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NORMAL FORM AND APPROXIMATE INTEGRALS OF TWO DIMENSIONAL HAMILTONIAN SYSTEMS

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Introduction

In the last decades the investigation of classical mechanics ground stimulated by discovery of dynamical systems chaotic motion have stated once more the fundamental question of quantun mechanics: what is the interplay between classical integrability and properties of the quantized system[1-5]. The transformation of classical Hamilton systems into Birkhoff-Gustavson normal form (BGNF)[6-7] provides a powerfull (even formal) tool for study this problem[8].

In this talk new approximate integrals of motion for two simple models for nuclei have written out: i) three α-particles on the line [9] ;ii) collective surface vibrations in the liquid drop model for superdeformed ^{74}Kr [10,11] nucleus . Making use of the corresponding quantum Hamiltonians both quasiclassical spectra and phase space structure have been analysed and compaired with the "exact" numerical results.

The first rather quick programs in FORTRAN for obtaining normal forms were realised in ref.[7,12] or that[13] suggested by Hori[14] and Deprit[15] In the present paper, we explore our REDUCE program GITA[16] for analytical construction of BGNF according to the Gustavson algorithm [7] as it is briefly represented in sec.2. This program is used.for finding the normal form of up to a given order s_{max} both for resonante and nonresonante cases.

1. The Birkhoff-Gustavson normal form

Every two-dimensional Hamiltonian near to equilibrium points can be represented in polynomial form as follows:

$$H(q,p)= H^{(2)}(q,p) + V(q) \quad (1)$$

where

$$H^{(2)}(q,p)= \sum_{\nu} 1/2\ \omega_\nu^2\ (p_\nu^2 + q_\nu^2), \quad (2)$$

$$V(q) = \sum_{|j|\geq 3} V_{j_1 j_2} q_1^{j_1} q_2^{j_2}\ ,q=(q_1,q_2),p=(p_1,p_2) \quad (3)$$

The procedure of reducing to BGNF form and its realization depend whether the frequencies ω_ν of the Hamiltonian (2) are incommensurable or not. If they are then there exists a canonical transformation $(q,p)\to(\xi,\eta)$ such that in variables (ξ,η) the Hamiltonian $\Gamma(\xi,\eta)$ will be a function of only two combinations $I_\nu=1/2(\xi_\nu^2+\eta_\nu^2)$, $\nu=1,2$. In other words, the Birkhoff normal form is an expansion of the initial Hamiltonian over two one-dimensional harmonic oscillators

$$H(q,p)\to \Gamma(\xi,\eta)= \omega_1 I_1+\omega_2 I_2+ \sum \alpha_{\mu\nu} I_\mu I_\nu+\dots \quad (4)$$

If the frequencies ω_ν are commensurable, i.e. if there exist resonance relations of the type $n\omega_1+m\omega_2= 0, (n,m \in \mathbb{N} \setminus \{0\})$ the Hamiltonian (1) can not be reduced to the normal Birkhoff form due to the appearence of zero denominators $n\omega_1+m\omega_2=0$. Therefore, one schould not cancel some terms in the Hamiltonian (1), i.e. the normal form becomes more complicated and will contain apart from I_ν other combinations of variables ξ_ν and η_ν as well. In our cases with resonance condition $\omega_1=\omega_2$ these combinations are $\xi_1\eta_2- \xi_2\eta_1$ and $\eta_1\eta_2+ \xi_1\xi_2$. Such an extended normal form is called the Birkhoff- Gustavson normal form

Besides that, since $I_1(\xi,\eta)$ and $\Gamma_s(\xi,\eta)$ are indepenent integrals of motion up to s-approximation, expressing ξ and η through the initial coordinates q,p, we obtain the approximate integral of motion $I_1(q,p)$ mentioned in the Introduction.

The procedure of reducing Hamiltonian (1) to BGNF contain a sequence of canonical transformations

$$\xi_\nu= q_\nu+ \frac{\partial W^{(j)}(q,\eta)}{\partial\eta_\nu};\ p_\nu= \eta_\nu+ \frac{\partial W^{(j)}(q,\eta)}{\partial q_\nu} \quad ,\ \nu= 1,2 \quad (5)$$

with the help of the s-order polynomial generating function, $W^{(s)}$

$$F_2(q,\eta)= \sum q_\nu\eta_\nu + W^{(s)}(q,\eta) \quad (6)$$

The integer s labels the power of that Hamiltonian's part which is reduced to the normal form .

Under canonical transformation (5) the Hamiltonian H(q,p) turns into a new one $\Gamma(\xi,\eta)$ so that

$$H(q,\eta+ \frac{\partial W^{(s)}}{\partial q})= \Gamma(q + \frac{\partial W^{(s)}}{\partial \eta} ,\eta). \quad (7)$$

Expanding this expression in Taylor series near $q=\xi,\eta=p$ we equate the terms of the same power. For i=2 we get the known result that the Hamiltonian (2) is still in a normal form

$$\Gamma^{(2)}(\xi,\eta)= H^{(2)}(q,p);\quad q=\xi,p=\eta \quad (8)$$

For higher terms (i.e. $i=s\geq3$) the procedure starts by, we derive a generic equation in the form

$$D(q,\eta)W^{(s)}(q,\eta)= -H^{(s)}(q,\eta) + \Gamma^{(s)}(q,\eta) \quad (9)$$

where D is an "operator of the normal form":

$$D(\xi,\eta) = \sum \omega_\nu(\eta_\nu \frac{\partial}{\partial\xi_\nu} - \xi_\nu\frac{\partial}{\partial\eta_\nu}). \quad (10)$$

One should solve eq.(9) at each step i=3,..s-2 reducing the Hamiltonian to the normal form of order s. Eq(9) we have to find two unknown functions: the generating polynomial $W^{(s)}$ and the normal form Hamiltonian $\Gamma^{(s)}$. To solve Eq.(9) a canonical transformation to complex variables x, y is performed

$$q_\nu= 1/\sqrt{2}\ (x_\nu+iy_\nu),\eta_\nu= i/\sqrt{2}\ (x_\nu- iy_\nu), \quad (11)$$

diagonalizing the operator

$$\tilde{D}(x,y) = i \sum \omega_\nu(x_\nu\frac{\partial}{\partial x_\nu} - y_\nu \frac{\partial}{\partial y_\nu}). \quad (12)$$

Then the basic equation becomes

$$\tilde{D}\tilde{W}^{(s)}(x,y)= - \tilde{H}(x,y) + \tilde{\Gamma}^{(s)}(x,y). \quad (13)$$

The basis in the Hilbert space where H is self-adjont consists of monomials

$$\Phi^{(s)}= x_1^{l_1}x_2^{l_2}\ y_1^{m_1}\ y_2^{m_2} \quad (14)$$

being eigenfunctions of the oprator D ,namely

$$\tilde{D}\ \Phi^{(s)}= \left[i \sum \omega_\nu(m_\nu-l_\nu)\right]\Phi^{(s)}. \quad (15)$$

Then ,the solution of the generic equation can be written as follows

$$\tilde{W}^{(s)}= \tilde{D}^{-1}[\tilde{\Gamma}^{(s)}- \tilde{H}^{(s)}]. \quad (16)$$

Now the quantity $\tilde{H}^{(s)}$ can unambiguosly be represented as a sum of polynomials $\tilde{H}^{(s)} = \tilde{N}^{(s)} + \tilde{R}^{(s)}$ with $\tilde{D}\tilde{N}^{(s)} = 0$. Choosing $\tilde{\Gamma}^{(s)} = \tilde{N}^{(s)}$ (the normal form) we see that the terms of the Hamiltonian $\tilde{H}^{(s)}$ leading to the condition $\sum\omega_\nu(m_\nu - l_\nu) = 0$ cancels by the same terms in the Hamiltonian $\Gamma^{(s)}$ .Moreover, the equation $\sum\omega_\nu(m_\nu - l_\nu) = 0$ holds either if ω_ν are incommensurable and $m_\nu = l_\nu$, or if $m_\nu \neq l_\nu$ but there exists a resonance relation of the form $\sum n_\nu\omega_\nu = 0$. In the last case, the structure of the normal form Γ becomes more complicated due to the presence of resonant terms $\xi_1\eta_2 + \xi_2\eta_1$ and $\eta_1\eta_2 + \xi_1\xi_2$. Making an inverse transformation from the complex variables x and y to variables ξ, η we finally find the Hamiltonian in the normal form $\Gamma^{(s)}(\xi,\eta)$ and the generating function $W^{(s)}$ in the s order of approximation in terms of the initial variables.

A certain modification [17] of the reducing procedure is much more appropriate for finding the BGNF. One introduces a suitable cannonical transformation $(q,p) \Rightarrow (\eta,\xi)$ which would transform the operator D (10) into the diagonal one. In the resonance case $\omega_1 = \omega_2$ for Hamiltonian (1) this transformation is of the form

$$q_1 = i/2(-\xi_1 + \xi_2 + \eta_1 + \eta_2),\ p_1 = 1/2(\xi_1 - \xi_2 + \eta_1 - \eta_2),$$
$$q_2 = 1/2(\xi_1 + \xi_2 + \eta_1 + \eta_2),\ p_2 = i/2(\xi_1 + \xi_2 - \eta_1 - \eta_2). \tag{17}$$

Now the Hamiltonian (1) becomes

$$K(\xi,\eta) = K^{(2)}(\xi,\eta) + \sum_{j>2} K^{(j)}(\xi,\eta), \tag{18}$$

the quantities

$$K^{(2)}(\xi,\eta) = i(\xi_1\eta_1 + \xi_2\eta_2),$$

being canonically conjugate coordinates and momenta $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1\ \ \eta_2)$. Note that the transformation (17) is a generalized canonical one [18].

2. Description of the program in REDUCE

The GITA program consists of 6 basic and 4 auxiliary blocks, some of them being created as procedures.

Block 1. Transformation of the coordinates $p_\nu \to \sqrt{\omega_\nu} p_\nu$, $q_\nu \to q_\nu/\sqrt{\omega_\nu}$ for reducing $H^{(2)}(p,q)$ to the normal form. The nonlinear part of the Hamiltonian (3) is simultaniously transformed too.

Block 2.Here one starts a cycle with respect to $s=3,.s_{max}$

s_{max} being the maximal order of reducing to the normal form. The procedure SUPQXY performs transformation to the complex variables (11) of the homogeneous part of the s-order Hamiltonian $\tilde{H}^{(s)}(x,y) = H^{(s)}(p,q)$.

Block 3. The SEPA procedure makes division of $\tilde{H}^{(s)}(x,y)$ into monomials (15).

Block 4. The BASIS procedure solves Eq(13) and finds the generating function $\tilde{W}^{(s)}(x,y)$ and the normal form of the Hamiltonian $\tilde{\Gamma}^{(s)}(x,y)$ of the *sth* order.

Block 5. The inverse transformation is fulfilled from the complex variable x, y to the initial variables q,p .

Block 6. Here the remaining part of the Hamiltonian containing higher than s orders is transformed. At the end of the cycle one goes back to block 2.

Upon completion of the cycle with respect to s one gets a Hamiltonian in the normal form up to order and the "abnormal" part of the order higher than s_{max}.

Block 7.One transforms the normal form $\Gamma^{(s)}(q,p)$ from the Cartesian coordinates to the action-angle variables I, φ.

Block 8. Output for further convenience.

3. Applications of BGNF in two physical models

I. Approximate integrals

The general potential form corresponding to the surface quadrupole vibrations of the spherical liquid drop can be parametrized as follows [19-20]

$$V(q_1,q_2) = \sum C_{nm}(q_1^2 + q_2^2)^n (q_1^2 q_2 - \frac{1}{3} q_2^3)^m \tag{19}$$

which is clearly C_{3v} invariants.

Consider the physically interesting case of kryptonium nucleus ^{74}Kr, the parameters C_{nm} (n+m=6) being determined in[10]. This potential surface possesses four minima and three saddle points. On the Poincare sections for the model (done in [11] and depicted on the left column in Fig.1 for energies $0.25E_D$, $0.5E_D$, $0.9E_D$.) one sees that the larger energy the larger portion of chaotic phase space .

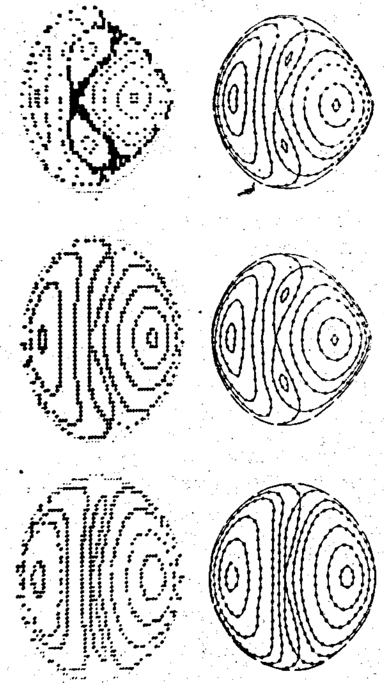

Fig.1.

On the right column were drown the energy isoclines of the approximate sixth-order integral of motion. The similarity of both picture is impressive up to energies where stochastic motion is comming up.

As a second model we choosed three α-particles moving on the line with Hamiltonian:

$$H=\sqrt{3}/2(q_1^2+p_1^2)+1/2(q_2^2+p_2^2)+1/(2\sqrt{3})(q_1^2q_2+\frac{1}{3}q_2^3)+\frac{V_4}{72}(q_1^4+6q_1^2q_2^2+q_2^4). \qquad (20)$$

The corresponding sixth-order approximate integral of motion $I(q_1,q_2,p_1,p_2)$ equals to $I=I^{(4)}+I^{(5)}+I^{(6)}$, where

$$I^{(4)}=10^{-2}[.59687(q_2^2+p_2^2)^2+2.16942(q_1^2+p_1^2)^2+3.69791(q_1^2+p_1^2)(q_2^2+p_2^2)]$$

$$I^{(5)}=10^{-3}[6.50658p_1^4q_2-3.75658p_1^3p_2q_1-5.13399\,p_1^2p_2q_2+36.97906p_1^2q_1^2q_2$$
$$+2.96411p_1p_2q_1q_2^2+1.98264p_1^2q_2^3-3.75658p_1p_2q_1^3+2.96411p_1p_2^3q_1+$$
$$+17.07155p_2^2q_1^2q_2+2.29738p_2^2q_2^3+2.29738q_1^5+20.47248q_1^4q_2+24.18817q_1^2q_2^3],$$
$$I^{(6)}=10^{-3}[6.79063p_1^4q_1^2+5.85592p_1^3p_2q_1q_2+5.22507p_1^2p_2^2q_1^2+3.22043p_1p_2q_1^3q_2$$
$$+7.60642p_1^2q_1^4+9.42845p_1^2q_1^2q_2^2+3.64681p_1^2q_2^4-2.41142p_1p_2^3q_1q_2-$$
$$-1.84092p_1p_2q_1q_2^3++2.70779p_2^4q_1^2+3.71579p_2^2q_1^4+6.60353p_1^2q_1^2q_2^2+3.07933q_1^6+$$
$$+1.87203q_1^4q_2^2+7.71598q_1^2q_2^4+0.565414q_2^6].$$

Again its topography map fits very well the corresponding Poincarè section up to chaos is becomming on. Summing up one sees that the approximate integrals reproduce to large extent details of the Poincaré sections below the transition energies to chaos.

II.Quasiclassical spectra

The normal Birkhoff-Gustavson form for the Hamiltonian H(p,q):

$$H(q,p)=1/2(p_1^2+p_2^2)+V(q_1,q_2) \qquad (21)$$

with potential

$$V(q_1,q_2)=\frac{1}{2}(q_1^2+q_2^2)+b(q_1^2q_2-\frac{1}{3}q_2^3)+c(q_1^2+q_2^2)^2 \qquad (22)$$

in 4th approximation order (n+m=4) equals to

$$\Gamma_4=2I_1+(-\frac{5b^2}{4}+\frac{9}{2}c)I_1^2+(-\frac{7b^2}{24}+\frac{c}{4})I_2^2+\frac{1}{4}(\frac{7b^2}{6}-c)I_2^2\cos\varphi_2 \qquad (23)$$

the I_ν,φ_ν, (ν=1,2) being the action-angle variables. Note that the variable φ_1 doesn't appear explicitly in(23), the corresponding canonically conjugate momentum I_1 beingthe integral of motion. The explicite dependence of (28) on the angle φ_2 destroy the standard Bohr-Somerfeld quantization forcing out to use of another methods (for example WKB procedure [8,20-22]).The above mentioned (Sec.2) modified BGNF allows us to perform the standard quantization in terms of the integral of motions $N=\xi_1\eta_1+\xi_2\eta_2$ and $M=\xi_1\eta_1-\xi_2\eta_2$,

The modified normal form for the same Hamiltonian (21,22) in the 4-th approximation (obtained by the GITN program) equals to

$$\Gamma_4=i(\xi_1\eta_1+\xi_2\eta_2+(\frac{b^2}{6}+c)(\xi_1^2\eta_1^2+\xi_2^2\eta_2^2)-(2b^2-4c)\xi_1\xi_2\eta_1\eta_2). \qquad (24)$$

Rewritting the Weyl quantization in the form :

$$W(\xi_\circ^n,\eta_\circ^n)=1/2^n\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\prod_{j=1}^{n}(\hat{\xi}_\circ\hat{\eta}_\circ-n+l+j) \qquad (25)$$

for the quantum Hamiltonian one gets the following form

$$\hat{\Gamma}_4=(\hat{\xi}_1\hat{\eta}_1+\hat{\xi}_2\hat{\eta}_2+1)+b^2/6((\hat{\xi}_1\hat{\eta}_1)^2+(\hat{\xi}_2\hat{\eta}_2)^2-5\hat{\xi}_1\hat{\eta}_1-5\hat{\xi}_2\hat{\eta}_2-12\hat{\xi}_1\hat{\eta}_1\hat{\xi}_2\hat{\eta}_2)$$

$$+\ c((\hat{\xi}_1\hat{\eta}_1)^2+(\hat{\xi}_2\hat{\eta}_2)^2+3(\hat{\xi}_1\hat{\eta}_1+\hat{\xi}_2\hat{\eta}_2)+4\hat{\xi}_1\hat{\eta}_1\hat{\xi}_2\hat{\eta}_2+2). \qquad (26)$$

Here the operators $\hat{\xi},\hat{\eta}$ are determined as in Eq.(17) after q and p being replaced by the standard coordinate and momentum operators. In the oscillator basis

$$|NL\rangle=[((N+L)/2)!((N-L)/2)!]^{-1/2}\ \hat{\xi}_2^{\frac{N-L}{2}}\ \hat{\xi}_1^{\frac{N+L}{2}}\ |0\rangle \qquad (27)$$

where the vacuum is defined by

$$\hat{\eta}_1|0\rangle=\hat{\eta}_2|0\rangle=0, N=0,1,2.., \ L=\pm N,\pm(N-2),..1 \text{ or } 0 .$$

The spectrum of Hamiltonian $\hat{\Gamma}_4$ becomes

$$E(N,L)=N+1+b^2/12\left[7L^2-5(N+1)^2+1\right]+c/2\left[3(N+1)^2-L^2+1\right]. \qquad (28)$$

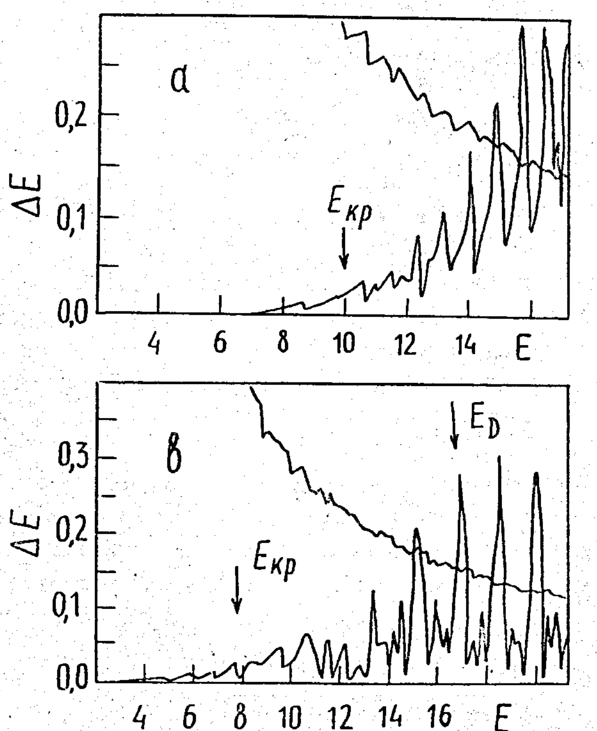

Fig.2.

Note that the formula (28) gives a quite good approximation for the initial Hamiltonian (21,22) spectrum provided that the stochasticity is not highly developed one. This fact is visualized in Fig.2a, 2b. The lower curve in Fig.2a is depicting the difference between exact and approximate eigenvalues. The upper curve on the same figure shows the "exact" level distribution for Hamiltonian H. Herein the values of parameters b and c equal to 0.12347 and 0.00135 respectively, wilst E_{cr} denotes the characteristic energy for transition to chaos.Fig.2b gives us the same portrait as the previous one for parameters b= 0.1, c=0., whereas E_D stands for the saddle point energy.

There is no point concerning the quasiclassical spectrum of the Hamiltonian (20). Its sixth- order BGNF in terms of action variables I_1, I_2 equals to

$$\Gamma_6(I_1,I_2)=1.73205\ I_1+I_2-0.01[8.6776\ I_1^2+2.3876\ I_2^2+14.7916\ I_1I_2]-$$
$$0.001[2.313\ I_1^3+0.233\ I_2^3+1.128\ I_1^2I_2+7.744\ I_1I_2^2] \qquad (29)$$

and the standard Bohr-Somerfeld correspondence $I_\varphi=n_\nu+1/2$ gives us

$$E(n_1,n_2)=\Gamma(n_1+1/2,n_2+1/2) .$$

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