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PHASE SPACE REPRESENTATIONS FOR SPIN $\frac{3}{2}$

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Представления фазового пространства для спина $\frac{3}{2}$

Рассмотрены общие свойства спиновых матриц и матриц плотности для произвольного спина s. Для спина $\frac{3}{2}$ построены представления фазового пространства. Для коррелятора проекций двух спинов $\frac{3}{2}$, находящихся в синглетном состоянии, получены представления, похожие на то, которое предполагал Белл на основе классической теории вероятностей. Найдены квантовые аналоги неравенства Белла для данного случая.

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Phase Space Representations for Spin $\frac{3}{2}$

General properties of spin matrices and density ones are considered for any spin s. For spin $\frac{3}{2}$ phase space representations are constructed. Representations, similar to the Bell one, for the correlator of projections of two spins $\frac{3}{2}$ in the singlet state are found. Quantum analogs of the Bell inequality are obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. The probability theory of the quantum mechanics differs essentially from the classical probability theory first of all by its rule of addition of probabilities [1]. The next difference is demonstrated by the remarkable result of Bell [2,3]: the hypothesis that the correlator of projections of two spins $\frac{1}{2}$ in the singlet state can be written according to the classical probability theory as an integral with respect to some hidden variables λ does contradict quantum mechanics.

At the same time, in quantum mechanics, translated into phase spase representations (PSR's), we can also obtain similar expressions for the correlator, however, either with probability densities, which are not positive definite, or with positive probability densities accompanied by some additional numerical factors [12]. Variables $\vec{s}$ of the phase space S^2 (i.e., $\vec{s}^2=1$) enter instead of λ. The corresponding analogs of the Bell inequality are of course uncontradictory.

For one-spin $\frac{1}{2}$ states (or states of many spins, noninteracting with each other, unlike in the singlet state) some quantities (expectation values, equations of motion) can be reduced in PSR's to their classical counterparts.

In PSR's for higher spins a relationship with the classics is somewhat more complex. Here, we transform the spin $\frac{3}{2}$ theory in the PSR, which is based directly on coherent states, and in other PSR's.

2. General properties of spin matrices. First of all the spin matrices $\hat{s}_m$ $(m = 1,2,3)$ for any spin s are subjected to

$$\hat{s}_m \hat{s}_m = s(s+1)\mathbb{1} \quad , \tag{1}$$

$$[\hat{s}_k, \hat{s}_l] = i\varepsilon_{klm}\hat{s}_m \, . \tag{2}$$

Further, since the spectrum of any spin projection matrix is known, we can easily form the minimal annihilating polynomial $\Pi_s(y)$

$$\Pi_s(\vec{x}\hat{\vec{s}}) = \\ = [(\vec{x}\hat{\vec{s}})^2 - \vec{x}^2 s^2][(\vec{x}\hat{\vec{s}})^2 - \vec{x}^2(s-1)^2]\cdots \begin{cases} \vec{x}\hat{\vec{s}} & \text{for integer } s \\ [(\vec{x}\hat{\vec{s}})^2 - \vec{x}^2\cdot(\frac{1}{2})^2] & \text{for half-integer } s \end{cases} \equiv 0, \tag{3}$$

where $\vec{x}$ is an arbitrary unnormalized 3-vector. Differentiating eq.(3) $2s+1$ times, one obtains the spin algebra relations

spin $\frac{1}{2}$

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left[(\vec{x}\hat{\vec{s}})^2-\vec{x}^2\left(\tfrac{1}{2}\right)^2\right]=0,$$

$$\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i=\tfrac{1}{2}\delta_{ij}\mathbb{1} \qquad \left(\hat{s}_i=\tfrac{1}{2}\sigma_i\right); \tag{4}$$

spin 1

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left[(\vec{x}\hat{\vec{s}})^2-\vec{x}^2\right](\vec{x}\hat{\vec{s}})=0,$$

$$\{\hat{s}_i\hat{s}_j\hat{s}_k\}=2\delta_{jk}\hat{s}_i+2\delta_{ik}\hat{s}_j+2\delta_{ij}\hat{s}_k \tag{5}$$

Using eq.(2) one can reduce eq.(5) to the Duffin-Kemmer algebra

$$\hat{s}_i\hat{s}_j\hat{s}_k+\hat{s}_k\hat{s}_j\hat{s}_i=\delta_{ij}\hat{s}_k+\delta_{jk}\hat{s}_i. \tag{5.a}$$

spin $\frac{3}{2}$

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_\ell}\left[(\vec{x}\hat{\vec{s}})^2-\vec{x}^2\left(\tfrac{3}{2}\right)^2\right]\left[(\vec{x}\hat{\vec{s}})^2-\vec{x}^2\left(\tfrac{1}{2}\right)^2\right]=0,$$

$$\begin{aligned}\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_\ell\}&=5\delta_{ij}\{\hat{s}_k\hat{s}_\ell\}+5\delta_{ik}\{\hat{s}_j\hat{s}_\ell\}+5\delta_{i\ell}\{\hat{s}_j\hat{s}_k\}+\\&+5\delta_{jk}\{\hat{s}_i\hat{s}_\ell\}+5\delta_{j\ell}\{\hat{s}_i\hat{s}_k\}+5\delta_{k\ell}\{\hat{s}_i\hat{s}_j\}-\\&-\tfrac{9}{2}\left(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\right)\mathbb{1}\end{aligned} \tag{6}$$

....

Here and in what follows $\{\ldots\}$ means the total symmetrization without division by $n!$:

$$\{\hat{s}_i\hat{s}_j\}=\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i \tag{7}$$

$$\{\hat{s}_i\hat{s}_j\hat{s}_k\}=(\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i)\hat{s}_k+\hat{s}_k(\hat{s}_i\hat{s}_j+\hat{s}_j\hat{s}_i)+\hat{s}_i\hat{s}_k\hat{s}_j+\hat{s}_j\hat{s}_k\hat{s}_i \tag{8}$$

Let us give some useful formulae

$$\hat{s}_k\hat{s}_m\hat{s}_k=[s(s+1)-1]\hat{s}_m, \tag{9}$$

$$\hat{s}_k\hat{s}_m\hat{s}_n\hat{s}_k=[s(s+1)-2]\hat{s}_m\hat{s}_n-\hat{s}_n\hat{s}_m+s(s+1)\delta_{mn}\mathbb{1}, \tag{10}$$

$$\hat{s}_k\{\hat{s}_m\hat{s}_n\}\hat{s}_k=[s(s+1)-3]\{\hat{s}_m\hat{s}_n\}+2s(s+1)\delta_{mn}\mathbb{1}. \tag{11}$$

Traces of any power of the spin projection matrix $(\vec{x}\hat{\vec{s}})$ is obvious

$$\begin{aligned}
tr(\vec{x}\hat{\vec{s}})&=|\vec{x}|\,tr(\vec{n}\hat{\vec{s}})=|\vec{x}|\sum_{-s,\ldots,s}m &&=0,\\
tr(\vec{x}\hat{\vec{s}})^2&=\vec{x}^2\sum_{-s,\ldots,s}m^2 &&=\tfrac{1}{3}s(s+1)(2s+1)\vec{x}^2,\\
tr(\vec{x}\hat{\vec{s}})^3&=|\vec{x}|^3\sum_{-s,\ldots,s}m^3 &&=0,\\
tr(\vec{x}\hat{\vec{s}})^4&=|\vec{x}|^4\sum_{-s,\ldots,s}m^4 &&=\tfrac{1}{15}s(s+1)(2s+1)(3s^2+3s-1)|\vec{x}|^4,
\end{aligned}$$

....

$$tr(\vec{x}\hat{\vec{s}})^k=|\vec{x}|^k\sum_{-s,\ldots,s}m^k=\begin{cases}=0 & \text{for odd } k\\ \neq 0 & \text{for even } k\end{cases},$$

.... (12)

Differentiating with respect to $\vec{x}$ k times, we get

$$\begin{aligned}
&tr\,\hat{s}_i=0,\\
&tr\{\hat{s}_i\hat{s}_j\}=2tr(\hat{s}_i\hat{s}_j)=2\delta_{ij}\sum_{-s,\ldots,s}m^2\\
&\qquad=\tfrac{2}{3}s(s+1)(2s+1)\delta_{ij},\\
&tr\{\hat{s}_i\hat{s}_j\hat{s}_k\}=0,\\
&tr\{\hat{s}_i\hat{s}_j\hat{s}_k\hat{s}_\ell\}=8(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\sum_{-s,\ldots,s}m^4\\
&\qquad=\tfrac{8}{15}s(s+1)(2s+1)(3s^2+3s-1)(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\\
&\qquad=2\cdot 41(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}) \qquad \text{for } s=\tfrac{3}{2},
\end{aligned}$$

.... (13)

The trace of the symmetrized product of an odd number of spin matrices equals zero, and of an even number of spin matrices is multiple of the sum of all possible products of the Kronecker symbols δ_{ij}.

Completeness relations. The matrices $\mathbb{1}, \hat{s}_i, \{\hat{s}_i\hat{s}_j\}, \{\hat{s}_i\hat{s}_j\hat{s}_k\}, \ldots$ up to the symmetrized products of $2s$ factors for the spin s form the total basis of $(2s+1)\times(2s+1)$ matrices. For spin $\frac{3}{2}$ the total basis consists of the 4x4 matrices $\mathbb{1}$, $\hat{s}_i$, $\{\hat{s}_i\hat{s}_j\}$, and $\{\hat{s}_i\hat{s}_j\hat{s}_k\}$ and they satisfy the completeness relation

$$-\frac{17}{32}|\mathbb{1}|\otimes\|\mathbb{1}\| - \frac{293}{2^3\cdot 3^3}|\hat{s}_i|\otimes\|\hat{s}_i\| + \frac{1}{2^3\cdot 3}|\{\hat{s}_i\hat{s}_j\}|\otimes\|\{\hat{s}_i\hat{s}_j\}\| +$$

$$+\frac{1}{2\cdot 3^4}|\{\hat{s}_i\hat{s}_j\hat{s}_k\}|\otimes\|\{\hat{s}_i\hat{s}_j\hat{s}_k\}\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \qquad (14)$$

$|\cdots|$ and $\|\cdots\|$ denote matrices. In terms of indices, e.g., $|\hat{s}_i|\otimes\|\hat{s}_i\|$ means $(\hat{s}_i)_{\alpha\beta}(\hat{s}_i)_{\gamma\delta}$. Then $|\mathbb{1}\|\otimes\|\mathbb{1}|$ means $(\mathbb{1})_{\alpha\delta}(\mathbb{1})_{\gamma\beta} = \delta_{\alpha\delta}\delta_{\gamma\beta}$.

In the course of calculations we shall often use the spin matrices $\hat{s}_i$ in the canonical representation. Simple products and symmetrized ones in this representation are given explicitly in Appendix A.

3. The density matrices of interest $\hat{\rho}(s,m,\vec{a})$

$$(\vec{a}\hat{\vec{s}})\hat{\rho}(s,m,\vec{a}) = \hat{\rho}(s,m,\vec{a})(\vec{a}\hat{\vec{s}}) = m\,\hat{\rho}(s,m,\vec{a}) \qquad (15)$$

($\vec{a}^2 = 1$) for any spin s can be constructed out immediately as the Lagrange-Silvester polynomials which are obvious and can be expressed via the minimal annihilating polynomial

$$\hat{\rho}(s, m\,\mathrm{on}\,\vec{a}) \equiv \hat{\rho}(s,m,\vec{a}) = |s,m,\vec{a}\rangle\langle s,m,\vec{a}| =$$

$$= N\,\frac{\Pi_s(y)}{y-m}\Big|_{y=\vec{a}\hat{\vec{s}}} \qquad (\vec{a}^2=1) \qquad (16)$$

where $N = N(s,m)$ are the normalization factors which ensure the condition

$$\mathrm{tr}\,\hat{\rho} = 1 \qquad (17)$$

In the case of spin $\frac{3}{2}$ we have

$$\hat{\rho}\left(\tfrac{3}{2},\vec{a}\right) = \frac{1}{6}\left[(\vec{a}\hat{\vec{s}})+\tfrac{3}{2}\mathbb{1}\right]\left[(\vec{a}\hat{\vec{s}})^2-\left(\tfrac{1}{2}\right)^2\mathbb{1}\right],$$

$$\hat{\rho}\left(\tfrac{1}{2},\vec{a}\right) = -\frac{1}{2}\left[(\vec{a}\vec{s})^2-\left(\tfrac{3}{2}\right)^2\mathbb{1}\right]\left[(\vec{a}\vec{s})+\tfrac{1}{2}\mathbb{1}\right]. \qquad (18)$$

The density matrices for negative m are obtained from these by simply replacing $\vec{a}$ by $-\vec{a}$. Note that

$$\sum_{m=-s,\dots,s}\hat{\rho}(m,\vec{a}) = \mathbb{1}. \qquad (19)$$

The completeness relation in terms of the density matrices $\hat{\rho}(m,\vec{a})$ (as basis ones) can be written as follows:

$$\int d\mu(\vec{s})\sum_{m=\frac{1}{2},\frac{3}{2}} c_m|\hat{\rho}(m,\vec{s})|\otimes\|\hat{\rho}(m,\vec{s})\| = |\mathbb{1}|\otimes\|\mathbb{1}\| + |\mathbb{1}\|\otimes\|\mathbb{1}| \qquad (20)$$

where $d\mu(\vec{s}) = \frac{1}{2\pi}\delta(\vec{s}^2-1)\,d^3s$ is the measure on S^2, and

$$c_{\frac{1}{2}} = 3\cdot 5, \qquad c_{\frac{3}{2}} = 5. \qquad (21)$$

Eq.(20) follows from

$$\int d\mu(\vec{s})\,\hat{\rho}\left(\tfrac{3}{2},\vec{s}\right)\otimes\hat{\rho}\left(\tfrac{3}{2},\vec{s}\right) = \frac{1}{6^2}\Big(\frac{27}{32}\mathbb{1}\otimes\mathbb{1} + \frac{9\cdot 23}{5\cdot 7\cdot 8}\hat{s}_i\otimes\hat{s}_i +$$

$$+\frac{9}{2\cdot 3\cdot 4\cdot 5}\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\} + \frac{1}{3\cdot 5\cdot 6\cdot 7}\{\hat{s}_i\hat{s}_j\hat{s}_k\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\}\Big), \qquad (22)$$

$$\int d\mu(\vec{s})\,\hat{\rho}\left(\tfrac{1}{2},\vec{s}\right)\otimes\hat{\rho}\left(\tfrac{1}{2},\vec{s}\right) = \frac{1}{4}\Big(\frac{3}{32}\mathbb{1}\otimes\mathbb{1} - \frac{5\cdot 13}{3\cdot 7\cdot 8}\hat{s}_i\otimes\hat{s}_i +$$

$$+\frac{1}{4\cdot 5\cdot 6}\{\hat{s}_i\hat{s}_j\}\otimes\{\hat{s}_i\hat{s}_j\} + \frac{1}{3\cdot 5\cdot 6\cdot 7}\{\hat{s}_i\hat{s}_j\hat{s}_k\}\otimes\{\hat{s}_i\hat{s}_j\hat{s}_k\}\Big) \qquad (23)$$

and the completeness relation (14).

The states $|s,m,\vec{a}\rangle$, mentioned in eq.(16), can be treated as coherent states /7-9/. The completeness relations for them can be written as follows:

$$(2s+1)\int d\mu(\vec{a})\,\hat{\rho}(s,m,\vec{a}) = \mathbb{1} \qquad (24)$$

for any spin s and any value m.

Expectation values of spin projections in the states $\hat{\rho}(s,m,\vec{a})$ for any spin s equal

$$\langle s,m,\vec{a}|(\vec{b}\hat{\vec{s}})|s,m,\vec{a}\rangle = \mathrm{tr}[(\vec{b}\hat{\vec{s}})\hat{\rho}(s,m,\vec{a})] = m(\vec{a}\vec{b}). \qquad (25)$$

To prove this relation let us decompose the vector $\vec{b}$ into two components parallel and orthogonal to $\vec{a}$

$$\vec{b} = (\vec{a}\vec{b})\vec{a} + [\vec{b}-(\vec{a}\vec{b})\vec{a}] \equiv (\vec{a}\vec{b})\vec{a}+\vec{c}.$$

The second component gives zero

$$\mathrm{tr}[(\vec{c}\hat{\vec{s}})\hat{\rho}(s,m,\vec{a})] = 0, \qquad (26)$$

since

$$\mathrm{tr}[(\vec{c}\hat{\vec{s}})(\vec{a}\hat{\vec{s}})^k] = 0 \quad \text{for any } k. \qquad (27)$$

Indeed the latter trace can be written as a trace of the symmetrized product

$$a_{m_1}a_{m_2}\cdots a_{m_k}c_{m_{k+1}}\,tr\{\hat{s}_{m_1}\hat{s}_{m_2}\cdots\hat{s}_{m_k}\hat{s}_{m_{k+1}}\}\,.$$

If the number of matrices k+1 is odd, the trace equals zero. If k+1 is even, the trace is multiple of the sum of all possible products of the Kronecker $\delta_{m_i m_j}$. Therefore, each term of the sum contains the factor $(\vec{a}\vec{c})=0$. Thus, eqs. (27) and (26) are valid. Then

$$tr[(\vec{b}\hat{\vec{s}})\hat{\rho}(s,m,\vec{a})]=(\vec{a}\vec{b})tr[(\vec{a}\hat{\vec{s}})\hat{\rho}(s,m,\vec{a})]=(\vec{a}\vec{b})m$$

by definition (15), and eq.(25) is proved. From eq.(25) it follows that

$$tr[\hat{s}_j\,\hat{\rho}(s,m,\vec{a})]=m\,a_j. \qquad (28)$$

The probability of finding the spin projection n along $\vec{b}$ in the state with the projection m along $\vec{a}$ is given by

$$\rho(s,n,\vec{b};m,\vec{a})=\rho(s,n\,\text{on}\,\vec{b};m\,\text{on}\,\vec{a})=tr[\hat{\rho}(s,n,\vec{b})\,\hat{\rho}(s,m,\vec{a})] \qquad (29)$$

It is difficult to calculate these traces directly since the traces $tr[(\vec{b}\hat{\vec{s}})^k(\vec{a}\hat{\vec{s}})^l]$ for $k>1$, $l>1$ are not ones of symmetrized products, and decomposition into symmetrized products is complex. Another way is to choose the special frame of reference with the axis Oz along $\vec{b}$ and to use the canonical representation of the spin matrices (Appendix A). For spin $\frac{3}{2}$ the probabilities $\rho(n,\vec{b};m,\vec{a})$ are calculated in Appendix B and are given in Table 1. Of course,

$$\sum_{n=-s,\dots,s}\rho(n,\vec{b};m,\vec{a})=\sum_{m=-s,\dots,s}\rho(n,\vec{b};m,\vec{a})=1 \qquad (30)$$

4. The singlet state of two spins $\frac{3}{2}$ can be written as

$$|singlet\rangle=\tfrac{1}{2}\Big(u^a(\tfrac{3}{2})\otimes u^b(-\tfrac{3}{2})-u^a(\tfrac{1}{2})\otimes u^b(-\tfrac{1}{2})+u^a(-\tfrac{1}{2})\otimes u^b(\tfrac{1}{2})-u^a(-\tfrac{3}{2})\otimes u^b(\tfrac{3}{2})\Big), \qquad (31)$$

$$(\hat{s}^a_m+\hat{s}^b_m)|singlet\rangle=0, \qquad (m=1,2,3) \qquad (32)$$

$$\hat{\rho}^{\,singlet}=|singlet\rangle\langle singlet|, \qquad (33)$$

$$(\hat{s}^a_m+\hat{s}^b_m)\hat{\rho}^{\,singlet}=\hat{\rho}^{\,singlet}(\hat{s}^a_m+\hat{s}^b_m)=0. \qquad (34)$$

In eq. (31) any axis of quantization can be adopted since, in fact, the state $|singlet\rangle$ is independent of the choice of the axis. The density

Table 1. Spin $\frac{3}{2}$. $8\rho(n,\vec{b};m,\vec{a})$.

m \ n	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3}{2}$	$(1+\vec{a}\vec{b})^3$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$	$3(1+\vec{a}\vec{b})(1-\vec{a}\vec{b})^2$	$(1-\vec{a}\vec{b})^3$
$\frac{1}{2}$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$	$(1+\vec{a}\vec{b})(1-3\vec{a}\vec{b})^2$	$(1-\vec{a}\vec{b})(1+3\vec{a}\vec{b})^2$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$
$-\frac{1}{2}$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$	$(1-\vec{a}\vec{b})(1+3\vec{a}\vec{b})^2$	$(1+\vec{a}\vec{b})(1-3\vec{a}\vec{b})^2$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$
$-\frac{3}{2}$	$(1-\vec{a}\vec{b})^3$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$	$3(1-\vec{a}\vec{b})(1+\vec{a}\vec{b})^2$	$(1+\vec{a}\vec{b})^3$

Table 2. Spin $\frac{3}{2}$. $32\rho(m,\vec{a};n,\vec{b}\,|\,singlet)$.

m \ n	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
$\frac{3}{2}$	$(1-\vec{a}\vec{b})^3$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$	$3(1-\vec{a}\vec{b})(1+\vec{a}\vec{b})^2$	$(1+\vec{a}\vec{b})^3$
$\frac{1}{2}$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$	$(1-\vec{a}\vec{b})(1+3\vec{a}\vec{b})^2$	$(1+\vec{a}\vec{b})(1-3\vec{a}\vec{b})^2$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$
$-\frac{1}{2}$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$	$(1+\vec{a}\vec{b})(1-3\vec{a}\vec{b})^2$	$(1-\vec{a}\vec{b})(1+3\vec{a}\vec{b})^2$	$3(1-\vec{a}\vec{b})^2(1+\vec{a}\vec{b})$
$-\frac{3}{2}$	$(1+\vec{a}\vec{b})^3$	$3(1+\vec{a}\vec{b})^2(1-\vec{a}\vec{b})$	$3(1+\vec{a}\vec{b})(1-\vec{a}\vec{b})^2$	$(1-\vec{a}\vec{b})^3$

matrix can be written to be manifestly independent of this axis, e.g., $\frac{1}{4}[1^a \otimes 1^b - \sigma_m^a \otimes \sigma_m^b]$ for spin $\frac{1}{2}$ /12/ and $\frac{1}{3}[-1^a \otimes 1^b + (s_m^a \otimes s_m^b)^2]$ for spin 1. However, it is the density matrix in the form (33), which is convenient for calculations in the special frame of reference with $Oz \parallel \vec{b}$.

<u>The correlator of projections of two spins in the singlet state</u>

$$c(\vec{a},\vec{b}) = \underset{a}{tr}\,\underset{b}{tr}\left[(\vec{a}\hat{\vec{s}}^a)(\vec{b}\hat{\vec{s}}^b)\hat{\rho}^{singlet}\right] =$$

$$= (\vec{a}\hat{\vec{s}}^a)_{\alpha'\alpha}(\vec{b}\hat{\vec{s}}^b)_{\beta'\beta}\,\hat{\rho}^{singlet}_{\alpha\beta,\alpha'\beta'} \tag{35}$$

Let us choose the frame of reference with $Oz \parallel \vec{b}$. If Oz is assumed to be the axis of quantization in eq.(31), and if the canonical representation of the spin $\frac{3}{2}$ matrices (see Appendix A) is adopted, we have

$$u\left(\tfrac{3}{2}\right) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad u\left(\tfrac{1}{2}\right) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad u\left(-\tfrac{1}{2}\right) = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad u\left(-\tfrac{3}{2}\right) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{36}$$

Then, with the density matrix (33) we obtain

$$\begin{aligned} c(\vec{a},\vec{b}) &= \\ &= \tfrac{1}{4}(1\,0\,0\,0)(\vec{a}\hat{\vec{s}}^a)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \otimes (0\,0\,0\,1)\,\hat{s}_3^b\begin{bmatrix}0\\0\\0\\1\end{bmatrix} + \tfrac{1}{4}(0\,1\,0\,0)(\vec{a}\hat{\vec{s}}^a)\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \otimes (0\,0\,1\,0)\,\hat{s}_3^b\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \\ &+ \tfrac{1}{4}(0\,0\,1\,0)(\vec{a}\hat{\vec{s}}^a)\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \otimes (0\,1\,0\,0)\,\hat{s}_3^b\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \tfrac{1}{4}(0\,0\,0\,1)(\vec{a}\hat{\vec{s}}^a)\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \otimes (1\,0\,0\,0)\,\hat{s}_3^b\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \end{aligned} \tag{37}$$

Eq. (37) contains only the diagonal elements of $\hat{s}_3^b$. Other 12 terms, which the direct product $|singlet\rangle\langle singlet|$ gives, vanish because they contain all 12 nondiagonal elements of the matrix $\hat{s}_3^b$ but $\hat{s}_3$ is purely diagonal. Eq. (37) contains only the diagonal elements of $(\vec{a}\hat{\vec{s}}^a)$ too, in fact of $a_3\hat{s}_3^a$, since $\hat{s}_1$ and $\hat{s}_2$ have zeros on their diagonals. Hence,

$$c(\vec{a},\vec{b}) = \frac{1}{4}a_3\left[\frac{3}{2}\left(-\frac{3}{2}\right) + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{2} - \frac{3}{2}\cdot\frac{3}{2}\right] = -\frac{5}{4}a_3 \tag{38}$$

and in the general frame of reference

$$c(\vec{a},\vec{b}) = -\frac{5}{4}(\vec{a}\vec{b}). \tag{39}$$

This calculation is obviously applicable for any spin s. Only dimensions of matrices and columns (rows) are different ($(2s + 1)\times(2s + 1)$ and $2s + 1$, respectively). Thus, we obtain

$$c(\vec{a},\vec{b}) = \frac{1}{2s+1}a_3\cdot(-1)\cdot\sum_{-s,\dots,s} m^2 = -\frac{1}{3}s(s+1)a_3 \tag{40}$$

and in an arbitrary frame of reference

$$c(\vec{a},\vec{b}) = -\frac{1}{3}s(s+1)(\vec{a}\vec{b}) \tag{41}$$

For another derivation see ref. /6/.

<u>The probability to find definite projections of two spins in the singlet state</u> equals

$$\begin{aligned} \rho(m,\vec{a};n,\vec{b}|singlet) &= \underset{a}{tr}\,\underset{b}{tr}\left(\hat{\rho}^a(m,\vec{a})\,\hat{\rho}^b(n,\vec{b})\,\hat{\rho}^{singlet}\right) = \\ &= \hat{\rho}^a_{\alpha'\alpha}(m,\vec{a})\,\hat{\rho}^b_{\beta'\beta}(n,\vec{b})\,\hat{\rho}^{singlet}_{\alpha\beta,\alpha'\beta'} = \\ &= \frac{1}{2s+1}\rho(m,\vec{a};-n,\vec{b}) = \frac{1}{2s+1}\rho(m,\vec{a};n,-\vec{b}) = \\ &= \frac{1}{2s+1}\rho(-m,\vec{a};n,\vec{b}) = \frac{1}{2s+1}\rho(m,-\vec{a};n,\vec{b}) \end{aligned} \tag{42}$$

for any spin s. Eq. (42) means that the probabilities of interest are reduced to one-spin probabilities (such as in Table 1 for spin $\frac{3}{2}$). Indeed, if we measure first a projection of one of the spins, say of a, along $\vec{a}$, we can find any possible value m with the probability $\frac{1}{2s+1}$. Then, spin b will have the projection $-m$ along $\vec{a}$ with the probability 1 (due to the angular momentum conservation), and hence, the projection n along $\vec{b}$ with the probability $\rho^b(-m,\vec{a};n,\vec{b})$. Thus, eq. (42) is proved. This can be confirmed by the direct calculation (see Appendix C).

The probabilities $\rho(m,\vec{a};n,\vec{b}|singlet)$ for spin $\frac{3}{2}$ are given in Table 2.

It is clear that

$$\sum_{m,n=-s,\dots,s}\sum \rho(m,\vec{a};n,\vec{b}|singlet) = 1. \tag{43}$$

5. <u>A phase space representation</u> can be defined as follows. Each operator $\hat{F}$ of the spin $\frac{3}{2}$ theory can be represented by the pair of functions

$$\left\{F(\tfrac{1}{2},\vec{s}),F(\tfrac{3}{2},\vec{s})\right\} \doteq \left\{tr\left(\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{F}\right), tr\left(\hat{\rho}(\tfrac{3}{2},\vec{s})\hat{F}\right)\right\} \tag{44}$$

The trace of $\hat{F}$ is expressed via any of its PSR components

$$tr\,\hat{F} = (2s+1)\int d\mu(\vec{s})F(m,\vec{s}) \tag{45}$$

This follows from eq. (24).

In particular, representatives of the density matrices are

$$\begin{aligned} &\left\{\rho(\tfrac{1}{2},\vec{s};\tfrac{1}{2},\vec{a}),\rho(\tfrac{3}{2},\vec{s};\tfrac{1}{2},\vec{a})\right\} = \left\{\tfrac{1}{8}(1+\vec{a}\vec{s})(1-3\vec{a}\vec{s})^2, \tfrac{3}{8}(1+\vec{a}\vec{s})^2(1-\vec{a}\vec{s})\right\} \\ &\left\{\rho(\tfrac{1}{2},\vec{s};\tfrac{3}{2},\vec{a}),\rho(\tfrac{3}{2},\vec{s};\tfrac{3}{2},\vec{a})\right\} = \left\{\tfrac{3}{8}(1+\vec{a}\vec{s})^2(1-\vec{a}\vec{s}), \tfrac{1}{8}(1+\vec{a}\vec{s})^3\right\} \end{aligned} \tag{46}$$

The normalization condition (17) is written as follows:

$$\mathrm{tr}\,\hat{\rho}(m,\vec{a}) = (2s+1)\int d\mu(\vec{s})\,\rho(n,\vec{s};m,\vec{a}) = 1. \qquad (47)$$

<u>Restoration theorem</u>. The completeness relation (20) guarantees that any operator can be restored via its representative

$$\hat{F} = -\mathbb{1}\cdot\mathrm{tr}\,\hat{F} + \sum_{m=\frac{1}{2},\frac{3}{2}} c_m \int d\mu(\vec{s})\,\hat{\rho}(m,\vec{s})\,F(m,\vec{s}), \qquad (48)$$

where $\mathrm{tr}\,\hat{F}$ is given by eq. (45).

The trace of the product of two operators is written as

$$\mathrm{tr}(\hat{F}\hat{G}) = -\mathrm{tr}\,\hat{F}\cdot\mathrm{tr}\,\hat{G} + \sum_{m=\frac{1}{2},\frac{3}{2}} c_m \int d\mu(\vec{s})\,F(m,\vec{s})\,G(m,\vec{s}). \qquad (49)$$

In particular, the expectation value of $\hat{F}$ is

$$\mathrm{tr}(\hat{F}\hat{\rho}) = -\mathrm{tr}\,\hat{F} + \sum_{m=\frac{1}{2},\frac{3}{2}} c_m \int d\mu(\vec{s})\,F(m,\vec{s})\,\rho(m,\vec{s}). \qquad (50)$$

Expectation values of the spin projection $(\vec{b}\hat{\vec{s}})$ can be represented as

$$\mathrm{tr}\big((\vec{b}\hat{\vec{s}})\hat{\rho}(m,\vec{a})\big) = \sum_{n=\frac{1}{2},\frac{3}{2}} c_n \int d\mu(\vec{s})\,\mathrm{tr}\big((\vec{b}\hat{\vec{s}})\hat{\rho}(n,\vec{s})\big)\cdot\mathrm{tr}\big(\hat{\rho}(n,\vec{s})\hat{\rho}(m,\vec{a})\big) =$$

$$= \frac{15}{2}\int d\mu(\vec{s})\,(\vec{b}\vec{s}) \sum_{n=\frac{1}{2},\frac{3}{2}} \rho(n,\vec{s};m,\vec{a}) \qquad (51)$$

The singlet state is represented by four functions

$$\rho(m,\vec{a};n,\vec{b}\,|\,\mathrm{singlet}) \qquad m,n = \frac{1}{2},\frac{3}{2}. \qquad (52)$$

The normalization condition of the singlet state is expressed via any of these functions

$$(2s+1)^2 \int d\mu(\vec{a}) \int d\mu(\vec{b})\,\rho(m,\vec{a};n,\vec{b}\,|\,\mathrm{singlet}) = 1. \qquad (53)$$

The correlator of projections of two spins in the singlet state can be represented

$$c(\vec{a},\vec{b}) = \mathrm{tr}_a\,\mathrm{tr}_b\big((\vec{a}\hat{\vec{s}}^a)(\vec{b}\hat{\vec{s}}^b)\,\hat{\rho}^{\,\mathrm{singlet}}\big) =$$

$$= \sum\sum_{m,n=\frac{1}{2},\frac{3}{2}} c_m c_n \int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,\mathrm{tr}\big((\vec{a}\hat{\vec{s}})\hat{\rho}(m,\vec{s}^a)\big)\,\mathrm{tr}\big((\vec{b}\hat{\vec{s}})\hat{\rho}(n,\vec{s}^b)\big)\cdot$$

$$\cdot\,\rho(m,\vec{s}^a;n,\vec{s}^b\,|\,\mathrm{singlet}) =$$

$$= \int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,(\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b)\sum\sum_{m,n=\frac{1}{2},\frac{3}{2}} c_m c_n\, m n\,\rho(m,\vec{s}^a;n,\vec{s}^b\,|\,\mathrm{singlet}) =$$

$$= \left(\frac{15}{2}\right)^2 \int d\mu(\vec{s}^a)\int d\mu(\vec{s}^b)\,(\vec{a}\vec{s}^a)(\vec{b}\vec{s}^b)\sum\sum_{m,n=\frac{1}{2},\frac{3}{2}} \rho(m,\vec{s}^a;n,\vec{s}^b\,|\,\mathrm{singlet}). \qquad (54)$$

This expression for the correlator is similar to that proposed by Bell with the reference to the classical probability theory (for two spins $\frac{1}{2}$). Following his derivation /2,4,5/ we obtain for the correlator (54) the inequality

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}') + c(\vec{a}',\vec{b})| \le 2\cdot\left(\frac{15}{4}\right)^2, \qquad (55)$$

which, of course, is noncontradictory.

6. <u>Other PSR's</u> can be obtained, if we represent the completeness relation in the forms

$$4\int d\mu(\vec{s})\;|\hat{\rho}(\tfrac{3}{2},\vec{s})| \otimes \|\hat{Y}(\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \qquad (56)$$

$$4\int d\mu(\vec{s})\;|\hat{\rho}(\tfrac{1}{2},\vec{s})| \otimes \|\hat{Z}(\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|, \qquad (57)$$

$$4\int d\mu(\vec{s})\;|\hat{X}(\vec{s})| \otimes \|\hat{X}(\vec{s})\| = |\mathbb{1}\|\otimes\|\mathbb{1}|. \qquad (58)$$

Being considered as equations in $\hat{Y}$, $\hat{Z}$ and $\hat{X}$ they can be solved x) to obtain

$$\hat{Y}(\vec{a}) = \alpha\mathbb{1} + \beta(\vec{a}\hat{\vec{s}}) + \gamma(\vec{a}\hat{\vec{s}})^2 + \delta(\vec{a}\hat{\vec{s}})^3 \qquad (59)$$

with $\alpha = -\frac{21}{16}$, $\beta = -\frac{5^2\cdot 11}{2^3\cdot 3}$, $\gamma = \frac{5}{4}$, $\delta = \frac{5\cdot 7}{2\cdot 3}$, (59.a)

$$\hat{Z}(\vec{a}) = \alpha\mathbb{1} + \beta(\vec{a}\hat{\vec{s}}) + \gamma(\vec{a}\hat{\vec{s}})^2 + \delta(\vec{a}\hat{\vec{s}})^3 \qquad (60)$$

with $\alpha = \frac{29}{16}$, $\beta = \frac{5\cdot 79}{2^3\cdot 3^2}$, $\gamma = -\frac{5}{4}$, $\delta = -\frac{5\cdot 7}{2\cdot 3^2}$, (60.a)

$$\hat{X}(\vec{a}) = \alpha\mathbb{1} + \beta(\vec{a}\hat{\vec{s}}) + \gamma(\vec{a}\hat{\vec{s}})^2 + \delta(\vec{a}\hat{\vec{s}})^3 \qquad (61)$$

with $\alpha = \frac{-5\gamma\pm 1}{4}$, $\beta = \frac{1}{20}\left[-41\delta \pm 2\sqrt{15}\right]$, $\gamma = \pm\frac{\sqrt{5}}{4}$, $\delta = \pm\frac{\sqrt{35}}{6}$. (61.a)

Any combinations of signs are acceptable (the condition $\mathrm{tr}\,\hat{X}(\vec{a}) = 4\alpha + 5\gamma = 1$ excludes the minus in α).

It is clear, that the completeness relations (56)-(58) permit to introduce one-component representatives. Arbitrary operator $\hat{F}$ can be represented by any of the following functions

$$F_1(\vec{s}) = \mathrm{tr}[\hat{\rho}(\tfrac{3}{2},\vec{s})\hat{F}], \quad F_{1'}(\vec{s}) = \mathrm{tr}[\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{F}],$$

$$F_2(\vec{s}) = \mathrm{tr}[X(\vec{s})\hat{F}],\; F_3(\vec{s}) = \mathrm{tr}[\hat{Y}(\vec{s})\hat{F}], \quad F_4(\vec{s}) = \mathrm{tr}[\hat{Z}(\vec{s})\hat{F}] \qquad (62)$$

x) Note that for the integer spins the equations, similar to eq. (56) (or (57)) for m = 0 cannot be solved. It is clear from eq. (25).

In these terms we can write the restorarion theorems:

$$\hat{F} = 4\int d\mu(\vec{s})\,\hat{Y}(\vec{s})F_1(\vec{s}) = 4\int d\mu(\vec{s})\,\hat{Z}(\vec{s})F_{1'}(\vec{s}) =$$
$$= 4\int d\mu(\vec{s})\,\hat{X}(\vec{s})F_2(\vec{s}) =$$
$$= 4\int d\mu(\vec{s})\,\hat{\rho}(\tfrac{3}{2},\vec{s})F_3(\vec{s}) = 4\int d\mu(\vec{s})\,\hat{\rho}(\tfrac{1}{2},\vec{s})F_4(\vec{s}), \qquad (63)$$

the trace of the product of two operators

$$\mathrm{tr}(\hat{F}\hat{G}) = 4\int d\mu(\vec{s})F_1(\vec{s})G_3(\vec{s}) = 4\int d\mu(\vec{s})F_{1'}(\vec{s})G_4(\vec{s}) =$$
$$= 4\int d\mu(\vec{s})F_2(\vec{s})G_2(\vec{s}), \qquad (64)$$

the expectation values

$$\mathrm{tr}(\hat{F}\hat{\rho}(\tfrac{3}{2},\vec{\alpha})) = 4\int d\mu(\vec{s})F_3(\vec{s})\rho_1(\vec{s};\tfrac{3}{2},\vec{\alpha}) = 4\int d\mu(\vec{s})F_4(\vec{s})\rho_{1'}(\vec{s};\tfrac{3}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_2(\vec{s})\rho_2(\vec{s};\tfrac{3}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_1(\vec{s})\rho_3(\vec{s};\tfrac{3}{2},\vec{\alpha}) = 4\int d\mu(\vec{s})F_1(\vec{s})\rho_{3'}(\vec{s};\tfrac{3}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_{1'}(\vec{s})\rho_4(\vec{s};\tfrac{3}{2},\vec{\alpha}), \qquad (65)$$

$$\mathrm{tr}(\hat{F}\hat{\rho}(\tfrac{1}{2},\vec{\alpha})) = 4\int d\mu(\vec{s})F_3(\vec{s})\rho_1(\vec{s};\tfrac{1}{2},\vec{\alpha}) = 4\int d\mu(\vec{s})F_4(\vec{s})\rho_{1'}(\vec{s};\tfrac{1}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_2(\vec{s})\rho_2(\vec{s};\tfrac{1}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_1(\vec{s})\rho_3(\vec{s};\tfrac{1}{2},\vec{\alpha}) =$$
$$= 4\int d\mu(\vec{s})F_{1'}(\vec{s})\rho_4(\vec{s};\tfrac{1}{2},\vec{\alpha}) = 4\int d\mu(\vec{s})F_{1'}(\vec{s})\rho_{4'}(\vec{s};\tfrac{1}{2},\vec{\alpha}). \qquad (66)$$

The densities entering are given by

$$\rho_1(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \mathrm{tr}[\hat{\rho}(\tfrac{3}{2},\vec{s})\hat{\rho}(\tfrac{3}{2},\vec{\alpha})] = \tfrac{1}{8}(1+\vec{s}\vec{\alpha})^3, \qquad (67)$$

$$\rho_{1'}(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \mathrm{tr}[\hat{\rho}(\tfrac{1}{2},\vec{\alpha})\hat{\rho}(\tfrac{3}{2},\vec{\alpha})] = \tfrac{3}{8}(1+\vec{s}\vec{\alpha})^2(1-\vec{s}\vec{\alpha}), \qquad (68)$$

$$\rho_2(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \mathrm{tr}[\hat{X}(\vec{s})\hat{\rho}(\tfrac{3}{2},\vec{\alpha})] =$$
$$= \alpha + \tfrac{3}{4}\gamma + (\tfrac{3}{2}\beta + \tfrac{21}{8}\delta)(\vec{\alpha}\vec{s}) + \tfrac{3}{2}\gamma(\vec{\alpha}\vec{s})^2 + \tfrac{3}{4}\delta(\vec{\alpha}\vec{s})^3, \qquad (69)$$

$$\rho_3(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \mathrm{tr}[\hat{Y}(\vec{s})\hat{\rho}(\tfrac{3}{2},\vec{\alpha})] = \tfrac{1}{8}[-3-15(\vec{s}\vec{\alpha})+15(\vec{s}\vec{\alpha})^2+35(\vec{s}\vec{\alpha})^3] =$$
$$= \tfrac{1}{4}[1+3P_1(\vec{s}\vec{\alpha})+5P_2(\vec{s}\vec{\alpha})+7P_3(\vec{s}\vec{\alpha})], \qquad (70)$$

$$\rho_{3'}(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \tfrac{1}{4}\delta_{S^2}(\vec{s},\vec{\alpha}) = \tfrac{1}{4}\lim_{\eta\to 1-}\frac{1-\eta^2}{1-2\eta(\vec{s}\vec{\alpha})+\eta^2}$$
$$= \tfrac{1}{4}[1+\sum_{\ell=1}^{\infty}(2\ell+1)P_\ell(\vec{s}\vec{\alpha})], \qquad (71)$$

$$\rho_4(\vec{s};\tfrac{3}{2},\vec{\alpha}) = \mathrm{tr}[\hat{Z}(\vec{s})\hat{\rho}(\tfrac{3}{2},\vec{\alpha})] = \tfrac{1}{8}[7+25(\vec{s}\vec{\alpha})-15(\vec{s}\vec{\alpha})^2-\tfrac{35}{3}(\vec{s}\vec{\alpha})^3], \qquad (72)$$

$$\rho_1(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \mathrm{tr}[\hat{\rho}(\tfrac{3}{2},\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{\alpha})] = \tfrac{3}{8}(1+\vec{s}\vec{\alpha})^2(1-\vec{s}\vec{\alpha}), \qquad (73)$$

$$\rho_{1'}(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \mathrm{tr}[\hat{\rho}(\tfrac{1}{2},\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{\alpha})] = \tfrac{1}{8}(1+\vec{s}\vec{\alpha})(1-3\vec{s}\vec{\alpha})^2, \qquad (74)$$

$$\rho_2(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \mathrm{tr}[\hat{X}(\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{\alpha})] =$$
$$= \alpha + \tfrac{7}{4}\gamma + (\tfrac{1}{2}\beta + \tfrac{19}{4}\delta)(\vec{s}\vec{\alpha}) - \tfrac{3}{2}\gamma(\vec{s}\vec{\alpha})^2 - \tfrac{9}{4}\delta(\vec{s}\vec{\alpha})^3, \qquad (75)$$

$$\rho_3(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \mathrm{tr}[\hat{Y}(\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{\alpha})] = \tfrac{1}{8}[7+5\cdot 13(\vec{s}\vec{\alpha})-15(\vec{s}\vec{\alpha})^2-3\cdot 5\cdot 7(\vec{s}\vec{\alpha})^3], \qquad (76)$$

$$\rho_4(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \mathrm{tr}[\hat{Z}(\vec{s})\hat{\rho}(\tfrac{1}{2},\vec{\alpha})] = \tfrac{1}{8}[-3-15(\vec{s}\vec{\alpha})+15(\vec{s}\vec{\alpha})^2+35(\vec{s}\vec{\alpha})^3] =$$
$$= \tfrac{1}{4}[1+3P_1(\vec{s}\vec{\alpha})+5P_2(\vec{s}\vec{\alpha})+7P_3(\vec{s}\vec{\alpha})], \qquad (77)$$

$$\rho_{4'}(\vec{s};\tfrac{1}{2},\vec{\alpha}) = \tfrac{1}{4}\delta_{S^2}(\vec{s},\vec{\alpha}), \qquad (78)$$

where $\alpha,\beta,\gamma,\delta$ in eqs. (69) and (75) are given by eq. (61.a), $P_\ell(\vec{s}\vec{\alpha})$ are the Legendre polynomials, $\delta_{S^2}(\vec{s},\vec{\alpha})$ is the δ-function on the sphere S^2 (the Poisson kernel). The densities (70) and (77) can be replaced by the δ-function. All the densities are normalized according to eq. (47). Only the densities $\rho_1, \rho_{1'}, \rho_{3'}(\vec{s};\tfrac{3}{2},\vec{\alpha})$ and $\rho_{4'}(\vec{s};\tfrac{1}{2},\vec{\alpha})$ are positive definite. For expectation values of the spin projection $(\vec{b}\hat{\vec{s}})$ we get

$$\mathrm{tr}[(\vec{b}\hat{\vec{s}})\hat{\rho}(m,\vec{\alpha})] = 4c_i\int d\mu(\vec{s})(\vec{b}\vec{s})\rho_i(\vec{s};m,\vec{\alpha}), \qquad (79)$$

where $m=\tfrac{1}{2},\tfrac{3}{2}$; $i=1,1',2,3,3',4,4'$,

$$c_1=\tfrac{5}{2},\ c_{1'}=\tfrac{15}{2},\ c_2=\pm\tfrac{1}{2}\sqrt{15},\ c_3=c_{3'}=\tfrac{3}{2},\ c_4=c_{4'}=\tfrac{1}{2}. \qquad (80)$$

The factors c_i follow from

$$\mathrm{tr}\{(\vec{b}\hat{\vec{s}})[\hat{X}(\vec{s});\hat{Y}(\vec{s});\hat{Z}(\vec{s})]\} = [\pm\tfrac{1}{2}\sqrt{15};\tfrac{5}{2};\tfrac{15}{2}](\vec{b}\vec{s}). \qquad (81)$$

The singlet state of two spins $\tfrac{3}{2}$ is represented by any of the functions

$$\rho_1(\vec{\alpha},\vec{b}|singlet) = \rho(\tfrac{3}{2},\vec{\alpha};\tfrac{3}{2},\vec{b}|singlet) = \tfrac{1}{32}(1-\vec{\alpha}\vec{b})^3, \qquad (82)$$

$$\rho_{1'}(\vec{\alpha},\vec{b}|singlet) = \rho(\tfrac{1}{2},\vec{\alpha};\tfrac{1}{2},\vec{b}|singlet) = \tfrac{1}{32}(1-\vec{\alpha}\vec{b})(1+3\vec{\alpha}\vec{b})^2, \qquad (83)$$

$$\rho_2(\vec{\alpha},\vec{b}|singlet) = \mathrm{tr}_a\,\mathrm{tr}_b[\hat{X}^a(\vec{\alpha})\hat{X}^b(\vec{b})\hat{\rho}^{singlet}] =$$
$$= \tfrac{1}{16}[1-3P_1(\vec{\alpha}\vec{b})+5P_2(\vec{\alpha}\vec{b})-7P_3(\vec{\alpha}\vec{b})], \qquad (84)$$

$$\rho_{2'}(\vec{\alpha},\vec{b}|singlet) = \tfrac{1}{16}\delta_{S^2}(\vec{\alpha},-\vec{b}), \qquad (85)$$

$$\rho_3(\vec{\alpha},\vec{b}|singlet) = \mathrm{tr}_a\,\mathrm{tr}_b[\hat{Y}^a(\vec{\alpha})\hat{Y}^b(\vec{b})\hat{\rho}^{singlet}] =$$
$$= \tfrac{1}{32}[-23+5^2\cdot 29(\vec{\alpha}\vec{b})+75(\vec{\alpha}\vec{b})^2-5^2\cdot 7^2(\vec{\alpha}\vec{b})^3], \qquad (86)$$

$$\rho_4(\vec{\alpha},\vec{b}|singlet) = \mathrm{tr}_a\,\mathrm{tr}_b[\hat{Z}^a(\vec{\alpha})\hat{Z}^b(\vec{b})\hat{\rho}^{singlet}] =$$
$$= \tfrac{1}{32}[-23-\tfrac{25}{3}(\vec{\alpha}\vec{b})+75(\vec{\alpha}\vec{b})^2-(\tfrac{5\cdot 7}{3})^2(\vec{\alpha}\vec{b})^3]. \qquad (87)$$

All the densities are normalized according to eq. (53). The density ρ_2 can be replaced by $\rho_{2'}$. Only the densities $\rho_1, \rho_{1'}$ and $\rho_{2'}$ are positive definite. In terms of the above densities the correlator of projections of two spins $\frac{3}{2}$ in the singlet state takes the form

$$c(\vec{a},\vec{b}) = 4^2 c_i^2 \int d\mu(\vec{\xi}^a) \int d\mu(\vec{\xi}^b) (\vec{a}\vec{\xi}^a)(\vec{b}\vec{\xi}^b)\, \rho_i(\vec{\xi}^a, \vec{\xi}^b|\text{singlet}), \tag{88}$$

where i = 1, 1', 2, 2', 3, 4, the constants c_i are given by eq. (80), $c_2 = c_{2'} = \pm\frac{1}{2}\sqrt{15}$. These expressions for $c(\vec{a},\vec{b})$ with the positive ρ lead to the following inequalities

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}') + c(\vec{a}',\vec{b})| \leq 2\cdot c_i^2 = 2\cdot\begin{cases} \left(\frac{5}{2}\right)^2 & i=1 \\ \left(\frac{15}{2}\right)^2 & i=1' \\ \frac{15}{4} & i=2' \end{cases} \tag{89}$$

Among all the estimations (55) and (89) the last is the best. It can be easily obtained for any spin s as follows. Indeed, according to eq. (41)

$$c(\vec{a},\vec{b}) = -\frac{1}{3}s(s+1)(\vec{a}\vec{b}) =$$
$$= s(s+1)\int d\mu(\vec{\xi}^a)\int d\mu(\vec{\xi}^b)(\vec{a}\vec{\xi}^a)(\vec{b}\vec{\xi}^b)\,\delta_{S2}(\vec{\xi}^a, -\vec{\xi}^b). \tag{90}$$

This expression leads to the inequality

$$|c(\vec{a},\vec{b}) - c(\vec{a},\vec{b}')| + |c(\vec{a}',\vec{b}') + c(\vec{a}',\vec{b})| \leq 2\cdot s(s+1) \tag{91}$$

for any spin s. From eq. (91) it follows

$$|\vec{a}\vec{b} - \vec{a}\vec{b}'| + |\vec{a}'\vec{b}' + \vec{a}'\vec{b}| \leq 2\cdot 3 \tag{92}$$

with no dependence on the spin. Note that $\delta_{S2}(\vec{\xi}^a, -\vec{\xi}^b)$ may be treated as a classical counterpart of the singlet state, being the density $\delta_{S2}(\vec{\xi}^a, \vec{\xi}_0)\delta_{S2}(\vec{\xi}^b, -\vec{\xi}_0)$ for two "spins", integrated over all the initial directions $\vec{\xi}_0$ $\delta_{S2}(\vec{\xi}^a, -\vec{\xi}^b) = \int d\mu(\vec{\xi}_0)\delta_{S2}(\vec{\xi}^a, \vec{\xi}_0)\,\delta_{S2}(\vec{\xi}^b, -\vec{\xi}_0)$. However the quantum correlator $c(\vec{a},\vec{b})$ contains the superfluous factor

Appendix A. Canonical representation of spin $\frac{3}{2}$ matrices (see ref. /13/).

$$\hat{s}_1 = \frac{1}{2}\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & \sqrt{3} \\ \cdot & \cdot & \sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_2 = \frac{i}{2}\begin{pmatrix} \cdot & -\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & -2 & \cdot \\ \cdot & 2 & \cdot & -\sqrt{3} \\ \cdot & \cdot & \sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_3 = \frac{1}{2}\begin{pmatrix} 3 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -3 \end{pmatrix}$$

The dots stand for zeros.

Simple products of the spin $\frac{3}{2}$ matrices $\hat{s}_i$.

$$\hat{s}_1^2 = \frac{1}{4}\begin{pmatrix} 3 & \cdot & 2\sqrt{3} & \cdot \\ \cdot & 7 & \cdot & 2\sqrt{3} \\ 2\sqrt{3} & \cdot & 7 & \cdot \\ \cdot & 2\sqrt{3} & \cdot & 3 \end{pmatrix}, \quad \hat{s}_1\hat{s}_2 = \frac{i}{4}\begin{pmatrix} 3 & \cdot & -2\sqrt{3} & \cdot \\ \cdot & 1 & \cdot & -2\sqrt{3} \\ 2\sqrt{3} & \cdot & -1 & \cdot \\ \cdot & 2\sqrt{3} & \cdot & -3 \end{pmatrix}, \quad \hat{s}_1\hat{s}_3 = \frac{1}{4}\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 3\sqrt{3} & \cdot & -2 & \cdot \\ \cdot & 2 & \cdot & -3\sqrt{3} \\ \cdot & \cdot & -\sqrt{3} & \cdot \end{pmatrix},$$

$$\hat{s}_2\hat{s}_1 = \frac{i}{4}\begin{pmatrix} -3 & \cdot & -2\sqrt{3} & \cdot \\ \cdot & -1 & \cdot & -2\sqrt{3} \\ 2\sqrt{3} & \cdot & 1 & \cdot \\ \cdot & 2\sqrt{3} & \cdot & 3 \end{pmatrix}, \quad \hat{s}_2^2 = \frac{1}{4}\begin{pmatrix} 3 & \cdot & -2\sqrt{3} & \cdot \\ \cdot & 7 & \cdot & -2\sqrt{3} \\ -2\sqrt{3} & \cdot & 7 & \cdot \\ \cdot & -2\sqrt{3} & \cdot & 3 \end{pmatrix}, \quad \hat{s}_2\hat{s}_3 = \frac{i}{4}\begin{pmatrix} \cdot & -\sqrt{3} & \cdot & \cdot \\ 3\sqrt{3} & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & 3\sqrt{3} \\ \cdot & \cdot & -\sqrt{3} & \cdot \end{pmatrix},$$

$$\hat{s}_3\hat{s}_1 = \frac{1}{4}\begin{pmatrix} \cdot & 3\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & 2 & \cdot \\ \cdot & -2 & \cdot & -\sqrt{3} \\ \cdot & \cdot & -3\sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_3\hat{s}_2 = \frac{i}{4}\begin{pmatrix} \cdot & -3\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & -2 & \cdot \\ \cdot & -2 & \cdot & \sqrt{3} \\ \cdot & \cdot & -3\sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_3^2 = \frac{1}{4}\begin{pmatrix} 9 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 9 \end{pmatrix},$$

$$\hat{s}_1^3 = \frac{1}{8}\begin{pmatrix} \cdot & 7\sqrt{3} & \cdot & 6 \\ 7\sqrt{3} & \cdot & 20 & \cdot \\ \cdot & 20 & \cdot & 7\sqrt{3} \\ 6 & \cdot & 7\sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_1^2\hat{s}_2 = \frac{i}{8}\begin{pmatrix} \cdot & \sqrt{3} & \cdot & -6 \\ 7\sqrt{3} & \cdot & -8 & \cdot \\ \cdot & 8 & \cdot & -7\sqrt{3} \\ 6 & \cdot & -\sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_1^2\hat{s}_3 = \frac{1}{8}\begin{pmatrix} 9 & \cdot & -2\sqrt{3} & \cdot \\ \cdot & 7 & \cdot & -6\sqrt{3} \\ 6\sqrt{3} & \cdot & -7 & \cdot \\ \cdot & 2\sqrt{3} & \cdot & -9 \end{pmatrix},$$

$$\hat{s}_1\hat{s}_2\hat{s}_1 = \frac{i}{8}\begin{pmatrix} \cdot & -\sqrt{3} & \cdot & -6 \\ \sqrt{3} & \cdot & -4 & \cdot \\ \cdot & 4 & \cdot & -\sqrt{3} \\ 6 & \cdot & \sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_1\hat{s}_2^2 = \frac{1}{8}\begin{pmatrix} \cdot & 7\sqrt{3} & \cdot & -6 \\ -\sqrt{3} & \cdot & 8 & \cdot \\ \cdot & 8 & \cdot & -\sqrt{3} \\ -6 & \cdot & 7\sqrt{3} & \cdot \end{pmatrix}, \quad \hat{s}_1\hat{s}_2\hat{s}_3 = \frac{i}{8}\begin{pmatrix} 9 & \cdot & 2\sqrt{3} & \cdot \\ \cdot & 1 & \cdot & 6\sqrt{3} \\ 6\sqrt{3} & \cdot & 1 & \cdot \\ \cdot & 2\sqrt{3} & \cdot & 9 \end{pmatrix},$$

$$\hat{s}_1\hat{s}_3\hat{s}_1 = \frac{1}{8}\begin{pmatrix} 3 & \cdot & 2\sqrt{3} & \cdot \\ \cdot & 5 & \cdot & -2\sqrt{3} \\ 2\sqrt{3} & \cdot & -5 & \cdot \\ \cdot & -2\sqrt{3} & \cdot & -3 \end{pmatrix}, \quad \hat{s}_1\hat{s}_3\hat{s}_2 = \frac{i}{8}\begin{pmatrix} 3 & \cdot & -2\sqrt{3} & \cdot \\ \cdot & -13 & \cdot & 2\sqrt{3} \\ 2\sqrt{3} & \cdot & -13 & \cdot \\ \cdot & -2\sqrt{3} & \cdot & 3 \end{pmatrix}, \quad \hat{s}_1\hat{s}_3^2 = \frac{1}{8}\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 9\sqrt{3} & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & 9\sqrt{3} \\ \cdot & \cdot & \sqrt{3} & \cdot \end{pmatrix},$$

$$\hat{s}_2\hat{s}_1^2=\frac{i}{8}\begin{pmatrix}\cdot & -7\sqrt{3} & \cdot & -6\\ -\sqrt{3} & \cdot & -8 & \cdot\\ \cdot & 8 & \cdot & \sqrt{3}\\ 6 & \cdot & 7\sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_2\hat{s}_1\hat{s}_2=\frac{1}{8}\begin{pmatrix}\cdot & \sqrt{3} & \cdot & -6\\ \sqrt{3} & \cdot & 4 & \cdot\\ \cdot & 4 & \cdot & \sqrt{3}\\ -6 & \cdot & \sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_2\hat{s}_1\hat{s}_3=\frac{i}{8}\begin{pmatrix}-9 & \cdot & 2\sqrt{3} & \cdot\\ \cdot & -1 & \cdot & 6\sqrt{3}\\ 6\sqrt{3} & \cdot & -1 & \cdot\\ \cdot & 2\sqrt{3} & \cdot & -9\end{pmatrix},$$

$$\hat{s}_2^2\hat{s}_1=\frac{1}{8}\begin{pmatrix}\cdot & -\sqrt{3} & \cdot & -6\\ 7\sqrt{3} & \cdot & 8 & \cdot\\ \cdot & 8 & \cdot & 7\sqrt{3}\\ -6 & \cdot & -\sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_2^3=\frac{i}{8}\begin{pmatrix}\cdot & -7\sqrt{3} & \cdot & 6\\ 7\sqrt{3} & \cdot & -20 & \cdot\\ \cdot & 20 & \cdot & -7\sqrt{3}\\ -6 & \cdot & 7\sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_2^2\hat{s}_3=\frac{1}{8}\begin{pmatrix}9 & \cdot & 2\sqrt{3} & \cdot\\ \cdot & 7 & \cdot & 6\sqrt{3}\\ -6\sqrt{3} & \cdot & -7 & \cdot\\ \cdot & -2\sqrt{3} & \cdot & -9\end{pmatrix},$$

$$\hat{s}_2\hat{s}_3\hat{s}_1=\frac{i}{8}\begin{pmatrix}-3 & \cdot & -2\sqrt{3} & \cdot\\ \cdot & 13 & \cdot & 2\sqrt{3}\\ 2\sqrt{3} & \cdot & 13 & \cdot\\ \cdot & -2\sqrt{3} & \cdot & -3\end{pmatrix},\quad \hat{s}_2\hat{s}_3\hat{s}_2=\frac{1}{8}\begin{pmatrix}3 & \cdot & -2\sqrt{3} & \cdot\\ \cdot & 5 & \cdot & 2\sqrt{3}\\ -2\sqrt{3} & \cdot & -5 & \cdot\\ \cdot & 2\sqrt{3} & \cdot & -3\end{pmatrix},\quad \hat{s}_2\hat{s}_3^2=\frac{i}{8}\begin{pmatrix}\cdot & -\sqrt{3} & \cdot & \cdot\\ 9\sqrt{3} & \cdot & -2 & \cdot\\ \cdot & 2 & \cdot & -9\sqrt{3}\\ \cdot & \cdot & \sqrt{3} & \cdot\end{pmatrix},$$

$$\hat{s}_3\hat{s}_1^2=\frac{1}{8}\begin{pmatrix}9 & \cdot & 6\sqrt{3} & \cdot\\ \cdot & 7 & \cdot & 2\sqrt{3}\\ -2\sqrt{3} & \cdot & -7 & \cdot\\ \cdot & -6\sqrt{3} & \cdot & -9\end{pmatrix},\quad \hat{s}_3\hat{s}_1\hat{s}_2=\frac{i}{8}\begin{pmatrix}9 & \cdot & -6\sqrt{3} & \cdot\\ \cdot & 1 & \cdot & -2\sqrt{3}\\ -2\sqrt{3} & \cdot & 1 & \cdot\\ \cdot & -6\sqrt{3} & \cdot & 9\end{pmatrix},\quad \hat{s}_3\hat{s}_1\hat{s}_3=\frac{1}{8}\begin{pmatrix}\cdot & 3\sqrt{3} & \cdot & \cdot\\ 3\sqrt{3} & \cdot & -2 & \cdot\\ \cdot & -2 & \cdot & 3\sqrt{3}\\ \cdot & \cdot & 3\sqrt{3} & \cdot\end{pmatrix},$$

$$\hat{s}_3\hat{s}_2\hat{s}_1=\frac{-i}{8}\begin{pmatrix}9 & \cdot & 6\sqrt{3} & \cdot\\ \cdot & 1 & \cdot & 2\sqrt{3}\\ 2\sqrt{3} & \cdot & 1 & \cdot\\ \cdot & 6\sqrt{3} & \cdot & 9\end{pmatrix},\quad \hat{s}_3\hat{s}_2^2=\frac{1}{8}\begin{pmatrix}9 & \cdot & -6\sqrt{3} & \cdot\\ \cdot & 7 & \cdot & -2\sqrt{3}\\ 2\sqrt{3} & \cdot & -7 & \cdot\\ \cdot & 6\sqrt{3} & \cdot & -9\end{pmatrix},\quad \hat{s}_3\hat{s}_2\hat{s}_3=\frac{i}{8}\begin{pmatrix}\cdot & -3\sqrt{3} & \cdot & \cdot\\ 3\sqrt{3} & \cdot & 2 & \cdot\\ \cdot & -2 & \cdot & -3\sqrt{3}\\ \cdot & \cdot & 3\sqrt{3} & \cdot\end{pmatrix},$$

$$\hat{s}_3^2\hat{s}_1=\frac{1}{8}\begin{pmatrix}\cdot & 9\sqrt{3} & \cdot & \cdot\\ \sqrt{3} & \cdot & 2 & \cdot\\ \cdot & 2 & \cdot & \sqrt{3}\\ \cdot & \cdot & 9\sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_3^2\hat{s}_2=\frac{i}{8}\begin{pmatrix}\cdot & -9\sqrt{3} & \cdot & \cdot\\ \sqrt{3} & \cdot & -2 & \cdot\\ \cdot & 2 & \cdot & -\sqrt{3}\\ \cdot & \cdot & 9\sqrt{3} & \cdot\end{pmatrix},\quad \hat{s}_3^3=\frac{1}{8}\begin{pmatrix}27 & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & \cdot & -27\end{pmatrix},$$

Symmetrized products of the spin $\frac{3}{2}$ matrices $\hat{s}_i$.

$$\{\hat{s}_1\hat{s}_2\}=\hat{s}_1\hat{s}_2+\hat{s}_2\hat{s}_1=i\sqrt{3}\begin{pmatrix}\cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & \cdot & -1\\ 1 & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot\end{pmatrix},\quad \{\hat{s}_2\hat{s}_3\}=\hat{s}_2\hat{s}_3+\hat{s}_3\hat{s}_2=i\sqrt{3}\begin{pmatrix}\cdot & -1 & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & 1\\ \cdot & \cdot & -1 & \cdot\end{pmatrix},\quad \{\hat{s}_3\hat{s}_1\}=\hat{s}_3\hat{s}_1+\hat{s}_1\hat{s}_3=\sqrt{3}\begin{pmatrix}\cdot & 1 & \cdot & \cdot\\ 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & -1 & \cdot\end{pmatrix},$$

$$\frac{1}{2}\{\hat{s}_1\hat{s}_2\hat{s}_2\}=\hat{s}_1\hat{s}_2^2+\hat{s}_2^2\hat{s}_1+\hat{s}_2\hat{s}_1\hat{s}_2=\frac{1}{8}\begin{pmatrix}\cdot & 7\sqrt{3} & \cdot & -18\\ 7\sqrt{3} & \cdot & 20 & \cdot\\ \cdot & 20 & \cdot & 7\sqrt{3}\\ -18 & \cdot & 7\sqrt{3} & \cdot\end{pmatrix},\quad \frac{1}{2}\{\hat{s}_1\hat{s}_3\hat{s}_3\}=\hat{s}_1\hat{s}_3^2+\hat{s}_3^2\hat{s}_1+\hat{s}_3\hat{s}_1\hat{s}_3=\frac{1}{8}\begin{pmatrix}\cdot & 13\sqrt{3} & \cdot & \cdot\\ 13\sqrt{3} & \cdot & 2 & \cdot\\ \cdot & 2 & \cdot & 13\sqrt{3}\\ \cdot & \cdot & 13\sqrt{3} & \cdot\end{pmatrix},$$

$$\frac{1}{2}\{\hat{s}_2\hat{s}_1\hat{s}_1\}=\hat{s}_2\hat{s}_1^2+\hat{s}_1^2\hat{s}_2+\hat{s}_1\hat{s}_2\hat{s}_1=\frac{i}{8}\begin{pmatrix}\cdot & -7\sqrt{3} & \cdot & -18\\ 7\sqrt{3} & \cdot & -20 & \cdot\\ \cdot & 20 & \cdot & -7\sqrt{3}\\ 18 & \cdot & 7\sqrt{3} & \cdot\end{pmatrix},\quad \frac{1}{2}\{\hat{s}_2\hat{s}_3\hat{s}_3\}=\hat{s}_2\hat{s}_3^2+\hat{s}_3^2\hat{s}_2+\hat{s}_3\hat{s}_2\hat{s}_3=\frac{i}{8}\begin{pmatrix}\cdot & -13\sqrt{3} & \cdot & \cdot\\ 13\sqrt{3} & \cdot & -2 & \cdot\\ \cdot & 2 & \cdot & -13\sqrt{3}\\ \cdot & \cdot & 13\sqrt{3} & \cdot\end{pmatrix},$$

$$\frac{1}{2}\{\hat{s}_3\hat{s}_1\hat{s}_1\}=\hat{s}_3\hat{s}_1^2+\hat{s}_1^2\hat{s}_3+\hat{s}_1\hat{s}_3\hat{s}_1=\frac{1}{8}\begin{pmatrix}21 & \cdot & 6\sqrt{3} & \cdot\\ \cdot & 19 & \cdot & -6\sqrt{3}\\ 6\sqrt{3} & \cdot & -19 & \cdot\\ \cdot & -6\sqrt{3} & \cdot & -21\end{pmatrix},\quad \frac{1}{2}\{\hat{s}_3\hat{s}_2\hat{s}_2\}=\hat{s}_3\hat{s}_2^2+\hat{s}_2^2\hat{s}_3+\hat{s}_2\hat{s}_3\hat{s}_2=\frac{1}{8}\begin{pmatrix}21 & \cdot & -6\sqrt{3} & \cdot\\ \cdot & 19 & \cdot & 6\sqrt{3}\\ -6\sqrt{3} & \cdot & -19 & \cdot\\ \cdot & 6\sqrt{3} & \cdot & -21\end{pmatrix},$$

$$\{\hat{s}_1\hat{s}_2\hat{s}_3\}=\frac{3}{2}\sqrt{3}\,i\begin{pmatrix}\cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & \cdot & 1\\ 1 & \cdot & \cdot & \cdot\\ \cdot & -1 & \cdot & \cdot\end{pmatrix}.$$

Appendix B. Calculation of traces (29). If the special frame of reference with $Oz \parallel \vec{b}$ is chosen and if the canonical representation of the spin matrices is adopted (see Appendix A), the density matrices take obviously the form

$$\hat{\rho}(\tfrac{3}{2},\vec{b})=\begin{bmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{bmatrix},\qquad \hat{\rho}(\tfrac{1}{2},\vec{b})=\begin{bmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{bmatrix},$$

$$\hat{\rho}(-\tfrac{1}{2},\vec{b})=\begin{bmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{bmatrix},\qquad \hat{\rho}(-\tfrac{3}{2},\vec{b})=\begin{bmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1\end{bmatrix}. \tag{B.1}$$

Now the calculation of the trace (29) is reduced to finding of some diagonal element of the matrix $\hat{\rho}(m,\vec{a})$. Since $\hat{\rho}(m,\vec{a})$ consists of the products $(\vec{a}\hat{\vec{s}})^k$, we first calculate the traces $tr[\hat{\rho}(m,\vec{b})(\vec{a}\hat{\vec{s}})^k]$ ($k = 1,2,3$) for $Oz \parallel \vec{b}$. Let us decompose $(\vec{a}\hat{\vec{s}})^2$ and $(\vec{a}\hat{\vec{s}})^3$ into the symmetrized products as follows

$$(\vec{a}\hat{\vec{s}})^2 = a_1^2\hat{s}_1^2 + a_2^2\hat{s}_2^2 + a_3^2\hat{s}_3^2 + a_1a_2\{\hat{s}_1\hat{s}_2\} + a_2a_3\{\hat{s}_2\hat{s}_3\} + a_3a_1\{\hat{s}_3\hat{s}_1\},$$

$$\begin{aligned}(\vec{a}\hat{\vec{s}})^3 &= a_1^3\hat{s}_1^3 + a_2^3\hat{s}_2^3 + a_3^3\hat{s}_3^3 + a_1a_2a_3\{\hat{s}_1\hat{s}_2\hat{s}_3\} +\\ &+ a_1a_2^2(\hat{s}_1\hat{s}_2^2+\hat{s}_2^2\hat{s}_1+\hat{s}_2\hat{s}_1\hat{s}_2) + a_1a_3^2(\hat{s}_1\hat{s}_3^2+\hat{s}_3^2\hat{s}_1+\hat{s}_3\hat{s}_1\hat{s}_3) +\\ &+ a_2a_1^2(\hat{s}_2\hat{s}_1^2+\hat{s}_1^2\hat{s}_2+\hat{s}_1\hat{s}_2\hat{s}_1) + a_2a_3^2(\hat{s}_2\hat{s}_3^2+\hat{s}_3^2\hat{s}_2+\hat{s}_3\hat{s}_2\hat{s}_3) +\\ &+ a_3a_1^2(\hat{s}_3\hat{s}_1^2+\hat{s}_1^2\hat{s}_3+\hat{s}_1\hat{s}_3\hat{s}_1) + a_3a_2^2(\hat{s}_3\hat{s}_2^2+\hat{s}_2^2\hat{s}_3+\hat{s}_2\hat{s}_3\hat{s}_2).\end{aligned} \tag{B.2}$$

Using the table of the symmetrized products of Appendix A, we obtain

$$\begin{aligned}&tr[\hat{\rho}(\tfrac{3}{2},\vec{b})\mathbb{1}]=1,\\ &tr[\hat{\rho}(\tfrac{3}{2},\vec{b})(\vec{a}\hat{\vec{s}})]=\tfrac{3}{2}a_3,\\ &tr[\hat{\rho}(\tfrac{3}{2},\vec{b})(\vec{a}\hat{\vec{s}})^2]=\tfrac{3}{4}a_1^2+\tfrac{3}{4}a_2^2+\tfrac{9}{4}a_3^2=\tfrac{3}{4}+\tfrac{3}{2}a_3^2,\\ &tr[\hat{\rho}(\tfrac{3}{2},\vec{b})(\vec{a}\hat{\vec{s}})^3]=\tfrac{21}{8}a_3+\tfrac{3}{4}a_3^3;\end{aligned} \tag{B.3}$$

$$\begin{aligned}&tr[\hat{\rho}(\tfrac{1}{2},\vec{b})\mathbb{1}]=1,\\ &tr[\hat{\rho}(\tfrac{1}{2},\vec{b})(\vec{a}\hat{\vec{s}})]=\tfrac{1}{2}a_3,\\ &tr[\hat{\rho}(\tfrac{1}{2},\vec{b})(\vec{a}\hat{\vec{s}})^2]=\tfrac{7}{4}-\tfrac{3}{2}a_3^2,\\ &tr[\hat{\rho}(\tfrac{1}{2},\vec{b})(\vec{a}\hat{\vec{s}})^3]=\tfrac{19}{8}a_3-\tfrac{9}{4}a_3^3.\end{aligned} \tag{B.4}$$

Then we get the traces desired

$$\begin{aligned}&\rho(\tfrac{3}{2},\vec{b};\tfrac{3}{2},\vec{a})=tr(\hat{\rho}(\tfrac{3}{2},\vec{b})\hat{\rho}(\tfrac{3}{2},\vec{a}))=\tfrac{1}{8}(1+a_3)^3,\\ &\rho(\tfrac{1}{2},\vec{b};\tfrac{3}{2},\vec{a})=tr(\hat{\rho}(\tfrac{1}{2},\vec{b})\hat{\rho}(\tfrac{3}{2},\vec{a}))=\tfrac{3}{8}(1+a_3)^2(1-a_3),\\ &\rho(\tfrac{1}{2},\vec{b};\tfrac{1}{2},\vec{a})=tr(\hat{\rho}(\tfrac{1}{2},\vec{b})\hat{\rho}(\tfrac{1}{2},\vec{a}))=\tfrac{1}{8}(1+a_3)(1-3a_3)^2.\end{aligned} \tag{B.5}$$

In the general frame of reference a_3 converts into $(\vec{a}\vec{b})$. The list of these probabilities is given in Table 1 on p. 7.

Appendix C. The direct derivation of eq. (42). In the special frame of reference with $Oz \parallel \vec{b}$ we have

$$\begin{aligned}&\rho(m,\vec{a};n,\vec{b}\,|\,singlet)=\\ &=\tfrac{1}{4}(1\,0\,0\,0)\,\hat{\rho}^a(m,\vec{a})\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\otimes(0\,0\,0\,1)\,\hat{\rho}^b(n,\vec{b})\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+\\ &+\tfrac{1}{4}(0\,1\,0\,0)\,\hat{\rho}^a(m,\vec{a})\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\otimes(0\,0\,1\,0)\,\hat{\rho}^b(n,\vec{b})\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+\\ &+\tfrac{1}{4}(0\,0\,1\,0)\,\hat{\rho}^a(m,\vec{a})\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\otimes(0\,1\,0\,0)\,\hat{\rho}^b(n,\vec{b})\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+\\ &+\tfrac{1}{4}(0\,0\,0\,1)\,\hat{\rho}^a(m,\vec{a})\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\otimes(1\,0\,0\,0)\,\hat{\rho}^b(n,\vec{b})\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\end{aligned} \tag{C.1}$$

where the matrices $\hat{\rho}^b(n,\vec{b})$ are given by expressions (B.1). Other 12 terms of this sum equal zero since $\hat{\rho}^b(n,\vec{b})$ are purely diagonal. For a fixed n only one term remains in the expression (C.1) since only one matrix element $(\quad)\hat{\rho}^b(n,\vec{b})[\quad]$ of four differs from zero. The matrix elements $(\quad)\hat{\rho}^a(m,\vec{a})[\quad]$ are given by eqs. (B.5) of Appendix B or by Table 1 in the general frame of reference (of course, with $m \to -m$, or $\vec{a} \to -\vec{a}$, or $n \to -n$, or $\vec{b} \to -\vec{b}$).

It is clear that the same calculation is applicable for any spin s. Then $\frac{1}{4}$ is replaced by $\frac{1}{2s+1}$. There remain $2s+1$ terms in the sum similar to (C.1), and only one of them is nonzero for a given n.

Referenses

1. Feynman R.P., Hibbs A.R. Quantum mechanics and path integrals. McGraw-Hill, New York, 1965.

2. Bell J.S. Physica,1964, 1, p.195.
3. Bell J.S. Rev. Mod. Phys.,1966, 38, p.447.
4. Bell J.S. Foundations of Quantum Mechanics (ed. B. Espagnat), Academic Press, New York, 1971, p.171.
5. Clauser J.P., Shimoni A. Reports on Progress in Physics, 1978, 41, p.1881.
6. Mermin N.D. Phys. Rev.,1980, D22, p.356.
7. Arecchi F.T., Courtens E., Gilmore R., Thomas H. Phys. Rev.,1972, A6, p.2211.
8. Klauder J.P. Journ. Phys.,1983, A16, p. L317.
9. Переломов А.М. Обобщенные когерентные состояния и их приложения. "Наука", М., 1987.
10. Polubarinov I.V. JINR, E2-11478, 1978 ; E2-81-240, 1981 ; E2-82-800, 1982 ; E2-83-688, Dubna, 1983.
11. Полубаринов И.В. В кн.: "Теоретико-групповые методы в физике". "Наука", М., 1983, т.2, стр. 3 и 189.
Polubarinov I.V. In: Group Theoretical Methods in Physics, Harwood Academic Publishers, Chur, London, Paris, New York, 1985, v.1, pp. 167 and 577.
12. Polubarinov I.V. JINR, E2-88-80, Dubna, 1988.
13. Condon E.U., Shortley G.H. The theory of atomic spectra. Cambridge, 1935.
Edmonds A.R. In:"Деформация атомных ядер. ИЛ, М., 1958.
14. Cirelson B.S. Lett. Math. Phys. 1980, 4, p. 93.

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