90 - 439

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E4-90-439

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DIRAC QUANTIZATION OF THE ROTATOR

Submitted to "Journal of Physics A"

1990

1. INTRODUCTION

Suppose we look for a quantum system of A particles in which the Hamiltonian is changed by time depending parameters $\vec{R}(R_1(t), R_2(t), \ldots)$. Thus, the evolution of the system between times $t=0$ and $t=T$ can be understood as a motion around a clothed path in the parameter space if $\vec{R}(T)=\vec{R}(0)$. For any fixed time moment one can solve the eigenvalue equation $H(x,p;\vec{R})\Psi_\nu(x,\vec{R})=\epsilon_\nu(\vec{R})\Psi_\nu(x,\vec{R})$ and these states perform a natural basis to expand the exact time depending solution of the Schrödinger equation $H\psi=i\hbar\dot{\psi}$

$$\psi=\sum_\nu C_\nu(t)\,\Psi_\nu(x,\vec{R}) .$$

Especially, for adiabatic motion the system remains in one of the eigenstates, say Ψ_μ, and during time only a phase appears. Berry /1/ found out that in addition to the common dynamical phase $-i\int_0^T \epsilon_\mu(\vec{R}(t))dt$ a geometrical phase may appear and the correct wave function after a period is

$$\Psi_\mu(x,\vec{R}(T)) = \exp - i\int_0^T \epsilon_\mu\, dt \cdot \exp i \oint \vec{A}_\mu\, d\vec{R} \cdot \Psi_\mu(x,\vec{R}(0)) . \qquad (1)$$

The expression A_μ has formally the property of a vector potential and is determined by the relation /1,2/

$$\vec{A}_\mu = i\langle\Psi_\mu(\vec{R})|\vec{\nabla}_{\vec{R}}|\psi_\mu(\vec{R})\rangle = i\int dx\, \Psi^*_\mu(x,\vec{R})\vec{\nabla}_{\vec{R}}\psi_\mu(x,\vec{R}) . \qquad (2)$$

Mathematically /3/ the Berry phase in eq.(1) is an integral over the curvature of the parameter space $\vec{R}$ known as the Chern class of the connection. As a direct consequence of the vector potential property of $\vec{A}_\mu$ for the quantized motion in the parameter space instead of the momentum $\vec{P}=-i\vec{\nabla}_{\vec{R}}$ the covariant expression $\vec{D}^\mu=\vec{P}-\vec{A}_\mu$ appears.

At some points of the parameter space $\vec{R}$ the vector potential has singularities, the so-called diabolic points, and just these points correspond to degeneracies caused by intersections of levels of the spectrum $\epsilon_\nu(\vec{R})$. Calculating the phase $\oint A_\mu d\vec{R}$ the diabolic points are circumvented.

Instead of the restricted time-dependent description one may apply some of these ideas to stationary problems too /4,5/. Let the Hamiltonian of the system be

$$H = -\nabla^2_{\vec{R}}/2m + H^{(0)}(x,p) + V(\vec{R},x)$$

and look this time for stationary solutions

$$H\psi_\kappa(\vec{R},x) = E_\kappa \psi_\kappa(\vec{R},x) .$$

Expanding the eigenfunctions into a complete set of adiabatic functions

$$\psi_\kappa(\vec{R},x) = \sum_\nu \phi^\kappa_\nu(\vec{R})\, \psi_\nu(\vec{R},x) ,$$

with

$$(H^{(0)} + V)\psi_\nu(\vec{R},x) = \epsilon_\nu(\vec{R})\, \psi_\nu(\vec{R},x)$$

for the collective eigenfunctions, we obtain, remaining in an adiabatic approximation only diagonal contributions

$$\{(\vec{p} - \vec{A}_\nu)^2 /2M + \epsilon_\nu(\vec{R})\}\phi^\kappa_\nu = E_\nu \phi^\kappa_\nu ,$$

where $\vec{A}_\nu$ is the same vector potential, that appeared in eq.(2).

In this paper, we look for the application of the general idea of the Berry phase to the adiabatic rotation of an A-particle system (the atomic nucleus for the sake of a concrete model, comp. also /6/). The collective parameters are then the Euler angles $\Omega = \phi, \theta, \chi$ which span the collective space and for which we may take the three-dimensional sphere S^3 of R^4.

For adiabatic rotation the internal state $|k\rangle$ is characterized by the quantum number k of the projection of the total angular momentum of the A-particle system on the symmetry axis of the deformed nucleus

$$J_z|k\rangle = k|k\rangle .$$

We apply the method of generator coordinates /7/ to describe rotations in three dimensions. Instead of the Born - Oppenheimer wave function (1) one may choose oriented deformed states /4,6,7/, defined by

$$|\Omega,k\rangle = R(\Omega)|k\rangle , \tag{3}$$

where $R(\Omega)$ denotes the rotation operator

$$R(\Omega) = \exp(-iI_z\phi)\exp(-iI_y\theta)\exp(-iI_z\chi) , \tag{4}$$

and the total wave function is represented by

$$\psi = \int d\Omega\, \phi(\Omega)|\Omega,k\rangle . \tag{5}$$

The generator states (3) are not orthogonal and the matrix elements

$$\langle k|h\cdot R|k\rangle = \epsilon_k \langle k|R(\Omega)|k\rangle = \epsilon_k n_k(\theta) ,$$

(h denotes the s.p. Hamiltonian in second quantization)

$$n_k(0) = 1, \qquad n_k(\pi) = 1 ; \qquad (k - \text{integer})$$

are the analogue of the standard Born - Oppenheimer adiabatic eigenvalues $\epsilon_\nu(\vec{R})$. As we shall see in section 3, applying the definition of the vector potential (2) together with expression (3), and performing a projection to the relevant collective subspace S^2 spanned by the points of the unit sphere in R^3, one obtains a vector potential which is singular at the diabolic points $\theta = 0$, $\theta = \pi$. We have already noted the analogy between the rotation and the behaviour of spin -1/2 particles in a magnetic field. Just for the last case Berry /1/ showed that the motion is such, as if it is caused by the action of a magnetic monopole situated at the diabolic point. Therefore, also the vector potential appearing for the rotation of a top is exactly the same as for a magnetic monopole in R^3. Now, for the magnetic monopole the vector potential contains the charge /8/

$$\vec{A} = g\frac{c}{eh}\frac{1}{r(r \mp z)}(-y,x,0) .$$

In the case of rotation of a top instead of $g\cdot\frac{c}{eh}$ there appears the quantity k. As is well-known /9/, the string singularities of A lead to the topological quantization of the flux $\phi = \oint \vec{A}\,d\vec{R} = 4\pi g$ or, equivalently, of the magnetic charge g. Then, it is natural to ask how to formulate the Dirac quantization condition for the case of rotations. As we will show, the concept of Berry's effective vector potential, applied to rotations, gives a direct way to solve this task. As a result, the parameter k gets quantized and hence the spin of the system is quantized, too. Especially, to obtain half-integer spin values, one has to take into account explicitly the well-known D_2-symmetry of axial symmetric rotations /6/.

The second section contains some formal relations needed to perform the projection from S^3 to S^2. In Sect.3 we derive the collective Hamiltonian in S^2 and establish useful connections between the effective vector potential, Berry's phase and generator coordinate overlaps. In Sect. 4 the quantization is performed, the wave functions for the motion in the field of the singular vector potential is obtained, furthermore, the wave function is symmetrized taking care of D_2-invariance, and the Dirac quantization condition is obtained. Finally, in the last section the results are connected and interesting new relations of the theory of collective rotations to other problems, like Skyrmion quantization, are discussed.

2. THE COLLECTIVE SPACE, $S^3 \to S^2$ MAPPING

For the rotation of a top the three-dimensional sphere S^3 may be chosen as the space of collective coordinates. Really, each point $x\,(x_1, x_2, x_3, x_4)$, $\sum_{i=1}^{4} x_i^2 = 1$ on S^3 is parametrized by the three Euler angles ϕ, θ, χ through the relations

$$x_1 = \cos\left(\frac{\phi+\chi}{2}\right)\cos\frac{\theta}{2}, \quad x_2 = \sin\left(\frac{\phi+\chi}{2}\right)\cos\frac{\theta}{2},$$

$$x_3 = \cos\left(\frac{\chi-\phi}{2}\right)\sin\frac{\theta}{2}, \quad x_4 = \sin\left(\frac{\chi-\phi}{2}\right)\sin\frac{\theta}{2}. \tag{6}$$

Introducing complex variables $Z_0 = x_1 + ix_2$ and $Z_1 = x_3 + ix_4$ we get the compact form

$$Z_0 = e^{i\phi/2}\cos\frac{\theta}{2}\, e^{i\chi/2},$$

$$Z_1 = e^{-i\phi/2}\sin\frac{\theta}{2}\, e^{i\chi/2}. \tag{7}$$

As is well-known /10/ in classical mechanics the motion of the so-called symmetrical top is completely described by three degrees of freedom ϕ, θ, χ and generally for each of them there exists one frequency. Contrary to this the third Euler angle χ is an irrelevant variable in the quantum description. In the collective model of Bohr and Mottelson a rotation around the body's symmetry axis by the angle $\Delta\chi$ causes the appearance of a factor $\exp(i\Delta\chi J_3)$, and this factor is just compensated by a rotation of the internal system, leading to $\exp(-i\Delta\chi I_3)$, provided the condition $J_3 = I_3$ holds /11/. Therefore instead of S^3 one may use the two-dimensional sphere $S^2(\phi,\theta)$ as the collective coordinate space. The mathematical procedure for projecting from S^3 to S^2 is known as the Hopf mapping /2, 9/ and may be constructed explicity by introducing on S^2 the coordinates $\xi(\xi_1,\xi_2,\xi_3)$ by the relations

$$
\begin{aligned}
\xi_1 &= 2(x_1x_3 - x_2x_4) = \sin\theta\cos\phi\,, \\
\xi &= 2(x_2x_3 - x_1x_4) = \sin\theta\sin\phi\,, \\
\xi_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2\,.
\end{aligned}
\tag{8}
$$

Performing furthermore a stereographic mapping $S^2 \to R^2(x,y)$ for the complex variable $z = x + iy$ we obtain, using equations (7) and (8)

$$
z = \frac{\xi_1 + \xi_2}{1 - \xi_3} = \frac{z_0}{z_1}\, e^{i\phi}\,\mathrm{ctg}\,\frac{\theta}{2}\,. \tag{9}
$$

It is interesting to note that this formula is the standard relation in the spinor theory to express a point on S^2 by a pair of complex numbers z_0, z_1. Each transformation of points on S^2 corresponds then to a transformation of the spinor $\binom{z_0}{z_1}$. From the construction (9) it follows immediately that all points which in S^3 belong to a χ-circle, at the mapping to S^2 shrink into one point $z = z_0/z_1$. Furthermore, z_0 and z_1, considered as functions of χ, are double-valued, as follows from equation (7). This structure is overtaken on S^2, introducing the two so-called sections $\chi = \pm\phi$ for the corresponding lists. The possibility of employing the projection $\chi = -\phi$ for describing adiabatic rotation has been noted already by Bohr and Mottelson /11/. In the section $\chi = \phi$ one may represent z_0, z_1 as functions of the variable $z = \exp(i\phi)\,\mathrm{ctg}\,\theta/2$, which is defined on S^2 anywhere excluding the north-pole

$$z_0 = e^{i\phi} \cos\frac{\theta}{2} = \frac{z}{(1+|z|^2)^{1/2}} \,,$$
$$z_1 = \sin\frac{\theta}{2} = \frac{1}{(1+|z|^2)^{1/2}} \,. \tag{10}$$

Similarly, in the section $\chi = -\phi$ these quantities are represented as functions of $w = z^{-1}$ that has a singularity at the south-pole of S^2

$$z_0 = \cos\frac{\theta}{2} = \frac{1}{(1+|w|^2)^{1/2}} \,,$$
$$z \;= e^{-i\phi} \sin\frac{\theta}{2} = \frac{w}{(1+|w|^2)^{1/2}} \,. \tag{11}$$

Obviously, the functions z and w are connected by a reflection at the origin

$$z(\pi-\theta,\, \phi+\pi) = w(-\phi,\theta) \,.$$

This fact plays a role for the symmetrization of the rotor wave functions on S^2 as we shall see later. The spinors builded from z_0, z_1 in eq.(10) resp. (11) have a direct dynamical significance, being the eigenstates of the Hamiltonian for a spin -1/2 particle in a magnetic field $\vec{B} = Bn(t)$ ($n(t)$ - unit vector in R^3, /2/). Calculating with the help of this spinors the effective vector potential, the singularities just appear for $\chi = \phi$ at the north-pole, resp. for $\chi = -\phi$ at the south-pole.

3. DERIVATION OF THE COLLECTIVE HAMILTOMIAN

After introducing the collective space we will derive the collective Hamiltonian for rotations in S^2 applying the generator coordinate method /7/. Generator states are introduced acting with the rotational operator

$$R(\Omega) = \exp(-iJ_z\,\phi)\exp(-iJ_y\,\theta)\exp(-iJ_z\,\chi)$$

on a deformed A-nucleon state $|k\rangle$ assumed to be an eigen-state of the z-component of the total angular momentum $J_z|k\rangle = k|k\rangle$

as has already been stated in sect.1. For the sections $\chi = \pm\phi$ we obtain the generator states

$$|\Omega^{(\pm)}, k> \equiv R^{(\pm)}(\Omega)|k> = \\ = \exp(-iJ_z\phi)\exp(-iJ_y\theta)\exp(\mp iJ_z\phi)|k> \qquad (12)$$

from which the collective Hamiltonian will be derived by calculating the overlap integrals $<\Omega|\Omega'>$, $<\Omega|H|\Omega'>$ and expanding them into powers of the collective momentum operators

$$p_\theta = -i\frac{\partial}{\partial\theta}, \qquad p_\phi = -i\frac{\partial}{\partial\phi}, \qquad (13)$$

up to second order. Applying the relations

$$p_\theta R^{(\pm)} = R^{(\pm)}(-J_x\sin\phi + J_y\cos\phi), \\ p_\phi R^{(\pm)} = R^{(\pm)}(-(J_x\cos\phi + J_y\sin\phi)\sin\theta + J_z\cos\theta) \mp R^{(\pm)}J_z, \qquad (14)$$

and taking into account that $<k|J_x|k> = <k|J_y|k> = 0$ we have

$$<\Omega^{(\pm)}, k|p_\theta|\Omega^{(\pm)}, k> = 0, \qquad (15)$$

$$<\Omega^{(\pm)}, k|p_\phi|\Omega^{(\pm)}, k> = k(\cos\theta \mp 1) \equiv \\ \equiv A_\phi^{(\pm)}. \qquad (16)$$

The expectation value of the collective moment operator in generator states plays a central role in our description. As is shown by Berry /1/, the quantities constructed in this way for adiabatic motion of quantum systems have the property of a vector potential acting in the space of collective variables. In the rotational context this prescription was firstly used by Moody and Wilczek /12/. Concerning the vector potential A_ϕ defined by eq. (16), it is interesting to denote that the wave function overlap integral $<\Omega^{(\pm)}, \Omega'^{(\pm)}>$ that plays a crucial role for the description of collective properties, is directly connected with A_ϕ: Really, in the Gaussian overlap approximation

$$<\Omega^{(\pm)}, \Omega'^{(\pm)}> = \exp\{i(\phi-\phi')A_\phi + -\frac{1}{2}g_{\nu\mu}\delta^\nu\delta^\mu\}, \\ \delta^\nu = \xi^\nu - \xi'^\nu, \quad \xi^1 = \phi, \quad \xi^2 = \theta; \qquad (17)$$

the linear term contains just the vector potential A_ϕ. Formerly [7] linear contributions to $<\Omega,\Omega'>$ and $<\Omega, H\Omega'>$ have been dropped, because in a time-inversion symmetric theory they disappear.

Next, let us evaluate the metrical tensor $g_{\nu\mu}$ appearing in (17). Applying the definition [7], we have

$$g_{\nu\mu} \equiv \mathrm{Re} <\Omega^{(\pm)}, k \mid p_\nu p_\mu \mid \Omega^{(\pm)}, k>_c =$$

$$= \mathrm{Re}\,\{ <\Omega^{(\pm)}, k \mid p_\nu p_\mu \mid \Omega^{(\pm)}, k> - <\Omega^{(\pm)}, k \mid p_\nu \mid \Omega^{(\pm)}, k> \cdot$$

$$\cdot <\Omega^{(\pm)}, k \mid p_\mu \mid \Omega^{(\pm)}, k> \}.$$

Note that from this definition the contribution from the linear expectation values (15), (16) is excluded.

Using then the relations (14) we obtain

$$g_{ij} = \sum_{\kappa=1}^{2} L_i^\kappa L_j^\kappa <k \mid J_\kappa^2 \mid k>, \tag{18}$$

$$(J_1 \equiv J_x, \; J_2 \equiv J_y),$$

where the following 2x2 matrix:

$$L_i^\kappa = \begin{pmatrix} L_1^1 & L_1^2 \\ L_2^1 & L_2^2 \end{pmatrix} = \begin{pmatrix} \sin\phi & \cos\phi \\ \cos\phi \sin\theta & -\sin\phi\sin\theta \end{pmatrix} \tag{19}$$

was introduced. Furthermore, with the help of the corresponding reciprocal matrix K_a^μ, $\sum_\mu K_a^\mu L_\mu^b = \delta_{ab}$

$$K_a^\mu = \begin{pmatrix} \sin\phi & \cos\phi/\sin\theta \\ \cos\phi & -\sin\phi/\sin\theta \end{pmatrix} \tag{20}$$

and introducing the quantities

$$f_\kappa^i \equiv \frac{K_\kappa^i}{\sqrt{<k \mid J_\kappa^2 \mid k>}}, \tag{21}$$

we define the effective mass tensor

$$B_{ij} = \sum_\kappa f_\kappa^i f_\kappa^j \frac{<k \mid H J_\kappa^2 \mid k>}{<k \mid J_\kappa^2 \mid k>}. \tag{22}$$

Applying this expression and using

$$g = \det g_{\nu\mu} = \sin^2\theta <J_1^2><J_2^2>$$

the collective kinetic energy obtains the form

$$H_{coll} = \frac{1}{2}\frac{1}{g}\left(\frac{1}{i}\frac{\partial}{\theta_\mu} - A_\mu\right)\sqrt{g}\cdot B_{\mu\nu}\cdot$$

$$\left(\frac{1}{i}\frac{\partial}{\partial\theta_\nu} - A_\nu\right); \tag{23}$$

$$(\theta_1 = \phi,\ \ \theta_2 = \theta,\ \ A_1 = A_\phi,\ \ A_2 = A_\theta).$$

From equations (19), (20) and (18) we have

$$B_{\theta\theta} = J^{-1} = \frac{<k\,|\,H J_\kappa^2\,|\,k>}{<k\,|\,J_\kappa^2\,|\,k>^2}, \tag{24}$$

where J denotes the Yoccoz - Peierls moment of inertia; furthermore

$$B_{\theta\phi} = 0;$$

and

$$B_{\phi\phi} = \frac{1}{\sin^2\theta}\,\frac{1}{J}.$$

Collecting the results we finally obtain following expression for the collective Hamiltonian

$$H_{coll} = \Big\{-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \tag{25}$$

$$+\left(-i\frac{\partial}{\partial\phi} - A_\phi^{\pm}\right)^2\cdot\frac{1}{\sin^2\theta}\Big\}\frac{1}{2J}.$$

Obviously, the quantity A_ϕ enters into the collective Hamiltonian exactly in the same way as if on the ϕ-coordinate is acting an external vector potential A_ϕ leading to the appearance of the covariant derivative $D^{\pm}_{\phi} = -i\partial/\partial_\phi - A^{(\pm)}_{\phi}$. Furthermore, the quantities $A^{(+)}_{\phi}$ and $A^{(-)}_{\phi}$ are connected by the gauge transformation

$$A^{(+)}_{\phi} = A^{(-)}_{\phi} + iU(\phi)\frac{\partial}{\partial\phi}U^{-1}(\phi), \quad U(\phi) = \exp 2ik\phi . \tag{26}$$

Really not $A^{(\pm)}_{\phi}$, but rather $A^{(\pm)}/\sin\theta$ has the property of the ϕ-component of a vector-potential, and in spherical coordinates, requiring $\vec{A}d\vec{r} = A_\phi d\phi$ and using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$ we get

$$A^{(\pm)} = -\frac{1}{r(z \pm r)}(-y, x, 0)\cdot k, \quad (r = 1) . \tag{27}$$

Now, after using the identification

$$k = g\cdot\frac{c}{eh} ,$$

we have the interesting result that the vector potential, appearing in S^2 as the result of the Berry construction, is just equal to the vector potential of a magnetic monopole with the strength $g\frac{c}{eh}$. Then the structure of (26) reflects the well-known string singularities if the ± sections of S^2 are utilized /2/. The Hamiltonian (25) may be expressed with the help of unusual angular momentum operators which are known from the description of the magnetic monopole /5,12/

$$L_x = i(\sin\phi\,\partial/\partial\theta + \mathrm{ctg}\,\theta\cos\phi\,\partial/\partial\phi) + A^{(\pm)}_{\phi}\cos\phi/\sin\theta ,$$

$$L_y = i(-\cos\phi\,\partial/\partial\theta + \mathrm{ctg}\,\theta\sin\phi\,\partial/\partial\phi) + A^{(\pm)}_{\phi}\sin\phi/\sin\theta , \tag{28}$$

$$L_z = -i\partial/\partial\phi - k .$$

The first terms represent, respectively, the usual angular momentum component describing the quantized motion of the position of the top or of the particle for the magnetic monopole case. The additional terms are due to the field of the monopole. Classically the quantity

$$\vec{L} = \vec{r}\times(\vec{p} - \vec{A}) - k\vec{r}/r ,$$

plays the role of the angular momentum and quantization just gives eq. (28). Using these expressions, it is not difficult to verify directly that the usual commutational relations $[L_i, L_j] = i\epsilon_{ijk} L_k$ hold. Introducing then $L^2 = L_x^2 + L_y^2 + L_z^2$, from equation (28) we obtain

$$H_{coll} = (L^2 - k^2)/2J. \tag{29}$$

4. SOLUTION OF THE COLLECTIVE EIGENVALUE EQUATION

Let us now investigate the solution of the eigenvalue problem corresponding to the Hamiltonian (29)

$$H_{coll}\,\Psi^{(\pm)} = (L(L+1) - k^2)\;(L^2 - k^2)/2J\,\Psi^{(\pm)}. \tag{30}$$

The eigenfunctions $\Psi^{(\pm)}$ of L^2 and L_z are represented as

$$\Psi^{(\pm)} = \mathcal{N}\,\Theta^{(\pm)}_{L,M,k}(\theta)\cdot\Phi^{(\pm)}_{M,k}(\phi)\,|k>, \tag{31}$$

where

$$\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{(M' - A^{(\pm)}_{\phi})^2}{\sin^2\theta} + L(L+1) - k^2\}\Theta^{(\pm)}_{L,M,k}(\theta) = 0, \tag{32}$$

and

$$L_z\,\Phi_{M,k} \equiv M\Phi_{M,k} \equiv (M' - k)\cdot\Phi_{M,k}.$$

It is worth noting that eq.(32) contains, due to taking into account the vector potential A_ϕ, an effective potential that is exactly known from the classical theory of a symmetrical top /10/. Using $M' = M - k$ we have

$$U_{eff} = \frac{(M - k\cos\theta)^2}{2J\cdot\sin^2\theta}.$$

In the classical theory this term determines the periodical slope of the top in θ-direction known as nutation. The explicit solution of equation (32), as an example for $A^{(+)} = k(\cos\theta - 1)$ is given by $(z = \cos\theta)$

$$\Theta^{(+)}_{L,M,k}(\theta) = \left(\frac{z-1}{2}\right)^{-(M+k)/2}\left(\frac{z+1}{2}\right)^{-(M-k)/2} \cdot$$

$$\cdot\, F\left(\alpha\,\beta\,\gamma;\ \frac{1-z}{2}\right), \tag{33}$$

where $F(\alpha, \beta, \gamma, z)$ is a solution of the well-known hypergeometrical equation

$$F'' + \frac{-\gamma + (1+\alpha+\beta)z}{z(z-1)} \cdot F' + \frac{\alpha\beta}{z(z-1)} \cdot F = 0, \tag{34}$$

and

$\alpha = -L + M,$

$\beta = L + M + 1,$

$\gamma = 1 + M + k.$

Since the coefficients α and β do not depend on the quantity k, $\Theta_{L,M,k}$ is a polynomial and terminates at $L - M = n = 0, 1, \ldots L$; 0, 1, ... ; explicitly

$$\Theta^{(+)}_{L,M,k} = \frac{1}{2^L}(-)^{(L-M)}\frac{(M+k)!}{(L+k)!} \cdot$$

$$\cdot \frac{d^{L-M}}{(dz)^{L-M}}\left[(1-z)^{L+k}(1+z)^{L-k}\right](1-z)^{-(M+k)/2}(1+z)^{-(M-k)/2}. \tag{35}$$

A solution in the form (31) violates the time inversion symmetry because it singles out the states $|k\rangle$. To obtain a symmetric description, we introduce time-inverse internal states defining

$$|\tilde{k}\rangle = R_i^{-1}\,|k\rangle, \quad R_i = \exp(-iJ_y\pi),$$

$$J_z\,|\tilde{k}\rangle = -k\,|\tilde{k}\rangle; \tag{36}$$

and introduce the spinor representation

$$|k> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\tilde{k}> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vector potential (16) then obtains the form

$$A_{\phi}^{\pm} = k(\cos\theta \mp 1)\,\sigma_z \tag{37}$$

and also the components of the angular momentum operators (28) get Pauli operators, especially

$$L_z = -i\,\partial/\partial\phi - k\sigma_z\,. \tag{38}$$

As a consequence of D_2-symmetry (also R-invariance [11]) it follows that the total wave function remains unchanged if we perform a rotation in the collective space around the y-axis by 180° together with going from $|k>$ to $|\tilde{k}>$; hence

$$\begin{aligned} &\Theta^{(+)}_{L,M,k}(\theta)\,\Phi_{M,k}(\phi)\,|k> = \\ &= \Theta^{(+)}_{L,M,k}(\pi-\theta)\,\phi_{M,k}\,(\phi+\pi)\,|\tilde{k}>. \end{aligned} \tag{39}$$

From the definition (16) it follows that

$$A_{\phi}^{(+)}(\pi-\theta) = -A_{\phi}^{(-)}(\theta)\,,$$

therefore, from formula (32) we have the symmetry relation

$$\Theta^{(+)}_{L,M,k}(\pi-\theta) = \Theta^{(-)}_{L,M,-k}(\theta)\,, \tag{40}$$

and finally, taking account of eq. (40), we get the symmetrized wave-function of the rotor in S^2

$$\begin{aligned} \Psi_{LM} = \frac{N}{\sqrt{2}}\{&\Theta^{(+)}_{L,M,k}(\theta)\cdot e^{iM'\phi}\,|k> + \\ &+\Theta^{(-)}_{L,M,-k}(\theta)\cdot e^{iM'\phi}\cdot e^{-2ik\phi}\,e^{i(M+k)\pi}\,|\tilde{k}>\}. \end{aligned} \tag{41}$$

The factor $\exp -i2k\phi$ in the second term guarantees that with taking into account (38) the relation $L_z\,\Psi_{LM} = M\Psi_{LM}$ holds. Requiring now that Ψ_{LM} is a unique function of ϕ

$$\Psi_{LM}(\theta,\phi+2\pi) = \Psi_{LM}(\theta,\phi)\,, \tag{42}$$

from equation (41) it follows that $\exp(-2ik\cdot 2\pi) = 1$ and therefore, we obtain finally the quantization condition for k.

$$k = 0, \pm 1/2, \pm 1, \ldots (\hbar = 1). \tag{43}$$

By applying the gauge condition (26) the quantization condition may be represented as well in the form

$$\oint A_{\phi}^{(+)} d\phi - \oint A_{\phi}^{(-)} d\phi = 4\pi k \tag{44}$$

which is the Dirac quantization condition in the formulation of Wu and Yang [8]. From the requirement of uniqueness (42) it also follows that M' is an integer, and since $M = M' \pm k$, M and also L are integers or half-integers depending on k. Finally, the factor $\exp i(M+k)\pi$ causes the well-known interference effect in the spectrum.

5. CONCLUSIONS

As we have seen, the wave-function $\Psi^{(+)}$ contains the phase $\int A_{\phi}^{(+)} d\phi$ depending on the angle θ. This description, however, is not unique on S^2. Alternatively, one may introduce $\psi^{(-)}$ and connected with it the phase $\int A_{\phi}^{(-)} d\phi$, depending on $4\pi - \theta$. To solve this contradiction we have to note that the phases $\int A_{\phi}^{(\pm)} d\phi$ appear only in a description that breaks the time inversion symmetry, otherwise it is always possible to choose real functions, and also in the overlap expression (17) for time inversion symmetry the linear term would disappear. An alternative way is to introduce states violating this symmetry, and afterwards to construct from them symmetrized states. Just this is our procedure leading to the total wavefunction (41), which depends on the difference of Berry's phases, and the requirement of uniqueness gives finally the Dirac condition.

Let us denote another formulation which sheds light on different aspects. The states $|R^{(\pm)}, k\rangle$ (12) corresponding to the $\chi = \pm\phi$ sectors may be connected by the operation $\exp iJ_2\chi$, giving a rotation around the (body fixed) z′-axis. Furthermore, like for the motion of charged particles in a magnetic field [2], it is possible to show that the requirement $\Psi(\theta, \phi, \chi + 2\pi) = \exp 4ik\pi \cdot \psi(\theta, \phi, \chi)$ is an alternative way to get the Dirac quantization condition. It is interesting to denote that this procedure corresponds to the path integration quantization of rotation, introduced by Schulman [13], in which an even number of rotations around the z′-axis leads to an integer spin while an odd number of rotations gives half-integer values.

The main point of our investigation is the application of a new mathematical language to the known task of adiabatic rotation. The advantage of such approach is to establish contacts to different questions, which are closely connected with the quantum theory of rotation. Let us denote that the argument $A_\phi \cdot \dot{\phi}$ entering into the Berry phase, may be understood as an effective action $L_{eff} = A_\phi \cdot \dot{\phi}$. In the generator coordinate approach this expression is directly connected with the wave function overlap (17). Using the quantity $s = \begin{pmatrix} z_0 & z_1^* \\ z_1 & -z_0^* \end{pmatrix}$, where z_0 and z_1 have been introduced by formula (17), the effective Lagrangian L may be represented as $i/2\,\mathrm{tr}(\sigma_3 s^{-1} \dot{s})$ [2]. It is interesting to remark that just this structure of L_{eff} appears in the quantization of the Skyrmion and has its origin in the so-called Wess - Zumino term. Finally, we denote that the nonlinear sigma model in two space dimensions [14] contains the corresponding contribution known as the Chern - Simons term due to which there appears the remarkable property of the transmutation of statistics [15].

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Received by Publishing Department
on June 15, 1990.