)\tilde{\mathring{F}}'(X',\mathcal{P})\,d\mathcal{P} +$$
$$+ \sum_n^{N} \mathring{F}'(X, i\varkappa_n)\,M_n\,\tilde{\mathring{F}}'(X', i\varkappa_n) - \sum_n^{\mathring{N}} \mathring{F}'(X, i\mathring{\varkappa}_n)\,\mathring{M}_n\,\tilde{\mathring{F}}'(X', i\mathring{\varkappa}_n). \tag{50}$$

Here, the Jost solutions $\mathring{F}'_{\pm}(X, \mathcal{P})$ of Eqs.(27) without V_{123} are related with the Jost solutions $\mathring{F}_{\pm}(X,\mathcal{P})$ of the system (11b) by the unitary operator $\mathcal{U}$ (12) as (45)

$$\mathring{F}'_{\pm}(X, \mathcal{P}) = \mathcal{U}(X,\infty)\,\mathring{F}_{\pm}(X,\mathcal{P}) \tag{45a}$$

and they satisfy the boundary conditions

$$\lim_{X\to\infty} \mathring{F}'_{\pm}(X, \mathcal{P}) = \lim_{X\to\infty} \mathring{F}_{\pm}(X,\mathcal{P}) = \exp \pm i\left(X\mathcal{P} - 1\cdot\frac{\pi\gamma}{2}\right) \otimes 1$$

as follows from the definition (12a) $\mathcal{U}(\mathring{X}, \mathring{X}) \equiv 1$. Owing to the unitary arbitrariness in gauging the radial functions (45), the

$\hat{S}$-matrix corresponding to the system (27)

$$\hat{S}'(\mathcal{P}) = \mathcal{P}^{-1/2}\mathcal{F}'_-(\mathcal{P})[\mathcal{F}'_+(\mathcal{P})]^{-1}\mathcal{P}^{1/2}$$

coincides with the $\hat{S}$-matrix (23) for eqs.(11)

$$\hat{S}'(\mathcal{P}) = \hat{S}(\mathcal{P}). \tag{51a}$$

It is valid for the normalizing matrix of discrete states [24]

$$M_n' = M_n . \tag{51b}$$

With taking into account (51a), the scattering matrices $\hat{S}'(\mathcal{P})$ and $\hat{\mathring{S}}'(\mathcal{P})$, unitary on the open channels, are defined by the known amplitudes $f(\hat{X}, \underset{\sim}{P})$ and $\mathring{f}(\hat{X}, \underset{\sim}{P})$ on (33) and correspond to the system of Eqs.(27) with and without the three-particle potential V_{123}, respectively. The sets $\{E_n = -\varkappa_n^2, M_n\}$ and $\{\mathring{E}_n = -\mathring{\varkappa}_n^2, \mathring{M}_n\}$ are energies and normalizing matrices of the discrete states with and without V_{123}. Getting $\mathcal{V}'_{123}$ and F' from (47)-(49) in the fixed frame $|e\rangle$-representation and going back to the $|\Phi\rangle$-representation one obtains for the potential matrix $\mathcal{V}_{123}$ and Jost solutions of the system (11b) the following relations

$$\mathcal{V}_{123}(X) = \mathcal{V}(X) - \mathring{\mathcal{V}}(X) = -2\,\mathcal{U}^{-1}(X,\infty)\left[\frac{d}{dX}K(X,X)\right]\mathcal{U}(X,\infty), \tag{52}$$

$$F_{\pm}(X,\mathcal{P}) = \mathcal{U}^{-1}(X,\infty)\,F'_{\pm}(X,\mathcal{P}) .$$

It is worth remarking that the matrix elements of $\mathcal{U}$ in (45a) and (52) are the same provided they are defined by the same basis functions $\{\Phi_j\}$. Finally, the physical solutions of (11b) with $\mathcal{V}_{123}$ can be obtained as a linear combination of the Jost solutions (25).

In the generalized Gel'fand - Levitan formalism the integral kernel $Q(X,X')$ is defined by the spectral matrices ρ' and $\mathring{\rho}$ for the system (27) with and without V_{123}, respectively

$$Q(X,X') = \int_{-\infty}^{\infty} \mathring{\chi}'^{reg}(X,\mathcal{P})\, d(\rho' - \mathring{\rho}')\, \tilde{\mathring{\chi}}'^{reg}(X',\mathcal{P}) \tag{53}$$

$$\frac{d\rho}{dE} = \begin{cases} \frac{\mathcal{P}}{\pi}\,|\mathcal{F}'(\mathcal{P})|^{-2}, & E \geq 0 \\ \sum_n \delta(E-E_n)N_n^{-1}, & E < 0. \end{cases}$$

Here N_n are normalization matrices of the regular solutions. Getting the kernels K from (47) on known Q (53) and applying the appropriate multichannel Gel'fand-Levitan formulae (48) and (49), one obtains the potential matrix $\mathcal{V}'$ and regular solutions χ'^{reg} of the system (27). Then, $\mathcal{V}$ and χ for Eqs.(11b) are found by the inverse unitary transformation.

Note, the Bargmann-type potential V_{123} is obtained for the degenerated kernels $Q(X,X')$ with a factorised dependence on the variable X, X' and channel indices [22].

Let us consider the next generalization inverse scattering when both the three- and two-body potentials are unknown. It reduces to the determination of effective potentials $\mathring{V}(\underset{\sim}{X}) = \sum_\alpha V_\alpha$ and three-body potentials $V_{123}(\underset{\sim}{X})$ in M. It is solved at several stages: i) finding of the $\hat{S}$-matrix via the amplitude $f(\hat{X}, \underset{\sim}{P})$ in M, ii) reconstruction of the scalar potential matrix $\mathcal{V}(X)$, the matrix of the gauge vector potential $A(X)$ and matrix solutions of the system equations (11), ii) obtaining of the basis functions $\Phi_i(X,\hat{X})$ and finally the effective interaction potentials $\mathring{V}(X,\hat{X})$ and $V_{123}(X,\hat{X})$ and total solution $\Psi(\underset{\sim}{X})$.

Here, as well as in the previous case, two sets of scattering data $(\hat{S}(\mathcal{P}), \{E_n, M_n\})$, $(\hat{\mathring{S}}(\mathcal{P}), \{\mathring{E}_n, \mathring{M}_n\})$ corresponding to the system (27) with and without the potential V_{123} , are used. Starting from the second set of data one reconstructs the effective potential matrix $\mathring{\mathcal{V}}'(X) = \mathcal{U}(X)[\mathcal{E}(X)-\mathcal{E}(\infty)]\mathcal{U}^{-1}(X)$ and finds the solutions of (27) using the ordinary multichannel Gel'fand-Levitan-Marchenko formulae of the type (47)-(49).

Now in order to find the unitary transport operator $\mathcal{U}(X)$ and energy terms $\mathcal{E}_j(X)$ it is necessary to solve the algebraic eigenvalue problem

$$\mathring{\mathcal{V}}'(X)\,\mathcal{U}(X) = \mathcal{U}(X)\left[\mathcal{E}(X) - \mathcal{E}(\infty)\right].$$

The matrix $\mathring{\mathcal{V}}'(X)$ was reconstructed like (48) while in the previous case we solved the direct eigenvalue problem for the frame Eq.(7). Furthermore, the knowledge of $\mathcal{U}(X)$ permits one to reconstruct the matrix elements of the external gauge field A

$$A(X) = -i\left(\mathcal{U}(X)\frac{d}{dX}\,\mathcal{U}^{-1}(X) + \gamma X^{-1}\right)$$

responsible for the velocity-dependent potential $A(X)\frac{d}{dX}$ appearing in Eqs. (11).

Note, we cannot yet find the effective potential $\mathring{V}(X,\hat{X}) = \sum_{\alpha}^{3} V_{\alpha}(X,\hat{X}) = \sum_{ij} \varphi_i(X,\hat{X})\,\mathring{\mathcal{V}}_{ij}(X)\,\varphi_j^{*}(X,\hat{X})$ by the ordinary method because we do not know basis functions $\varphi_j(X,\hat{X})$ defined by the same potential $\mathring{V}(X,\hat{X})$ (7) at each fixed "slow" variable "X". Proceeding from the obtained spectral characteristics $\{\mathcal{E}_j(X), \gamma_j(X)\}$ which are parametric functions of X , we can formulate the parametric inverse Sturm-Liouville problem

for the restoration of the effective potential $\mathring{V}(X) = \sum_{\alpha}^{3} V_{\alpha}$ and the corresponding solutions $\varphi_j(X,\hat{X})$ [23,24].

Finally, using the afore-described scheme (47)-(52) we reconstruct the three-body potential V_{123} and relevant solutions X from $\{\hat{S}(\mathcal{P}), \{E_n, M_n\}\}$ and $(\hat{\mathring{S}}(\mathcal{P}), \{\mathring{E}_n, \mathring{M}_n\})$.

It is clear that the existence of the "global" adiabatic basis and the possibility of reconstructing effective gauge potentials $\mathcal{V}(X)$, $\mathcal{E}(X)$ and $A(X)$ from the three-particle scattering data are due to the fact that the information concerning of fragment interactions is contained both in the radial solutions and basis "quasiangular" functions $\{\varphi_j\}$. While in other approaches such as K-harmonics method, cluster function expansion, etc. this is missing in basis functions. That is why in our case with the restoration procedure of (47)-(52) we can formulate the complete 3-body inverse problem in the presence of 2- and 3-body interactions.

7. Conclusions

As far as we know, we are the first [1-3] to formulate a three-particle direct and inverse scattering problem involving both the processes with rearrangement and break-up on the basis of the suggested approach. In particular, we have established the adiabatic limit for the Faddeev equations within the total set of surface functions corresponding to two types, break-up and cluster states. The global adiabatic representation of the three-body wave function is investigated in terms of the local adiabatic representation of the Faddeev components. Within the unique adiabatic basis the three-body multichannel scattering problem is investigated by using the unitary gauge transformation. Thus, we give a new formulation of the three-body scattering problem on the basis of the global adiabatic representation and correct boundary conditions by taking account of break-up and rearrangement processes in contrast with ordinary hyperspherical and adiabatic approaches [22]. Owing to this obstacle in [1,3] we have stated and in [24] have formulated the inverse scattering problem and also suggested the method of construction of three-body Bargmann type potentials. Further investigations of exactly solvable three-body and multidimensional models will be considered elsewhere.

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Марковски Б.П., Сузько А.А., Виницкий С.И. E4-91-379
Новое адиабатическое представление трехчастичной задачи рассеяния

Исследуется новое адиабатическое представление трехчастичной волновой функции, полученное через локальные адиабатические разложения для фаддеевских компонент. В результате мы конструируем универсальный адиабатический базис, описывающий всевозможные каналы трехчастичной системы, включая полный развал системы и перераспределение ее фрагментов. Более того, мы даем полную классификацию состояний базиса и соответстующих поверхностных функций. Использование геометрического подхода и спектральной теории позволило нам разработать глобальный адиабатический подход к трехчастичной задаче рассеяния. На основе универсальности построенного базиса и учета физических граничных условий, автоматически следующих из интегральных уравнений Фаддеева, мы формулируем трехчастичную прямую и обратную задачи рассеяния с короткодействующими потенциалами, последовательно сводя их к многоканальной радиальной и параметрической квазиугловой.

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Markovski B.L., Suzko A.A., Vinitsky S.I. E4-91-379
New Adiabatic Representation for Three-Body Scattering Problem

A new adiabatic representation of the three-body wave function obtained in terms of the local adiabatic expansions of the Faddeev components is investigated. As a result, we construct a unique adiabatic basis describing all possible channels of the three-body system with inclusion of break-up and rearrangement ones. Moreover, we give a complete classification of the basis states and the corresponding surface functions. The implementation of simple geometrical and spectral arguments allows us to work out a global adiabatic approach to the three-body scattering problem for the class of pair short-range potentials. Therein we formulate the three-body scattering problem by consistently reducing it to the multichannel radial and parametric quasiangular ones, including the correct boundary conditions with account of break-up and rearrangement processes. We also state the inverse three-body problem.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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