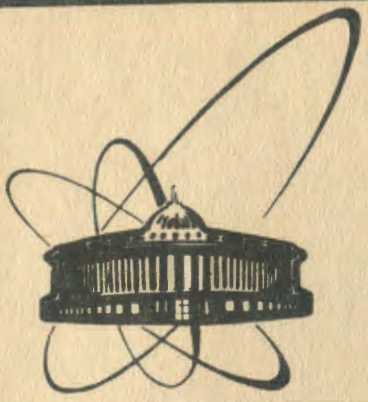

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E4-90-447

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EFFECTS OF ELECTRIC POLARIZABILITY OF NUCLEI IN LOW-ENERGY NUCLEAR COLLISIONS AND THE ACTION RADIUS OF THE POLARIZATION POTENTIAL

Submitted to Nuclear Physics A

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1990

1. Introduction

As is known[1], the most essential details of the super-low-energy scattering of a charged projectile by a target with a extended charge distribution may be established within the two-body model with a potential

$$V = V_c + V_p + V_s \quad . \tag{1}$$

In the system of units ($\hbar = c = \mu = 1$) used in the following V_c from (1) reads as $\pm 1/rR$, where r is the distance between the projectile and the c.m. of the target and $R > 0$ is the Bohr radius for the system "projectile plus target". When the target has a spherically symmetric charge distribution, which is assumed in the following, the rest of Coulomb interactions, i.e. the polarization potential denoted in (1) by V_p, has in the adiabatic approximation the asymptotics [2]:

$$V_p(r) \sim -\alpha/2Rr^4 \quad , \quad r \gg R \quad , \tag{2}$$

where α is the electric polarizability[3] of the target. The last term in (1), i.e. V_s, describes the non-Coulomb part of the effective interaction, and therefore

$$V_s(r) = o(V_p(r)) \quad , \quad r \gg R \quad . \tag{3}$$

Below we restrict our consideration to the S-wave collisions of two particles interacting via V(1) with V_p having asymptotics (2) and V_s having an arbitrary shape but satisfying the conditions:

$$\lim_{r \to 0} r^2 V_a(r) = 0 \quad , \quad V_a(r) \in C^0_{(0,\infty)} \quad , \tag{4a}$$

$$\lim_{r \to \infty} r^n \left(\exp\left(4\left(r \operatorname{sgn} V_c / R\right)^{1/2}\right) V_a(r)\right) = 0 \quad , \tag{4b}$$

where $a = s$ and $n = 0, 1, \ldots$

For brevity we use everywhere the following notation. We detone

by $\delta_{c\alpha}$ the total phase shift from $V_{c\alpha} \equiv V_c + V_\alpha$, where $\alpha = p, s, ps, \ldots$ and $V_{ps} \equiv V_p + V_s$. Symbol $\delta_{c,\alpha}$ stands for the phase-shift produced by V_α, $\alpha = p, s, ps, \ldots$ and additional to the phase shift δ_c from the Coulomb potential V_c. We represent by $\delta_{cp,s}$ the phase shift associated with V_s and additional to $\delta_{cp} = \delta_c + \delta_{c,p}$. In the above notation, we write $V_{cps} = V(1)$ and

$$\delta_{cps} = \delta_c + \delta_{c,ps} \quad , \tag{5a}$$

$$\delta_{cps} = \delta_{cp} + \delta_{cp,s} \tag{5b}$$

when V_p and V_s are present in (1). When V_p or V_s is absent in sum (1), we use $\delta_{cs} = \delta_c + \delta_{c,s}$ or δ_{cp} instead of δ_{cps} (5). When necessary, the analogous notation is used for some other functions corresponding to the above-mentioned cases and the shortened symbol A or $A(\kappa)$ stands for the limit as $r \to \infty$ of the phase function[4,5] $A(r)$ or $A(\kappa, r)$, where $\kappa^2 = E$ is the collision energy.

Now, to explain our main aims, we recall some results known in the potential scattering theory, a modern review of which one can find, for instance, in [6,7].

The low-energy $(\kappa \to 0)$ asymptotics (LEA) of δ_c [8], $\delta_{c,p}$ [9] and $\delta_{c,s}$ [5,10,11] are known explicitly and, respectively, read as

$$\delta_c(\kappa) \sim \eta(\ln\eta - 1) + \pi/4 \quad , \tag{6}$$

$$\tan\delta_{c,p} \sim \begin{cases} 4\alpha\kappa^5/15R^2 + o(\kappa R)^5 \quad , & V_c > 0 \, , \\ -a_{c,p}\kappa C^2(\eta)(1+o(1)) \quad , & V_c < 0 \end{cases} \tag{7a, 7b}$$

and

$$\tan\delta_{c,s}(\kappa) \sim -a_{c,s}\kappa C^2(\eta)\left(\left(1 - a_{c,s}\kappa^2\left(r_{c,s}/2 + R/3\right)\right) + o(\kappa R)^4\right) \tag{8}$$

In (6-8) the function $C(\eta)$ of $\eta \equiv \operatorname{sgn} V_c / 2\kappa R$ is the Coulomb barrier factor[8], $a_{c,p}$, $a_{c,s}$ and $r_{c,s}$ are the finite constants usually[1,6,7] called the scattering lengths and the effective radius.

Next, it is known[4,5] that the picture of threshold potential scattering is determined mainly by the behaviour of the potential tail. Due to (1-3) the tail of V_{cps} is $V_c + V_p$. When $V_c > 0$

and $\kappa \to 0$, then, according to (7a) and (8), $|\delta_{c,p}| \gg |\delta_{c,s}|$, and therefore,

$$\delta_{c,ps}(\kappa) \sim \delta_{c,p}(\kappa) \sim O(\alpha \kappa^5 R^{-2}) , \kappa \to 0. \quad (9)$$

Hence, in the above case the contribution from V_p to $\delta_{c,ps}$ dominates over the contribution from V_s, and therefore, V_p has to be taken into account in the super-low-energy problems of nuclear physics.

Attention to this physically apparent statement was renewed in /12,13/, where the pd -phase shift $\delta_{c,ps}$ was inserted into

$$K_{c,ps}(\kappa) \equiv \kappa C^2(\eta) \cot \delta_{c,ps}(\kappa) + h(\eta)/R , \quad (10)$$

and then by using (9) it was shown that the scattering length

$$a_{c,ps} \equiv - \lim_{\kappa \to 0} K^{-1}_{c,ps}(\kappa) \quad (11)$$

is infinite. Thus it was proved that, when $K_{c,ps}$ (10) and $a_{c,ps}$ (11) are defined analogously to the effective-range function /1,10/

$$K_{c,s}(\kappa) \equiv \kappa C^2(\eta) \cot \delta_{c,s}(\kappa) + h(\eta)/R \sim -a^{-1}_{c,s} + \kappa^2 r_{c,s}/2 + \ldots \quad (12)$$

and the scattering length

$$a_{c,s} \equiv - \lim_{\kappa \to 0} K^{-1}_{c,s}(\kappa) \quad (13)$$

for V_{cs} , one gets $|a_{c,ps}| = \infty$.

It is just the result that has been conjectured in /9/ and that has stimulated extensive research of the effects caused by electric polarizability of nuclei in elastic and inelastic nuclear reactions.

In Sec. 2 we give a critical and, as we believe, complete enough review of the works devoted to this object and published after the report /12/. We explain in detail mistakes and inaccurate points contained in some quoted papers. Also we prove that the methods employed in some works may be improved and developed for solving important problems of the low-energy potential scattering theory and the theory of astrophysical nuclear reactions.

In Sec. 3 we formulate four problems of that sort and outline possible methods to solve them.

One of these problems, namely the conception of the action radius of a polarization potential having the asymptotics (2), has as we

believe, a more fundamental significance than the others. Therefore, in Sec. 4 we analyse this conception in detail. In Sec. 5 we summarize the original results and discuss some prospects.

Before proceeding to a review, we want to stress the following facts. Physically, $|V_p| \ll |V_c|$ everywhere, therefore all the effects caused by V_p, i.e. the so-called polarization effects are essentially the minor corrections to the pure Coulomb picture of collision. These corrections have to be evaluated by mathematically rigorous methods to guarantee the justice of consecutive physical conclusions. It is the point that we shall try to follow throughout all the present work.

2. Critical review

2.1. The polarization effects in elastic nuclear reactions

There is a series of papers[14-23], where these effects were explored within the two-body model. It was assumed that the S-wave radial function $u(\kappa,r)$ describing the scattering of two nuclei obeys the equations

$$\left(\partial_r^2 + \kappa^2 - V(r)\right) u(\kappa,r) = 0 \quad , \tag{14a}$$

$$u(\kappa,0) = 0 \quad , \tag{14b}$$

$$u(\kappa,r) \sim \sin\left(\rho - \eta \ln 2\rho + \delta_{cps}(\kappa)\right), \ \rho \equiv \kappa r \to \infty, \tag{14c}$$

where V is the sum (1) with $V_c > 0$, V_s satisfying (4) and

$$V_p(r) \equiv -(\alpha/2Rr^4)\,\Theta(r - r_p) \quad , \tag{15}$$

where Θ is the Theta-function[8] and r_p is an arbitrary but fixed parameter such that (3) is valid when $r > r_p$.

In[14,15] the problem (14) was solved numerically for the pd-scattering. The phase shift $\delta_{c,ps}$ was extracted from the asymptotics (14c) and then was inserted into the function $\mathcal{D}$ (see eqs. (3-5) of ref.[15] which is actually the phase function

$$t_{c,a}(\kappa,r_0) = \tan\delta_{c,a}(\kappa,r_0) \quad , \tag{16}$$

where the other phase function $\delta_{c,a}(\kappa,r_0)$ is the phase shift produced by the potential $V_a(r)\theta(r_0 - r)$ and $a = ps$. The scattering length $a_{c,ps}(r_0)$ corresponding to this cut-off potential

was defined analogously to $a_{c,s}$ from (8), (12) and (13), i.e. as

$$a_{c,\alpha}(\tau_0) \equiv -\lim_{K\to 0} t_{c,\alpha}(K,\tau_0)/K C^2(\eta), \quad (17)$$

where $\alpha = ps$. Then, it was shown that function (17) has the asymptotics

$$a_{c,ps}(\tau_0) \sim -d(2R)^{-2}\exp(\gamma(\tau_0/R)^{1/2}), \quad 2<\gamma<4, \quad \tau_0 \to \infty \quad (18)$$

and formulae (10,11,16-18) were used to prove that $a_{c,ps}(\infty)(17)$ is the scattering length $a_{c,ps}(11)$ and $|a_{c,ps}| = \infty$. Thus, the main result ($|a_{c,ps}| = \infty$) of papers [12,13] was confirmed in [14,15] by direct numerical solution of the problem (14).

Note, all the numerical results reported in [14,15] were obtained with a too poor accuracy, because (18) disagrees with the asymptotics

$$a_{c,ps}(\tau_0) \sim -(\alpha R/16\pi\tau_0^3)\exp(4(\tau_0/R)^{1/2}), \quad \tau_0 \to \infty \quad (19)$$

derived in [18,21] explicitly.

In view of this critical remark and also for the following discussion we present the facts proving that a high accuracy calculation of u and $\delta_{c,ps}$ by a direct numerical solution of the problem (14) with $K \to 0$ is impossible.

First, even when $V(1)$ is V_c, the high accuracy calculation of regular $u(K,\tau) = F(\rho,\eta)$, as well as irregular $v(K,\tau) = G(\rho,\eta)$, solution of eq. (14a), i.e. the Coulomb functions[8], is possible [24,25] only by using special recipes based on the asymptotical expansions (AE's). For instance, when $\tau \ll \tau_c \equiv 1/K^2 R$, one may successfully use the finite series of the Bessel-Clifford expansions:

$$F(\rho,\eta) = K C(\eta) \sum_{n=0}^{\infty} K^{2n} f_n(\tau), \quad G(\rho,\eta) = C^{-1}(\eta) \sum_{n=0}^{\infty} K^{2n} g_n(\tau), \quad (20)$$

where f_n and g_n are the known[8,25] functions.

Second, when $K \to 0$, the asymptotics (14c) rapidly oscillates; due to (5), (6), (7a) and (9) $\delta_{c,ps}$ and δ_c are large whereas $\delta_{c,ps}$ vanishes; moreover, u is also a function rapidly decreasing in the region $\tau < \tau_c$, because in this region $u \sim K C(\eta)$ (see, for instance, refs.[20,21]).

And finally, to find $\delta_{c,ps}$ within the problem (14), one has, evidently, to calculate u with a relative accuracy $\varepsilon < |\delta_{c,ps}|$ for

any $r \leq B$, where B , i.e. the actual upper limit for integration of (14a), must be equal to a certain r_c /19/. Hence, $\varepsilon \to 0$, $B \to \infty$ as $K \to 0$.

For the above reasons a direct numerical treatment of the problem (14) becomes still more complex with decreasing energy. Evidently, one has to rewrite this problem in a form more adopted for numerical and analytical investigations.

Works/16-23/ have been performed along this line. Having actually used only the decompositions (5) and the known /8/ trigonometric identity, Bencze and Chandler /16/ have elegantly derived a simple formula for the Coulomb modified S-wave scattering length function $A_{c,a}(K)$ in the presence of the potential V_a that falls off faster than r^{-3} as $r \to \infty$. Also, they have numerically proved that for the pd-scattering length function $A_{c,ps}(K)$ this formula, i.e.

$$A_{c,ps}(K) \equiv K^{-1}_{c,ps}(K) \approx A_{cp,s}(K)\,[1 - \tan\delta_{c,p}(K)/A_{cp,s}(K)\,K\,C^2] / [1 + A_{cp,s}(K)\tan\delta_{c,p}(K)\,K\,C^2] \quad , \tag{21a}$$

where $K_{c,ps}$ is defined by (10) and

$$A_{cp,s}(K) \equiv -[K C^2(\eta)\tan\delta_{cp,s}(K) + h(\eta)/R]^{-1}, \tag{21b}$$

is more accurate than formula obtained in /12,13/ by using (10), the approximation

$$\delta_{c,ps} \approx \delta_{c,s} + \delta_{c,p} \tag{22}$$

and the first Born approximation/9/

$$\tan\delta_{c,a}(K) \approx \tan\delta^{B}_{c,a}(K) \equiv -K^{-1}\int_{r_a}^{\infty} V_a(t)\,F^2(Kt,\eta)\,dt, \tag{23}$$

for $\tan\delta_{c,a}$ with $a = p$ and $r_a < r_c$.

Since 1965 it is known/9/ that (23) correctly describes the threshould behaviour of the phase shift $\delta_{c,a}$ produced by the correction $V_a \sim -V_0 r^{-a}$, $a > 3$, $r \to \infty$ to $V_c > 0$. Recently, Kvitsinsky/17/ has generalised this statement to the case $a > 1$, analysed LEA of $\delta_{c,a}$ when $V_c < 0$ and decribed the threshould behaviour of the scattering amplitudes $f_{c,a}(\vec{K})$.

In particular, he proved the asymptotic $(K \to 0)$ relations

$$\delta_{c,p}(K) \sim \varphi(K, r_0) \equiv \int_{r_c}^{r_0} (V_a(t)/2p_c(K,t))\,dt \sim \tag{24a}$$

$$\sim (V_0/2R^{1-\alpha})\,\kappa^{2\alpha-3}\,B(\alpha-1,1/2)\,, \qquad (24b)$$

where $\tau_0=\infty$, B is the **Beta-function**[8] and φ_0 is the zero approximation for the solution φ of a very complex nonlinear problem

$$\varphi' = (V_\alpha/2p_c)(\cos 2(x_c+\varphi)-1) + (p_c'/p_c)\sin\varphi\cdot\cos(2x_c+\varphi)\,, \quad \tau>\tau_c\,, \qquad (25a)$$

with the single boundary condition

$$\varphi(\kappa,\tau_c)=0 \qquad (25b)$$

and the functions

$$p_c(\kappa,\tau)\equiv\kappa\,(1-\operatorname{sgn}V_c\cdot\tau_c/\tau)^{1/2}, \qquad (25c)$$

$$x_c(\kappa,\tau)\equiv\arctan\left(p_c(\kappa,\tau)\,F'(\rho,\eta)/F'(\rho,\eta)\right). \qquad (25d)$$

Unfortunately, Kvitsinsky[17] has not noticed three essential facts. First, when $V_c\equiv 0$, eq. (25a) becomes the equation firstly derived by MacCallum[26] and studied in detail by Dashen[27]. Second, due to (25c), the unhomogeneous term of (25a) diverges as $\tau\to\tau_c$. Hence [28], the problem (25) may have many solutions. And finally, owing to (25), for all $\kappa>0$ the phase shift $\delta_{c,\alpha}(\kappa)\equiv\varphi(\kappa,\infty)$ does not depend on the behaviour of V_α in region $\tau<\tau_c$. Physically, it is an absurd result. For the above reasons we are compelled to note that Kvitsinsky's proof of the formula (24a)is not quite corrent. Undoubtedly, the problem (25) has to be added by a second boundary condition, for instance, by the value of $\varphi'(\kappa,\tau_c)$, or regularised by Dashen's method. Although, having done this, we have verified (24), we want to stress that the problem (25) as well as its above modifications seem to be impractical. In fact, the solution $\varphi(\kappa,\tau_0)$ of (25) has no meaning of the phase shift from $V_\alpha\,\theta(\tau_0-\tau)$, if $\tau_0<\infty$, due to $V_\alpha/2p_c$-term, it tends to $\varphi(\kappa,\infty)$ as $\tau_0\to\infty$ too slowly; moreover, due to oscillating x_c (25d), it rapidly oscillates as $\kappa,\ \tau_0^{-1}\to 0$. Therefore, it is wise to deal with the phase functions having an apparent physical meaning for all $\tau\geq 0$ and defined as solutions of correct and simple problems. These phase functions are, for instance, $\delta_{c,\alpha}$ [18,19], cS and Sn [20], and C and S [21].

Bencze and Chandler[18] have used (16), (17) and the limit $(\kappa\to 0)$ forms of AE's (20) to derive from the known[4,5] equations

$$\delta'_{c,\alpha} = -\kappa^{-1} V_\alpha \left(F \cos\delta_{c,\alpha} + G \sin\delta_{c,\alpha}\right)^2, \quad r > 0, \tag{26a}$$

$$\delta_{c,\alpha}(\kappa, 0) = 0 \tag{26b}$$

the equations

$$a'_{c,\alpha} = V_\alpha \left(f_0 - a_{c,\alpha} g_0\right)^2, \quad r > 0, \tag{27a}$$

$$a_{c,\alpha}(0) = 0 \tag{27b}$$

for $a_{c,\alpha}(r)$ (17) and then to find the asymptotics of $a'_{c,\alpha}(r)$ as $r \to \infty$ explicitly.

By integrating this asymptotics one gets (19) when $\alpha = ps$. Although Bencze and Chandler[18] have pointed out that their main result ($|a_{c,ps}(\infty)| = \infty$) has been conjectured in[5], we refer to the original Babikov's paper[11] to formulate a useful criterion. By employing (20) and assuming for $t_{c,\alpha}$ (16) AE

$$t_{c,\alpha}(\kappa, r) = \kappa C^2(\eta) \sum_{n=0}^{\infty} \kappa^{2n} A_{\alpha,n}(r, h(\eta)), \tag{28}$$

Babikov[11] has reduced the problem

$$t'_{c,\alpha} = -\kappa^{-1} V_\alpha \left(F + t_{c,\alpha} G\right)^2, \quad r > 0, \tag{29a}$$

$$t_{c,\alpha}(\kappa, 0) = 0 \tag{29b}$$

to the recurrence equations for $A_{\alpha,n}$, $n = 0, 1, \ldots$ By analysis of these equations written as $r \gg R$ one can easily prove that $|A_{\alpha,m}(\infty, 0)| < \infty$ if and only if V_α meets (4b) for all $n \le m+1$. Due to this Babikov's criterion and the identity $a_{c,ps}(\infty) = A_{ps,0}(\infty, 0)$ that follows from (17) and (26-29), one has to keep in mind that the fact $|a_{c,ps}(\infty)| = \infty$ has also been conjectured in [11].

Clearly, when solving (26) or (29) for all $r > 0$, one may, by analysis of the ratio $\delta_{c,\alpha}(\kappa, r)/\delta_{c,\alpha}(\kappa, \infty)$, get a detailed information about contributions from various parts of V_α to the phase shift $\delta_{c,\alpha}$. It is just the reason, why for treatment of the $\pi^+ d$-scattering within the problem (14) we [19] have numerically integrated eqs. (26) rewritten in terms of the $\rho \equiv \kappa r$ variable. As we have found, the $\pi^+ d$-phase shifts $\delta_{c,p}$ and $\delta_{c,ps}$ are formed mainly in the interval $(0.8 r_c, 2 r_c)$ both limits of which are energy-dependent since $r_c \equiv 1/\kappa^2 R$; the relative accuracy of the approximations $\delta_{c,\alpha}(\kappa, \infty) \approx \delta_{c,\alpha}(\kappa, 15 r_c)$ is 10^{-4}; when $V_s \equiv 0$

the solution u_{cp} of the problem (14) with V_p (15) may be approximated by its asymptotics

$$u^{as}_{cp}(\kappa,r) \equiv F(\rho,\eta)\cos\delta_{c,p}(\kappa) + G(\rho,\eta)\sin\delta_{c,p}(\kappa) \quad (30)$$

or (14c) only for $r > 2r_c$ or $r > 10r_c$, respectively.
Also we have calculated the nonoscillating parts

$\sigma_{c,a}(\kappa) \equiv 4\pi\kappa^{-2}\sin^2\delta_{c,a}(\kappa)$, where $a = p, ps$ of the π^+d - cross sections $\sigma_{ca}(\kappa) \equiv 4\pi\kappa^{-2}\sin^2\delta_{ca}(\kappa)$ and proved that, due to the fact $\delta_{c,p}\cdot\delta_{c,s} < 0$ as $\kappa \to 0$, the cross section $\sigma_{c,ps}$ has a deep and sharp minimum. As we have pointed out, this effect is, in essence, generated by interplay of V_s and V_p in the π^+d - scattering and has therefore the same nature as the Ramsauer effect[2].

The above results of refs.[11,18,19] show that eqs. (26-29) are well adopted for analytical and numerical treatments of the contributions from V_p (15) to the scattering length (17), phase shifts and cross sections of elastic collisions. However, to get the solution of the problem (14) within eqs. (26) or (29), it is necessary to solve these equations, evaluate the corresponding amplitude functions[4,5] and only then one can construct u. Therefore, it is more practical to use another version of the variable phase approach, namely, a version in which u and $\delta_{c,a}(\kappa)$ may be found simultaneously after solving a simple linear system for two phase functions, for instance, cs, sn [20] or C, S [21].

The most sound Bencze et al.[20] results are the following. First, the suggestion to use cs and sn for a high accuracy calculation of the regular u_{ca} and irregular v_{ca} solutions of eq.(14a) with $V = V_{ca}$, $a = p, s, ps$, and $\kappa \to 0$. We remind that, following[4], Bencze et al. represented the function u_{ca} as

$$u_{ca} \sim U_{ca} \equiv (cs(F/\kappa C) + sn(CG)), \quad (31)$$

found a complete set of the equations

$$cs' = V_a U_{ca}(CG), \quad sn' = -V_a U_{ca}(F/\kappa C), \quad r > 0, \quad (32a)$$

$$cs(\kappa,0) = 1, \quad sn(\kappa,0) = 0 \quad (32b)$$

that uniquely define cs and sn, and showed that

$$\tan\delta_{c,a}(\kappa) = -\kappa C^2(\eta)\lim_{r\to\infty}(sn(\kappa,r)/cs(\kappa,r)) \quad (33)$$

Second, the formula

$$\tan\delta_{c,a}(\kappa) = B_{1a}(\kappa,0,\infty)/(1 + B_{2a}(\kappa,0,\infty)), \quad (34a)$$

where

$$B_{na}(\kappa,\tau_0,\tau) \equiv \kappa^{-1}\int_{\tau_0}^{\tau} V_a(t)\,u_{ca}(\kappa,t)\,[\,G(\kappa t,\eta)\,\delta_{n,2} - F(\kappa t,\eta)\,\delta_{n,1}\,]\,dt, \qquad (34b)$$

that has been proved by the known method of the general theory for the potential scattering[29]. Third, developing the known idea (see, for instance, refs.[9-11,30]) to use decomposition (5b) instead of (5a) and exploit the corresponding effective range-function

$$K_{cp,s}(\kappa) \equiv \kappa\,\tilde{C}^2(\eta)\cot\delta_{cp,s}(\kappa)+\tilde{h}(\eta)=-a_{cp,s}^{-1}+\kappa^2 r_{cp,s}/2+\ldots, \qquad (35)$$

where $\tilde{C}$ and $\tilde{h}$ are expressed[30] in terms of certain integrals but, in general, are unknown explicitly. And finally, the proof that the modified scattering length $a_{cp,s}$ from (35) is equal to

$$a_{cp,s} \equiv -\lim_{\kappa\to 0}\tan\delta_{c,ps}(\kappa)/\kappa\,\tilde{C}^2(\eta) = -\lim_{\kappa\to 0}\tan\delta_{c,ps}(\kappa)/\kappa\,C^2(\eta) \qquad (36)$$

is finite, has a physical meaning and may be easily evaluated by extrapolation of $\tan\delta_{c,ps}(\kappa)$ to $\kappa=0$ or by direct solution of eq. (14a) with $\kappa=0$.

Undoubtedly, it is urgent to add the Bencze et al. theory by an efficient method for evaluating $\tilde{C}$, $\tilde{h}$ and $r_{cp,s}$ entering into (35) and (36) and find the boundary conditions uniquely defining cs and sn for v_{ca}. In the present work we are compelled to restrict ourselves to presenting these conditions. So, we write $v_{ca}\sim U_{ca}$, where U_{ca} is defined by (31), and get (32a). Next, comparing the asymptotics of U_{ca} as $r\to 0$ with the required form ($v_{ca}=O(r^{-1})$) we find, by iterating (32a), that

$$cs \sim C^2(\eta)\int_{\tau_0}^{\tau} V_a(t)\,G^2(\kappa t,\eta)\,dt, \quad sn\to 1, \quad \tau_0>\tau, \quad \tau\to 0. \qquad (37)$$

In [21] we developed a version slightly different from (31-33). In this version u obeing (14) reads as

$$u_{ca} = N_{c,a}\,U_{ca} = N_{c,a}\,(cF+sG), \qquad (38)$$

the phase functions C and S related with t from (16), (29) and with cs and sn from (31-34) by relations $t\equiv S/C$, $C\equiv cs$, $S\equiv\kappa C^2(\eta)\,sn$, satisfy the equations

$$C' = \kappa^{-1} V_a\,U_{ca}\,G, \quad S'=-\kappa^{-1}V_a\,U_{ca}F, \quad \tau>0, \qquad (39a)$$

$$C(\kappa,0)=1, \quad S(\kappa,0)=0, \qquad (39b)$$

and the phase shift $\delta_{ca}(\kappa)$ and the norm factor $N_{c,a}(\kappa)$ from (38) are defined by limits as $\tau \to \infty$ of the corresponding functions:

$$\delta_{c,a}(\kappa,\tau) = \arctan\left(S(\kappa,\tau)/C(\kappa,\tau)\right), \quad (40)$$

$$N_{c,a}(\kappa,\tau) = \cos\delta_{c,a}(\kappa,\tau) \,/\, C(\kappa,\tau) . \quad (41)$$

Our treatment of system (39) with $\kappa \to 0$ and $a = \rho, S, \rho S$ was based on the Babikov's /11/ idea to use AE's (20) and on the Levy and Keller's /31/ idea to employ the iteration method /28/.

Below we generalise some results of the work /21/ discussed by using the complete AE's (20) and AE's:

$$C = \sum_{n=0}^{\infty} \kappa^{2n} C_n(\tau) \quad , \quad S = \kappa C^2(\eta) \sum_{n=0}^{\infty} \kappa^{2n} S_n(\tau) \quad (42)$$

instead of the truncated ones. So, inserting (20) and (42) into (38), (39) we get the analog of the first AE of (20):

$$u_{ca} = \kappa C(\eta) N_{c,a}(\kappa) \sum_{n=0}^{\infty} \kappa^{2n} U_n(\tau) \quad , \quad (43a)$$

where

$$U_n \equiv \sum_{\ell+m=n} \left(c_\ell f_m + s_\ell g_m\right) \quad (43b)$$

and C_n and S_n obey the recurrence equations

$$c' = V_a \sum_{\ell+m=n} U_\ell g_m \quad , \quad s' = -V_a \sum_{\ell+m=n} U_\ell f_m, \quad \tau > 0, \quad (44a)$$

$$c_n(0) = \delta_{n,0} \quad , \quad s_n(0) = 0 \quad , \quad n = 0, 1, \ldots \quad (44b)$$

and $N_{c,a}$ expands as /21/

$$N_{c,a}(\kappa) = \sum_{n=0}^{2} \kappa^{2n} N_{c,a}^{(n)} + \Delta N_{c,a}(\kappa) \quad , \quad (45)$$

where $N_{c,a}^{(n)}$, $n = 0, 1, 2$, are finite constants and $\Delta N_{c,s} = O(\kappa^6)$, but $\Delta N_{c,\rho}$, $\Delta N_{c,\rho s} = O(\kappa^{16/3})$.

In the region $\tau \gg \tau_c$ the AE's (20), (42) and (43) converge too slowly. Therefore, for approximation of C, S and u in this region we have used in /21/ the functions $C^{(m)}$, $S^{(m)}$ and $u^{(m)}$ originating from an m^{th}-iteration of system (39). We have got the condition on V_ρ (15), $a_{c,s}$ (13) and κ :

$$q \equiv (\alpha/3R\tau_\rho^2)\ \max\{\ 3(\pi/2)^{1/2},\ |a_{c,S}/\tau_\rho + O(\kappa R)^2|\} < 1/2 \qquad (46)$$

that ensures the uniform convergence of these iterations in the region $\tau \geq \tau_\rho$ and the validity of the representation

$$\tan\delta_{c,\rho}(\kappa) = \tan\delta^{(1)}_{c,\rho}(\kappa,\infty)\left[1 + O\left(B^{(1)}_{1\rho}(\kappa,\tau_\rho,\infty)\right)\right], \qquad (47a)$$

where $\kappa \to 0$,

$$\tan\delta^{(1)}_{c,\rho}(\kappa,\tau) \equiv B^{(1)}_{1\rho}(\kappa,\tau_\rho,\tau)/\left[1 + B^{(1)}_{2\rho}(\kappa,\tau_\rho,\tau)\right] \qquad (47b)$$

and $B^{(1)}_{n\rho}$ is given by (34b) when $u_{c\rho}$ is replaced by F. We have shown that, due to (16), (17), (33), (40) and (42), the scattering length $a_{c,\alpha}$, $\alpha = \rho, S, \rho S$ and the effective radius $\tau_{c,S}$ from (8) are the limits as $\tau \to \infty$ of the corresponding functions:

$$a_{c,\alpha}(\tau) = -S_0(\tau)/C_0(\tau) \quad , \qquad (48)$$

$$\tau_{c,\alpha}(\tau) = 2R/3 + 2\left[C_1(\tau) + S_1(\tau)/a_{c,\alpha}(\tau)\right]/S_0(\tau). \qquad (49)$$

Also we have proved that LEA (7a) follows from (47), the function $a_{c,\rho S}(\tau_0)$(17) represented as ratio (48) has asymptotics (19), and therefore, diverges as $\tau_0 \to \infty$ whereas the effective radius $\tau_{c,\rho S}(\tau_0)$ (49) corresponding to the potential $V_{\rho S}(\tau)\theta(\tau_0 - \tau)$ tends to $2R/3$ as $\tau_0 \to \infty$.

Undoubtedly, the κC^2-behaviour declared for $B^{(1)}_{1\rho}(\kappa,\tau_0,\tau_c)$ in (47b) is a gross mistake of our work /21/. In fact, using for F the appropriate AE's /25/ and applying the stationary phase method /32/, one can easily be convinced that $B^{(1)}_{1\rho}(\kappa,\tau_0,\tau_c) = O(\kappa^{16/3})$ when $\kappa \to 0$ and $\tau_0 < \tau_c$. Fortunately, this mistake, as we have established by a careful revision, does not alter the main results we have obtained in /21/, discussed before and shall use in the following.

The results of papers /17-21/ convincingly show that the variable phase approach is a well adopted method for exploring the problem (14) with V_ρ (15) and $\kappa \to 0$. Another efficient method is the WKB-approach /33/, because it also has an apparent physical interpretation /1/ and allows one to get some LEA explicitly.

These facts led us /22/ and, as we believe, also L'vov /23/ to investigate the problem (14) within the first order of the WKB-approach. We and L'vov reproved (7a) and (24) for $\alpha = 4$ and found, in essence, the WKB-approximation $\delta^{WKB}_{c,\rho}(\kappa,\tau)$ for the phase function $\delta_{c,\rho}(\kappa,\tau)$ from (16) and (25). This approximation reads as

$$\delta^{WKB}_{c,\rho}(\kappa,\tau) = \varphi_0(\kappa,\tau) = (4\alpha\kappa^5/15R^2)(1-t)^{1/2}\cdot (3t^2/8 + t/2 + 1) \qquad (50)$$

with φ_0 (24a) and $t \equiv \tau/\tau_c \geq 1$. When $\tau \to Q$, the WKB-approximation (u^{WKB}) for u becomes incorrect[33], therefore, in[22] we have constructed u_{cps} as

$$u_{cps} = N u^{as}_{cs} \equiv N(F\cos\delta_{c,s}(\kappa) + G\sin\delta_{c,s}(\kappa)),\ \tau \ll \tau_\rho, \qquad (51a)$$

$$u_{cps} = u_{c\rho}\cos\delta_{c\rho,s}(\kappa) + v_{c\rho}\sin\delta_{c\rho,s}(\kappa),\ \tau \gg \tau_\rho. \qquad (51b)$$

Replacing $F, G, u_{c\rho}$ and $v_{c\rho}$ by their found WKB-forms, we got the WKB-approximations u^{WKB}_{cps} for u_{cps} (51a) and u_{cps} (51b) and then from the usual[1] condition $u^{WKB}_{cps} \in C^1_{(0,\infty)}$ we found the WKB-approximations N^{WKB} and $\delta^{WKB}_{c\rho,s}$ for N and $\delta_{c\rho,s}$ from (51). Next, exploiting the idea of refs.[12,13] but using the approximation $\delta_{c,ps} \approx \delta^{WKB}_{c,\rho} + \delta^{WKB}_{c\rho,s}$ instead of (22) we have treated the threshold behaviour of the functions $A^{WKB}_{c,ps}(\kappa)$ approximating the scattering length functions $A_{c,ps}(\kappa) \equiv -K^{-1}_{c,ps}(\kappa)$, where $K_{c,ps}$ is defined by (10), for the scattering of $p, d, {}^3H$ and ^4He by deuterons and for the π^+d, $\pi^+{}^3H$ and $\pi^+{}^4He$ -scattering. We have made this to estimate the upper bounds of energy intervals, where, due to V_ρ (15), the functions $A^{WKB}_{c,ps}(\kappa)$ are nonlinear functions of energy and where, for this reason, V_ρ should be taken into account in the theoretical investigations of the above collisions.

Deriving in[22] the formula for $\delta^{WKB}_{c\rho,s}$ we have, without the slightest grounds, dropped some factors. For the collisions we have considered they are very close to unity. However, now using the functions $P_n(\kappa,\tau_\rho)$, $g(0,\tau_\rho)$ and $Q(0,\tau_\rho)$ found in[22] explicitly, we present the correct result,

$$\tan\delta^{WKB}_{c\rho,s}(\kappa) = -\kappa\cdot\exp(-\pi\eta)\cdot[a_{c,s}P_0 + \tau_\rho(a_{c,s}-\tau_\rho)P_0'] / [P_1 + (a_{c,s}-\tau_\rho)P_1'], \qquad (52)$$

to get the WKB-approximation

$$a^{WKB}_{c\rho,s} \equiv -\lim_{\kappa\to 0}(\tan\delta^{WKB}_{c\rho,s}(\kappa)/\kappa C^2(\eta)) = \exp(-2Q)\cdot [a_{c,s} + \tau_\rho(a_{c,s}-\tau_\rho)(g'/g - Q')/(1 + (a_{c,s}-\tau_\rho)\cdot (g'/g + Q'))] \qquad (53)$$

for the scattering length $a_{c\rho,s}$(36) and show that $|a^{WKB}_{c\rho,s}| < \infty$.

As we pointed out in [19], $A^{WKB}_{c,ps} \approx A_{c,ps}$ for the π^+d-scattering. Preparing the present work we have verified that $A^{WKB}_{c,ps} \approx A_{c,ps}$(21) for the pd -scattering. Although these facts show that the WKB-formalism of ref.[22] seems to be correct we must add it

by the known condition[33]:

$$p(\kappa,r) \equiv (\kappa^2/64) \int_{r_0}^{r} p^{-5/2}(\kappa,t)\,|(4p'(\kappa,t) - 5\ln p(\kappa,t))'|\,dt < \ln 3/2, \qquad (54a)$$

where

$$p^2(\kappa,r) \equiv (\kappa^2 - V_c(r) - V_p(r))/\kappa^4, \qquad (54b)$$

ensuring the validity of the inequalities

$$|z(\kappa,r)/z^{WKB}(\kappa,r) - 1| < 1,$$

where $z = u_{cp}, v_{cp}$ and $r > r_0 > 0$.

Completing this part of review we wish to point out some interesting details. The exact representation (34a) for $\tan\delta_{c,p}$ is reduced to (47b), where $r = \infty$, by approximating $u_{cp} \approx F$. When (46) is valid, the denominator of the fraction (47b) maybe approximated by unity. Then, one gets formula (23) which is reduced to (24a) or (7a) by using the WKB-asymptotics[25] $F \approx (4p_c)^{-1/2}$ with p_c (25c). Because of these connections between the results of different works[9,17,20,21], one should expect that for calculating of $\delta_{c,p}(K)$ the approximation (47b) is more accurate than (23) and (24 a).

2.2. The polarization effects in the nucleosynthesis reactions

Since 1986 when Belyaev et al.[34] predicted that the pd-potential V_p (15) violates the $p(d,\gamma)^3He$-reaction unitarity, the role of a polarization potential in the nucleosynthesis reactions is intensively treated. To demonstrate how understanding of this role has developed with time, we shortly review all the relevant papers[21-23, 34-47] in that order in which they have been published.

The extraordinary results of paper[34], in particular, the proof that V_p (15) causes the divergence of the pd-reaction cross section and its factor S (in our notation σ^{inel}_{cp} and S_{cp}) made us to present in [22] the WKB-estimation D^{WKB}_{cp} for the contribution

$$D(K) \equiv S_{cps}(K)/S_{cs}(K) - 1 \qquad (55)$$

from the pd-polarization potential (15) with $\alpha = 0.7$ fm[3][3] and $r_p = 4$ fm to S_{cps}. As we have shown, D^{WKB} is a smooth continuous function of the pd-energy and $D^{WKB} \approx 10^{-3}$ for $0 \le E \le 6$ keV. Hence, the factors S_{cps} and S_{cp} are also continuous and finite functions in the vicinity of $E = 0$, which contradicts to the above result: $S_{cp}(0) = \infty$ of ref.[34].

Later in[35,36] it was predicted that the pp-reaction cross section σ^{inel}_{cp} had a very large ($\sigma^{inel}_{cp}/\sigma^{inel}_{cs} \approx 10^{30}$) and sharp maximum caused by V_p (15).

The next work was the L'vov's preprint[23]. Using elements of the WKB-approach and perturbation theory in V_p (15), L'vov constructed the pd -scattering function u_{cps} within the problem (14). Unfortunately, to estimate $\mathcal{D}$ (55), he replaced the obtained u_{cps} only by two terms of its asymptotics as $r \to 0$. This approximation is too poor, because [48] the contribution from the region $r \leq r_d \approx 4$ fm, where r_d is the deuteron size, to S_{cs}, S_{cps} is smaller than that from the region $r > r_d$. Nevertheless, L'vov got $\mathcal{D} \approx 10^{-3}$ and, hence, he confirmed our result[22]. Further, in [37] it was predicted that the cross section σ^{inel}_{cp} of the $^3H(dn)^4He$-reaction had a narrow maximum ($\sigma^{inel}_{cp} / \sigma^{inel}_{cs} \approx 100$) produced by the $d\,^3H$ -polarization potential.

Note, in the rapporteur report[38] the results of papers[34-37] have been summarized without mention of L'vov's[23] and Levashev's[39] criticism. In this connection we must stress that L'vov is the first who has explained that the gross mistake of papers[34,35] is the groundless replacement of u_{cp} by u^{as}_{cp} (30) for all $r \geq 0$ and $K \to 0$.

Later, this mistake common to works[34-37] was pointed out by Levashev in theses[39] and then was discussed in detail in a series of his papers[40-43] actually repeating each other. According to the standard theory for nucleosynthesis reactions[48], Levashev has conjectured that the pp -reaction factors S_{cs} and S_{cps} are proportional to the square of the corresponding matrix element

$$\Lambda_{ca}(K) = (8\pi r_d^3 K^2 C^2(\eta))^{-1/2} \int_0^\infty u_d(r)\, u_{ca}(K, r)\, dr, \quad (56)$$

where $a = s, ps$ and u_d is the deuteron function. Then, he replaced the pp -scattering function u_{cps} by the function u^B_{cps} constructed by him within the first Born approximation over the pp-polarization potential. As a result of this replacement, he found Λ^B_{cps} and $\mathcal{D}^B$ instead of Λ_{cps} (56) and $\mathcal{D}$ (55). Hence, if one formulates the main Levashev result more correctly, this result means that the contribution $\mathcal{D}^B$ from V_p (15) to S^B_{cps} is negligible because $\mathcal{D}^B$ is determined by the value of V_p on the boundary of nuclear forces.

Although Levashev's treatment is more complete than that carried out in[22,23,44-46], and for the pd -reaction his estimation $\mathcal{D}^B \approx 10^{-3}$ of $\mathcal{D}$ (55) agrees with earlier[22,23] estimations, we are compelled to give two more critical remarks. First, Levashev has estimated only the part $\mathcal{D}^B$ of the total contribution $\mathcal{D}$ (55), namely it is (see ref.[21]) the part linear in the parameter $\alpha / R r_p^2$. Second, mathematically, his construction of the function u^B_{cps} is formal. Indeed, he, instead of proving that $u^B_{cps} \approx u_{cps}$ for all $r \geq 0$, has unsuccessfully referred to a numerical evidence[16,19] of the fact that

$\delta^B_{c,p}$ from (23) well approximates the phase shift $\delta_{c,p}$. Obviously, from this fact one may only conclude that $u_{cp} \approx u^B_{cp}$ for large enough r, namely /19/, for $r > 2r_c$.

Before the journal versions/41,42/ of preprint/40/ appeared, Bencze presented in/44/ an elegant and physically apparent method to estimate D (55). It should be stressed that this method based on the S-matrix theory is more general than that developed in/21-23,34-43, 45-47/. Unfortunately, the final Bencze's formula (eq.(14) in/44/) meaning that D (55) equals $(\tan \delta_{c,p})^2$ is incorrent. As Levashev /43/ noted, when this formula was derived one term was lost. Nevertheless, this regrattable fact does not alter the main Bencze's result showing the electric polarizability of nuclei to have a negligible effect on the inelastic total cross sections of super-low-energy reactions involving deuterons.

Now we give the critical remarks common to works/22,23,39-44/. Although, in these papers various approximation methods are employed, none of the authors have estimated the arising residual terms reliably and none have indicated the energy region, where employed approximations are correct within a given accuracy. Further, the results obtained in /22,23,39-44/ for D (55) are actually only estimations of D in the order of magnitude. Note also, all the authors of the above works concentrated on these estimations rather than on the treatment of LEA for the factors S_{cps}.

The above defects led us to construct in /21,45,46/ the mathematically rigorous and complete enough theory for the pp -reaction including the pp -polarization interaction (15).

In /45/ we exploited (31-32) and (55,56) and used the RSC-potential /49/ as V_s, the RSC-deuteron function u_d and V_p (15) with $\alpha = 2 \cdot 10^{-3}$ fm^{3}/3/ and $r_p = 4$ fm. As a result of a series of high accuracy calculations, for D (55) we established that $D(0) \approx 10^{-6}$ and $D(E)/D(0) \approx 1.02$ for $E = 6$ keV.

Working again in the framework of eqs.(31-32,55), we proved in thesis /46/ that

$$\Lambda_{c\alpha}(K) = \sum_{n=0}^{2} K^{2n} \Lambda^{(n)}_{c\alpha} + \Delta\Lambda_{c\alpha}(K), \; K \to 0, \quad (57)$$

where $\Lambda^{(n)}_{c\alpha}$ are constants independent of $\alpha = s$ or $\alpha = ps$ whereas $\Delta\Lambda_{cs} = O(K^6)$ and $\Delta\Lambda_{cps} = O(K^{16/3})$

In a subsequent preprint/21/ we explored the truncated system (44) and AE (43) with $N_{c,\alpha}$(45) in detail and suggested a reliable method to calculate the constants of LEA (12,45,57) without energy interpolation. By this method, that, in essence, is based on (38-45, 55-57), we carried out a detailed analytical and numerical analysis of $\Lambda_{c\alpha}$(56), LEA (57) and D (55).

In particular, for the pp -model used earlier in [46] we have got that $D \approx 10^{-6}$, D has a broad maximum at energy $E = E_0 \approx 400$ keV and D slowly falls with growing E in the region $E > E_0$. Also, we have proved that the contribution from the tail $V_p\,\theta(r - r_c)$ of V_p (15) to Λ_{cps} (56) has the $K^{16/3}$ -threshould behaviour and therefore is not dominant as compared with that from the part $V_p\,\theta(r_c - r)$.

To complete this part of review, we should mention the recent work [47]. In this work the known [4,5] first Born approximation $t^B_{c,p}$ for $t_{c,p}$ (16) obeying eqs. (29) has been used to construct the function u^B_{cp} approximating the $p\,^7Be$ -scattering function u_{cp} being a solution of the corresponding two-body problem (14). As a final result, it has been shown that the first Born approximation $D^B_{c,p}$ of the contribution D (55) from the $p\,^7Be$-polarization potential (15) to the factor S_{cp} of the $^7Be(p,\gamma)^8B$-reaction is about $2.6 \cdot 10^{-3}$.

2.3. Resume

First, we recall the main results of works we have reviewed. The result $|a_{c,ps}| = \infty$ conjectured in [5,9,11] and the fact that electric polarizability of nuclei has a **negligible** effect on the nucleosynthesis reactions are supported by various analytical and numerical evidence. The finite scattering length $a_{cp,s}$ having the physical meaning is defined by (36). Asymptotics (7a) found in [9] actually by inspection of the first Born approximation $t^B_{c,ps}$ for the phase function $t_{c,ps}$ from (29), may be evaluated within various approximations presented above as (24), (47) and (50). The upper bounds of energy intervals, where some nuclear-nuclear and pion-nuclear elastic collisions are described within the problem (14) mainly by the tail V_{cp} of V_{cps} (1), are estimated. It is shown that the polarization effect analogous to Ramsauer's effect should be expected in elastic collisions, if they may be considered within the two-body approximation (14) and if $\delta_{c,p} \cdot \delta_{c,s} < 0$ as $K \to 0$.

Now, we focus our attention on two facts. So, the main contribution from V_{ps} to the phase shift $\delta_{c,ps}$ and the corresponding elastic cross section $\sigma_{c,ps}$ is caused by the tail $V_p\,\theta(r - r_c)$ of V_p when $V_c > 0$. However, the main contribution from V_p to the norm factor $N_{c,ps}$ (45) of the scattering function u_{cps} (38) and to the factors S_{cps} of the nucleosynthesis reaction is produced by the other part of V_p, namely by the part $V_p\,\theta(r_N - r)$, where r_N is of the order of magnitude of the nuclear size. Hence, the region, where V_p acts effectively and therefore cannot be replaced by identical zero, depends essentially on the kind of process and a function of interest.

Finally, we must stress that all the above results have been found within the effectively two-body approximation. However, there is no criterion ensuring that an arbitrary few-body observable may be established within this approximation with a desirable accuracy. As it follows from all the review and essentially from this remark, the modern theory of low-energy potential scattering and the theory for nuclear-polarizability effects on the super-low-energy nuclear-polarizability effects on the super-low-energy nuclear collisions are incomplete and there are many interesting and unsolved problems.

3. Four problems and possible methods to solve them

Below we formulate four problems of low-energy potential scattering theory, give some recipes to solve them and explain why a complete solution of these problems is urgent for both the theoretical and experimental treatment of interplay of short- and long-range forces in quantum systems.

As is known[1], all the functions characterising the collisions of quantum mechanical objects are expressed through a regular or an irregular wave function of these objects. Hence, 1) construction of the LEE's for regular ($u_{c\alpha}$) and irregular ($v_{c\alpha}$) solutions of eq. (14a) with $V = V_c + V_\alpha$, $V_c > 0$ and $\alpha = p, s, ps$ is the key-problem for construction of the LEE's for the phase shifts $\delta_{c,\alpha}$, elastic $\sigma_{c,\alpha}$ and inelastic $\sigma^{inel}_{c\alpha}$ cross sections and so forth. We suggest to solve this problem within eqs. (38-45) as follows. First, in the regions $r < r_c$, $r \sim r_c$ and $r > r_c$ one has to represent F and G as AE's (20), as the Olver's[50] AE's containing the Airey functions[8] and as the WKB-series [25,33], respectively. Second, one has find the appropriate AE's for the phase functions C and S to separate the variables K and r. And finally, it is necessary to reduce in this way the original problem to the solution of K-independent system.

It was just the recipe following which we got AE (41). Now, we derive similar AE for $v_{c\alpha}$ obeying (14a) and the boundary conditions:

$$v_{c\alpha} = O(r^{-1}) \quad , \quad r \to 0 \, , \tag{58a}$$

$$v_{c\alpha} \sim \cos(p - \eta \ln 2p + \delta_{c\alpha}) \quad , \quad p \to \infty . \tag{58b}$$

Due to (14c) and (58b), Wronskian[28] for u and v is identically equal to K. Hence, we may write $v_{c\alpha}$ as

$$v_{c\alpha} = N^{-1}_{c,\alpha} \, v = N^{-1}_{c,\alpha} (cF + sG) \tag{59}$$

with $N_{c,\alpha}$ being the norm factor of $u_{c\alpha}$ (38). Next, replacing u in (14a) by v (59) and using (58) we get for C and S again system (39a) but with the boundary conditions

$$C(\kappa,\tau) \sim \kappa^{-1} \int_{\tau_0}^{\tau} V_\alpha(t)\, G^2(\kappa t,\eta)\, dt\,, \quad S(\kappa,\tau) \to 1\,, \quad \tau_0 > \tau \to 0, \tag{60}$$

that follow also from (37) and the relations $C \equiv CS$ and $S = \kappa C^2 sn$.

Now, inserting F and G as AE's (20) and C and S as the assumed AE's

$$C = (\kappa C^2)^{-1} \sum_{n=0}^{\infty} \kappa^{2n} C_n(\tau)\,, \qquad S = \sum_{n=0}^{\infty} \kappa^{2n} S_n(\tau) \tag{61}$$

into (39a) we get the recurrence eqs. (44) which, owing to (60) and (61), should be added with the following boundary conditions

$$c_n(\tau) \sim \int_{\tau_0}^{\tau} V_\alpha(t)\, g_n^2(t)\, dt\,, \quad s_n(\tau) \to \delta_{n,0}\,, \quad \tau_0 > \tau, \tau \to 0. \tag{62}$$

As a next step, we substitute AE's (20) and (61) into (59) to write the searched AE as

$$v_{c\alpha} = (N_{c,\alpha} C)^{-1} \sum_{n=0}^{\infty} \kappa^{2n} v_n(\tau)\,, \tag{63a}$$

where $N_{c,\alpha}$ has LEA (45) and

$$v_n \equiv \sum_{\ell+m=n} (c_\ell f_m + s_\ell g_m)\,. \tag{63b}$$

Since the polarization effects arise in the region of very low energies usually inaccessible for direct experimental treatments, experimental data should be extrapolated into that region. To this end, it is necessary to know, in particular, the LEE's for the cross sections $\sigma_{c,p}$, $\sigma_{c,ps}$ and $\sigma_{cp,s}$. Due to the formulae

$$\sigma_{c,\alpha} \equiv (4\pi\kappa^{-2})(\sin\delta_{c,\alpha})^2,\ \alpha = p, ps;\ \sigma_{cp,s} \equiv (4\pi\kappa^{-2})(\sin\delta_{cp,s})^2,$$

the construction of these LEE's is actually reduced to the solution of the following important problem that we formulate as 2) construction of LEE's for the phase shifts $\delta_{c,p}$, $\delta_{c,ps}$ and $\delta_{cp,s}$.

We do not know a simple method to build the complete LEE's for $\delta_{c,p}$ and $\delta_{c,ps}$. As we believe, the first few terms of these LEE's may be found explicitly within an m^{th}-order of the WKB-approach or by a careful analysis of LEA for the functions $C^{(m)}$ and $S^{(m)}$ obtained by an m^{th}-iteration of the system (39) or by employing the

new method[51] for evaluating the phase shifts. A reliable method to find AE for $\delta_{cp,s}$ is based on the known idea[4,5,20] which allows us to write the solution of the problem (14) as

$$u_{cps} = N_{cp,s}\,(c\,u_{cp} + s\,v_{cp}) \qquad (64)$$

to get the system (39) in which $\alpha = s$ and F, G stands for u_{cp}, v_{cp}, respectively, and to introduce $\delta_{cp,s}$ and $N_{cp,s}$ by analogy with (40) and (41), i.e. as the limit at $r \to \infty$ of the corresponding functions

$$\delta_{cp,s}(\kappa, r) = \arctan\,(s(\kappa, r)/C(\kappa, r)) \quad , \qquad (65)$$

$$N_{cp,s}(\kappa, r) = (\cos\delta_{cp,s}(\kappa, r))/C(\kappa, r) \quad . \qquad (66)$$

Upon substitution u_{cp}(43), v_{cp} (63) and

$$c = \sum_{n=0}^{\infty} \kappa^{2n} c_n(r) \quad , \quad s = \kappa (N_{c,p} C)^2 \sum_{n=0}^{\infty} \kappa^{2n} s_n(r) \qquad (67)$$

into the system for c and s described above we have that c_n and s_n from (67) obey the system (44) in which $\alpha = s$ and

$$U_n \equiv \sum_{\ell+m=n} (c_\ell\, U_{mcp} + s_\ell\, v_{mcp}) \quad , \qquad (68)$$

where U_{mcp} and v_{mcp} are the functions U_m (43b) and v_m (63b) from AE's (43a) and (63a) for u_{cp} and v_{cp}, respectively.

Now we substitute C and S as AE's (67) into (65) and get the searched AE

$$\tan\delta_{cp,s}(\kappa) = \kappa (N_{c,p} C)^2 \sum_{n=0}^{\infty} \kappa^{2n} A_n \quad . \qquad (69)$$

Here we do not prove that $|A_n| < \infty$ for all $n = 0, 1, \ldots$, if and only if V_s obeys (4b) because this is tedious rather than interesting.

Note, problems 1) and 2) are mathematical rather than physical ones, while the following problem, i.e.

3) a correct definition of the action radius for potential V_p (15), has a more general and urgent physical significance. Indeed, this problem arises at the origin of treatment, when one wants, on the basis of physical intuition, to qualitatively predict the expected effects caused by V_p (15) and to compare these effects with others similar in nature but caused by V_s obeying (4b). Problem 3) arises again when all the desirable formulae are derived, and one wants to use them for analytical and numerical treatments of the polarization effects. The point is the following. The most part of analytical re-

lations are simplified by letting $V_p \equiv 0$ for any $r \geq B$, where $B < \infty$ is an appropriate radius. Moreover, in practice, the most part of equations, relations, etc. containing V_p may be numerically solved only on the interval $0 \leq r \leq B$, where B is large enough but finite. For these well-known reasons it is necessary to correctly define the radius B which, in essence, is an upper bound of the distance range, where V_p acts effectively.

In the physical literature[1,6,7] this bound is usually called the action radius meaning a point B on the r-axis on the right of which one can let $V_p \equiv 0$. Although, the conception of action radius seems to be intuitively clear, it should be specified in each particular problem. In the opposite case, i.e. when one relies only on his own intuition rather than on a rigorous proof, the errors and unphysical conclusions seem to be inevitable. As an example highly illuminating in this respect, we refer to works[35-39], where it has been groundless assumed that the action radius of V_p(15), as well as the action radius of V_s (4b), is energy-independent and comparable with the deuteron size.

In computational mathematics[52] the action radius B of the function V_p is an upper limit of the interval $0 \leq r \leq B$ on which the problem is solved numerically. This limit is chosen from the condition that the calculated function must be close, to a reasonable extent, to an exact solution of the treated problem. In the asymptotical methods for the theory of ordinary differential equations[32,33] the action radius B has a meaning similar to the one mentioned above and is defined as a lower limit of the interval $B \leq r < \infty$, where a considered function may be, again in a reasonable meaning, replaced by its asymptotics as $r \to \infty$. These two mathematical methods for finding B will be developed in the next section in which we report the results of our current treatments of problem 3). We believe, these results will be useful also for solving the problem that is is more general than the problem 3) and reads as 4) a correct definition of the action radius of an arbitrary two-body potential in few-body problems.

To give an interesting fact illustrating how urgent is this problem, we address to the three-body problem with the two-body potential V_s satisfying (4). Let $\{ \mathcal{Y}_{[L]} \}$ be an appropriate potential harmonic basis[53]. In general, the matrix elements

$$V_{s[L][L']} \equiv \langle \mathcal{Y}_{[L]} | V_s | \mathcal{Y}_{[L']} \rangle$$

fall not faster than ρ^{-3} when the hyperradius ρ tends to infinity [54]. Hence, V_s usually called a short-range potential[1,6,7] beco-

mes in the above three-body problem the potential usually called a long-range one[1,6,7]. Clearly, to avoid this discrepancy, the conception of the short- or long-range potential should be defined also with respect to the considered few-body problem.

4. The action radius of the polarization potential

Let us start with a gedanken experiment. Let us forget that V_p and V_s obey (2-4) and assume that we know only the first, i.e. leading, terms in LEA (7) and (8), and we do not known the threshold behaviour of $\Delta N_{c,\alpha}$ and $\Delta \Lambda_{c\alpha}$, $\alpha = s, ps$, in (45) and (57). Exploiting only this poor information we try to solve the following inverse problem: to establish the behaviour of V_p and V_s at large z. If $V_c > 0$, then applying the Babikov's criterion, using (24) and the known, by assumption, leading terms of LEA (7a) and (8) we unambiguously conclude that V_p and V_s satisfy (2), (3) and V_s obeys (4b), where $n = 1$. When $V_c < 0$, the leading terms of LEA (7b) and (8) have a similar K-dependence. Hence, using again the Babikov's criterion we may only show that $V_p, V_s = o(z^{-3})$ as $z \to \infty$. Note, in both the cases ($V_c > 0$ or $V_c < 0$) we are not able to prove relations (4b) for $n > 1$. Next, from LEA (45) and (57) with omitted terms $\Delta N_{c,\alpha}$ and $\Delta \Lambda_{c\alpha}$ we cannot show that V_p and V_s have different asymptotics as $z \to \infty$.

As follows from our experiment, the above inverse problem cannot be completely solved before solving problems 1) and 2), and the conception of a long- or short-range potential, usually used for V_p and V_s, respectively, should be defined with respect to the problem of interest. One can say that in the studied problem the potential V is of a more long-range nature than the potential V' only if it is proved that namely in this problem the action radii B and B' of V and V' obey the inequality $B > B'$. Below, by solving the problem 3), we wish to demonstrate the general strategy for finding the action radius of an arbitrary potential.

We define the action radius B of V_p (15) for a function A as the solution $B = B(A, \varepsilon, \{\gamma\})$ of the inequality

$$d(A, B) \equiv |A(B, \{\gamma\})/A(\infty, \{\gamma\}) - 1| \le \varepsilon \qquad (70)$$

with a fixed $\varepsilon \in [0, 1)$ and a number $\{\gamma\}$ of all the possible parameters arising in the problem (14).

As A we consider: the phase functions $\delta_{c,p}$ (26), $\delta^{(1)}_{c,p}$ (47b), and $\delta^{WKB}_{c,p}$ (50), the solutions $u_{c\alpha}(K, z, B)$, $\alpha = p, ps$, of the problem (14) when V_p is a cut-off at $z = B$, the norm factor $N_{c,ps}(K, B)$ (41) of $u_{cps}(K, z, B)$ and the matrix element $\Lambda_{cps}(K, B)$

defined as the integral (56) in which u is replaced by $u_{cps}(\kappa,\tau,B)$. We stress that in each case A, d, ε and relation (70) will have an apparent meaning. Indeed, $A(\kappa,B)$, by definition, is $A(\kappa)\equiv A(\kappa,\infty)$ when V_p (15) is replaced by $V_p\theta(B-\tau)$, and if B satisfies (70), then $A(\kappa,B)$ approximates $A(\kappa)$ within a relative accuracy d smaller than a required relative accuracy ε.

In practice, $A(\kappa,B)$ is always calculated instead of $A(\kappa,\infty)$ and B is determined so that a required M significant digits in the numbers $A(\kappa,B)$ and $A(\kappa,B')$ are the same for any $B'>B$. To find B as a function of M, one calculates the sequence $\{A(\kappa,B_i)\}_{i=1}^{N}$, corresponding to the sequence $\{V_p\theta(B_i-\tau)\}_{i=1}^{N}$ with $B_1<B_2<\ldots<B_N$. Then, one chooses from $\{B_i\}_{i=1}^{N}$ a growing subsequence $\{B_{i_M}\}_{M=1}^{N_0}$ with the property that for each $M=1,\ldots,N_0-1$ the numbers $A(\kappa,B_{i_M})$ and $A(\kappa,B_{i_{M+1}})$ have the same M significant digits. When that subsequence is found, ineq.(70) is reduced to the inequality

$$d(A,B_{i_M})=|A(\kappa,B_{i_M})/A(\kappa,B_{i_N})-1|<\varkappa_{i_M}\cdot 10^{-M},\quad \varkappa_{i_M}\in(1,10), \tag{71}$$

that is valid for all $M=1,\ldots,N_0-1$. Thus, one gets the solution $B(A,\varepsilon,\{\gamma\})=B_{i_M}$ of ineq. (71) for $\varepsilon=\varkappa_{i_M}\cdot 10^{-M}$ and fixed parameters $\{\gamma\}$ including κ.

So, the practical recipe to find B is simple enough. We have used it in our numerical analysis, the results of which are reported below. By complete analogy with our treatment [19] of the π^+d-scattering we explored numerically the π^-d-scattering within problems (14) and (26). The results obtained for the π^-d-phase functions $\delta_{c,p}$ and $\delta_{c,ps}$, in particular, those listed in Tables 1 and 2 show the following. The phase shift $\delta_{c,p}$ forms mainly on the interval (τ_p,τ_p+2R) and rapidly decreases with growing parameter τ_p of V_p (15). The error $d(\delta_{c,p},B)$ (70) becomes smaller than unity when $B>2R+\tau_p$. All the bounds mentioned above very slightly depend on κ if $E\leq 10$ keV and $0.1R\leq\tau_p\leq 10R$.

The results listed in Table 2 also demonstrate that for $V_c<0$ and $V_s\equiv 0$ the limit (11) defining the scattering length $a_{c,p}$ is finite in accordance with (7b).

The facts recalled in Subsection 2.1 for the π^+d-scattering ($V_c>0$) and those given above for the π^-d-scattering ($V_c<0$), clearly prove that potential V_p (15) is of a more long-range nature when $V_c>0$ than in the case $V_c<0$. More precisely this statement is clarified by the numerically obtained formulae

Table 1

The $\pi^- d$ -phase function $\delta_{c,\rho}(\kappa,\tau)$ of (26) as a function of E(keV) and $\gamma \equiv \tau/R$ when the parameters of V_ρ (15) are: $\alpha = 0.7$ fm^3 and $\tau_\rho = R = 104$ fm. All the values of $\delta_{c,\rho}(\kappa,\tau)$ are multiplied by 10^{11}.

E / γ	0.1	1	10
1.04	1216	1182	908
1.08	2287	2216	1685
1.2	4813	4312	3402
1.4	7462	7183	4947
1.6	8971	8592	5621
1.8	9846	9388	5886
2.0	10356	9837	5976
4.0	11033	10343	6213
6.0	11075	10440	6272
10.0	11161	10503	6295
15.0	11165	10512	6300
30.0	11173	10518	6308

Table 2

The $\pi^- d$ -phase shift $\delta_{c,\rho}(\kappa) \equiv \delta_{c,\rho}(\kappa, \tau_\rho + 30R)$ as a function of E(keV) and the parameter $\tau_\rho = \gamma R$ of V_ρ (15) when $\alpha = 0.7$ fm^3. The values of $\delta_{c,\rho}(\kappa)$ are multiplied by 10^{10}, 10^{11} and 10^{14} for $\gamma = 0.1, 1, 10$, respectively.

γ / E	0.1	1	10
10^{-4}	69845	11249	12449
10^{-3}	69845	11249	12447
10^{-2}	69841	11242	12372
10^{-1}	69748	11173	11711
1.0	69572	10518	11469
10.0	69387	6308	6557

$$B(\delta_{c,p}(\kappa),\varepsilon,\{\kappa,\mathrm{sgn}V_c,r_p\})\approx\begin{cases}(0.3+0.6\varepsilon^{-1/3})r_c\,, & V_c>0\,, \quad (72a)\\ 0.6r_p-0.2R+0.5R\varepsilon^{-1/3}, & V_c<0\,. \quad (72b)\end{cases}$$

It is valid for $\varepsilon = 10^{-M}$, $M=1,\dots,5$; $0.1R\le r_p\le 10R$ and $0.1\le E \le 10$ keV and shows that for the calculation of the $\pi^{\pm}d$-phase shifts with the error $d(\delta_{c,p},B)$(70) satisfying (71) one has to cut off V_p at point $r=B$ which essentially depends on ε and $\{\gamma\}$.

Since for the $\pi^{\pm}d$ -scattering $a_{c,s}$=0.079 fm /55/, α =0.7 fm /3/ and $r_p\ge$ 4 fm the condition (46) is fulfilled for $r_p\ge$ 0.1 R and $E\le$ 10 keV, and therefore in this region $\delta_{c,p}(K,r)$ of (26) is very close to $\delta^{(1)}_{c,p}$ from (47b). Indeed, as we found numerically,

$$|\delta^{(1)}_{c,p}(K,r)/\delta_{c,p}(K,r)-1|<10^{-9}\,, \quad (73)$$

where $r>r_p\in[0.1R,\ 10R]$, $E\le 10$ keV.

Further, from (71) we get that the action radius of V_p for $\delta^{(1)}_{c,p}(K,\infty)$ is well approximated by the formula

$$B(\delta^{(1)}_{c,p}(K,\infty),\varepsilon,\{\kappa,V_c>0,r_p\})\approx(0.3+0.6\varepsilon^{-1/3})r_c \quad (74)$$

when ε, r_p and E are the same as in (72).

Clearly, due to (73), the right-hand sides of (72a) and (74) must be the same, and the functional form of relation (74) is caused by the explicit formulae (34b) and (47b), i.e., finally by the structure of the Coulomb functions. Hence, for an arbitrary two-body problem (14) one has the relations (72a) and (74a), if q obeys (46). This means that for the problem (14) the relations (46), (72a) and (74a) are automodel, and q, α/r_p^2R, R and r_c are automodel parameters. They have an automodel meaning analogous, for instance, to that the **Reinolds parameter has**/57/.

Next, in the framework of eqs. (38-43) we have calculated the sequences $\{u^{\pm}_{cp}(K,r,B^{\pm}_i)\}_{i=1}^{N}$ of the $\pi^{\pm}d$ -scattering functions corresponding to the sequences of the potentials $\{V_p\Theta(B^{\pm}_i-r)\}_{i=1}^{N}$. Having solved ineqs. (71) for $A=u^{\pm}_{cp}$ and $B^{\pm}$ we have found that the relations

$$B(u_{cp}(K,r),\varepsilon,\{\kappa,\mathrm{sgn}V_c,r_p\})=B(\delta_{c,p}(K),\varepsilon,\{\kappa,\mathrm{sgn}V_c,r_p\}) \quad (75)$$

are valid, when ε, r_p and E are the same as in (72) and $r\le B^{\pm}_N$. We have got also that to calculate the functions $u^{\pm}_{cp}$ on the intervals $0\le r\le B^{\pm}_N$ within the accuracy of six significant digits, one should set $B^{+}_N=50r_c$ and $B^{-}_N=r_p+50R$. As follows from (72),(75),

to calculate the $\pi^{\pm}d$-scattering functions with the error (70), one has to cut off V_p (15) at the corresponding points $B^{\pm}$ depending on $\{\varepsilon, \{\kappa, sgn V_c, \tau_p\}$ analogously to B (72) for $\delta_{c,p}(\kappa)$. Clearly, this statement is automodel when (46) is valid.

As a next example, we explored the pp-reaction within the model and method described in [21]. Upon a series of calculations carried out within eqs. (38-45) and (56,57) we found the sequences $\{N_{c,ps}(\kappa, B_i)\}_{i=1}^{N}$ and $\{\Lambda_{cps}(\kappa, B_i)\}_{i=1}^{N}$ for $N_{c,ps}$ (41) and the function $\Lambda_{cps}(\kappa, B)$ defined earlier. Letting $A = N_{c,ps}$ and then $A = \Lambda_{cps}$ in (70) we established that the formulae

$$B(N_{c,ps}, \varepsilon, \{\kappa, \tau_p\}) = B(\Lambda_{cps}, \varepsilon, \{\kappa, \tau_p\}) \approx 1.3 \cdot 10^{-3} R \varepsilon^{-2/5} \quad (76)$$

well approximate the action radius of V_p (15) for $N_{c,ps}$ and Λ_{cps} when $\varepsilon < 0.1$, $E \leq 10$ keV and $\tau_p = 4$ fm.
Note, due to (72a) and (76), the action radius of V_p for the pp-reaction cross section σ^{inel} very slightly depends on E and is smaller than the action radius of V_p for the pp-scattering cross section $\sigma_{c,ps}$.

Unfortunately, the demonstrated numerical method for evaluating the action radius B of V_p (15) is not constructive because B may be obtained from (71) only after calculating a function A of interest and because to get B as a function of κ, it is necessary to calculate A for various κ. These circumstances are actually the main defect of the numerical method for finding B.

Another method to get B for the studied function $A(\kappa, \tau)$ is an analytical method. It is based on the knowledge of single (any κ and $\tau \to \infty$) or double ($\kappa \to 0$ and τ is large enough but fixed, or κ is small enough but fixed and $\tau \to \infty$) asymptotics of $A(\kappa, \tau)$. To demonstrate this method, we give below two examples.

Let in (70) A be the $\pi^{+}d$-function $\delta_{c,p}^{WKB}(\kappa, \tau)$ (50), then the action radius of V_p (15) for the $\pi^{+}d$-phase shift $\delta_{c,p}^{WKB}(\kappa)$ reads as

$$B(\delta_{c,p}^{WKB}(\kappa), \varepsilon, \{\kappa, \tau_p > 0\}) \approx (16\varepsilon/5)^{-1/3} \tau_c \approx 0.68 \varepsilon^{-1/3} \tau_c \quad (77)$$

Since for $E \leq 10$ keV and $\tau_p \geq 0.1R$ the condition (54a) is fulfilled, we have $\tilde{\delta}_{c,p}(\kappa) \approx \delta_{c,p}^{WKB}(\kappa)$, therefore, for that τ_p and energies B (77) has the same functional form as the second and largest term of the sum (72a). Hence, the first term, i.e. $0.3\tau_c$, describes approximately the contribution from $V_p \theta(\tau_c - \tau)$ to the action radius B (72a). Evidently (54a) and (77) are the automodel relations for similar reasons as for (46), (72) and (74).

Now, let in (70) A be the function $N_{c,ps}(0,\tau)$(41) which, owing to (7a), (41) and (45), is equal to $1/C_0(\tau)$. Using the asymptotics of $C_0(\tau)$ as $\tau \to \infty$ found in[21], one can easily solve (70) for B and analytically find the action radius,

$$B(N_{c,ps}, \varepsilon, \{\kappa, \tau_p\}) \approx \xi \varepsilon^{-2/5} \equiv R(\alpha/10R^3)^{2/5}\varepsilon^{-2/5} \quad (78)$$

of V_p(15) for norm factor $N_{c,ps}$ of u_{cps} (38) in the case $V_c > 0$ and $\kappa = 0$. It should be stressed that the explicit formula (78) actually gives the major estimate of B which is valid also for a non-zero but low enough energy when the relative accuracy for the approximation $N_{c,ps}(\kappa,\tau) \approx N_{c,ps}(0,\tau)$ is smaller than ε for any τ. Note, also that for pp-collisions (78) agrees with (76) and ξ from (78) is an automodel parameter.

So the analytical method allows determination of the action radius B of V_p (15) without calculation of the sequence $\{A(\kappa, B_i)\}_{i=1}^{N}$ required in the numerical method. However, the analytical method, as well as the numerical one, has one defect. Indeed, the exact lower limit $\bar{\tau}$ of the interval $\tau \geq \bar{\tau}$, where the explored function A may be replaced by its truncated AE as $\tau \to \infty$, is usually unknown. Hence, by the analytical method one can actually find the major estimate of B, i.e. B satisfying ineq. (70) rather than eq.(70).

The resume of our numerical and analytical treatments is: the lower bound B of the interval $B < \tau \leq \infty$, where V_p (15) may be replaced by identical zero, is, in general, the function $B = B(A, \varepsilon, \{\gamma\})$ depending on the studied function A, relative accuracy ε required for evaluation of A, collision energy, $sgn V_c$ and parameters of V_p.

We propose to use this bound which should be always defined with respect to the explored function A and which may be in principle found from (70) and (71), as the action radius of V_p (15) at given $\varepsilon, \kappa, sgn V_c, \alpha, \tau_p$ and R. Physically and mathematically, a similar definition of the action radius seems to be correct for an arbitrary potential. Using this definition one again has to keep in mind that the action radius should be defined with respect to a problem in question and may essentially depend on some other parameters.

5. Summary and prospects

Let us recall our main results. In Sec. 2 we indicated earlier unnoted mistakes and inaccurate points which we found in the quoted papers[14,15,17,21-23,40-44] and we showed how the methods of papers [17,20-23] may be improved and developed. In Sec. 3 we formulated four

important problems and suggested some ideas to solve them. In Sec. 4 we presented our numerical and analytical results of treatment of the problem 3), proposed a correct definition of the action radius of the polarization potential, and demonstrated two methods to find this radius. As we stressed, both these methods have defects. This fact and the remark given at the end of Sec. 3 show that the problem 4), i.e. the correct definition of the action radius of an arbitrary potential in few-body problems is an interesting and important problem which we suggest to carefully explore along the line indicated in Sec. 3.

Now we list some facts in favour of the formulae (38-53, 57-59, 72-78), we have obtained, being useful for future analytical and numerical investigations of the low-energy nuclear collisions. First, all these formulae may be easily generalized to the case when V (1) is $V = V_c + V_\alpha + V_s$, where V_s obeys (4) and V_α satisfies (4a), but has the asymptotics $V_\alpha = O(r^{-\alpha})$, $\alpha > 2$, $r \to \infty$. Second, as we have shown, when the system (44a) under conditions (44b) or (62) is solved, one may: construct the AE's (43) and (63) for regular and irregular solutions of eq. (14a), find the coefficients $a_{c,s}(\infty)$ (48) and $\gamma_{c,s}(\infty)$ (49) in LEE's (8) and (12), and also get the coefficients $\Lambda_{c\alpha}^{(n)}$ and A_n in the LEE's (57) and (69). Third, it is very important that eqs. (38-45,48,49,57-69) allow us to calculate all these coefficients with a high accuracy and without interpolation to a zero energy because the key-eqs. (44) are very simple for a numerical solution and are K-independent. Fourth, eqs. (38-49) and (56,57) may be generalized to the case of noncentral potential V_s containing spin-orbital and tensor terms. To perform this generalization one must separate all the discrete, i.e. spin and isospin variables and then derive the matrix analogs to eqs. (38-49) and (56,57).

In the framework of derived equations one can, for instance, successfully investigate the LEA for the factors $S_{4,3}(E)$ of astrophysically important[48] reactions $^3H(\alpha,\gamma)^7Li$ and $^3He(\alpha,\gamma)^7Be$ and reliably find the first few terms of these LEA without energy interpolation. In this connection we must point that the idea to use the first terms of AE's (20) for the calculation of the factors $S_{4,3}$ at $E=0$ was first realized in [57] within the two-body Schrödinger equation describing $\alpha\,^3He$ and $\alpha\,^3H$ -scattering. Recently this idea has been employed in [58] to calculate these factors also at $E=0$ within the algebraic version of the **resonanting group approach** described in [59].

Fifth, since for the nuclear collisions the ratio $a/r_p^2 R$ is always small enough[3], the approximate formulae (72-78) are **automodel** and may be used for a reliable definition of the upper limits of in-

tegration of various equations containing V_p (15) and originating from the problem (14).

In conclusion note that the scheme we have followed to derive AE's (43,63,69) and formulae (72-78) look promising for the solving the $(3 \to 3)$ low-energy scattering problem written within the potential harmonic approach. The results of these constructions form the contents of the subsequent paper.

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Received by Publishing Department
on June 15, 1990.