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ON THE RELEVANCE OF GAMOW STATES
FOR THE DESCRIPTION OF SYSTEMS
WITH TIME-DEPENDENT PARAMETERS

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1. INTRODUCTION

In the theoretical description of nuclear or atomic colli-
sions exhibiting a resonance pattern in the cross section
a discretization of the continuum can be achieved by introduc-
ing Gamow states, defined as solutions of the stationary
Schrödinger equation with purely outgoing waves in the asym-
ptotic region[1-4].

In the last years, the mathematical foundation and the
praticable techniques for the use of Gamow states have been
improved appreciably.

Even if not square integrable, one can define a scalar
product between Gamow states and a norm. Gamow states are
orthogonal to each other as well as to bound states. Together
with the bound states and specific scattering states with
complex wave number k, located at a certain contour in the
complex k-plane, Gamow states form a complete set of basis
states. Moreover, Gamow states have been shown to have a large
overlap with wave packets that are peaked at the resonance
energy[5]. Further, a rigorous mathematical basis for the im-
plantation of Gamow states in quantum theory has been founded
by an appropriate extension of the Hilbert space, see,
e.g.[6,7,8]. So, on a sufficiently explored basis, routine
calculations with Gamow states are possible now (compare,
e.g.[9]), in particular, if the well-suited momentum repre-
sentation is used[10,11].

This progress has stimulated the applications of Gamow
states also in RPA-calculations of ph-excitations in nuclei
taking into account quasibound configurations, so that the
particle decay of resonances can be related to nuclear struc-
ture properties[12], as well as in other branches as atomic
and molecular physics[13,14] or in an extension of the
Hellmann-Feynman theorem[15] known from solid-state theory
and quantum chemistry.

The concept of Gamow states has also been used for an im-
plicitely time-dependent semiclassical description of the
coupling of relative motion and intrinsic degrees of freedom
in heavy-ion reactions at incident energies near the Coulomb
barrier[11]. This approach involves a truncated set of adia-

batic single-particle states within a finite-depth two-center potential restricting the continuum to Gamow states. If in the initial situation a single particle occupies a bound state in one of the potentials, during the course of the collision Gamow states may be populated via inelastic excitations due to non-adiabatic effects in the relative motion of the collision partners along classical trajectories. Consequently, the system starts to emit particles analogous to the radioactive decay. In a closed expression the instantaneous emission spectrum may be composed out of Breit-Wigner type resonance contributions, which are determined by the occupation probabilities of the Gamow states and two real resonance parameters, the position E and the width of the Gamow states, with all quantities being time dependent. A similar formula has been employed by Cassing and Nörenberg [16].

The ansatz for the computation of emission spectra from quasistationary states in decaying systems suggested in [11,16] exhibits the lack of a rigorous foundation. In particular, the range of validity of this approach concerning the degree of non-adiabaticity of the process or the limit of the ratio Γ/E for which bumps in the emission spectrum are no longer directly related to the parameters of the Gamow state is unknown. This fact indicates that the relevance of Gamow states in the treatment of a quantum system which is driven by a time-dependent potential V is not yet explored satisfactorily.

In general, this question is also related to the physical interpretation which can be ascribed to a specific eigenvector decomposition of the continuous spectrum of the Hamiltonian used in the rigged Hilbert space formulation of the quantum mechanics ([7,8] and references therein). In this approach the basis system given by the scattering states connected with the Hamiltonian (real energy eigenvalues) is replaced by a pole part consisting of discrete Gamow states with complex energy eigenvalues $E - i\Gamma$ ($\Gamma > 0$) and a background integral of complex scattering states along a contour below the location of the poles. In an application for collision processes the Gamow state describes an exponential decay of the quasistationary (intermediate) state of the system (with a mean life time $\tau = \hbar/\Gamma$) for $t > 0$, independent of the mode of its excitation in the past ($t < 0$). Memory effects are contained only in the remaining background terms which express the specific manner in which the decaying state is created as well as deviations from the exponential decay law for times very short or very long as compared to the mean life time of the unstable state. So, the neglect of the complex scattering states in the expansion of a single-particle continuum wave function implies the assumption that the emission process is governed by the decay of the Gamow states according to an exponential decay law, while the particular process by which the decaying state has been prepared in the past and deviations from the exponential decay law can be ignored.

In order to eliminate the role of Gamow states in time-dependent problems a particular model for a single particle moving in a binding potential has been formulated for which the emission spectrum predicted from a closed formula, which is based on adiabatic Gamow states, can be compared to the emission spectrum resulting from an exact solution of the time-dependent Schrödinger equation. This model makes use of a representation of the potential in separable terms which turned out to be highly appropriate for finding exact solutions for time-dependent problems [17, 18, 19].

In Sec. 2 the underlying general problem and the formal expressions for the emission probability are explained in some detail. The Gamow states in a separable potential are discussed in Sec. 3 together with the techniques to solve the equations. Numerical results for two specific patterns of the time-dependence of the separable potential are presented in Sec. 4, followed by some concluding remarks in Sec. 5. The units for length and energy are adapted to nuclear physics problems.

2. FORMULATION OF THE PROBLEM

The Hamiltonian for a single particle moving in a finite-range potential V which depends on a parameter λ varying in time t during a limited interval $0 \leq t \leq T$

$$H(t) = H_0 + V(\lambda(t)) , \quad \lambda(t > T) = \text{const} \tag{1}$$

is supposed to have the property that for all values of λ the spectrum of H contains a continuous part so that the particle motion may be unbounded in time. In the initial state $|\psi(t_0)\rangle$ the particle occupies a bound state of $H(t_0)$. Then, in the course of time the solution $|\psi(t)\rangle$ of the time-dependent Schrödinger equation ($\hbar = 1$)

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = (H_0 + V(\lambda(t))|\psi(t)\rangle \tag{2}$$

gains components leaving the potential region. A differential probability $dW/d\epsilon$ for the emission of the particle with kinetic energy ϵ can be deduced from $|\psi(t)\rangle$ for $t \to +\infty$.

2.1. Emission from Gamow States

An approximation for the emission probability $dW/d\epsilon$ can be derived in terms of adiabatic states $|\phi_\nu\rangle$ which are defined as solutions of the time-independent Schrödinger equation for given values of λ,

$$(H_0 + V(\lambda))|\phi_\nu(\lambda)\rangle = E_\nu(\lambda)|\phi_\nu(\lambda)\rangle . \tag{3}$$

The solutions of the homogeneous Lippmann-Schwinger equation equivalent to (3),

$$|\phi_\nu(\lambda)\rangle = G_0(E_\nu(\lambda))V(\lambda)|\phi_\nu(\lambda)\rangle \tag{4}$$

correspond to bound states for

$$G_0(E_\nu(\lambda)) = (E_\nu(\lambda) - H_0)^{-1} \tag{5}$$

and to Gamow states with purely outgoing waves in the asymptotic region for

$$G_0^{(+)}(E_\nu(\lambda) - i\Gamma_\nu(\lambda)) = \tag{6}$$
$$= [\lim_{\epsilon \to 0}(E(\lambda) - H_0 + i\epsilon)^{-1}]_{E(\lambda) \to E_\nu(\lambda) - i\Gamma_\nu(\lambda)} .$$

The notation in (6) stresses the fact that an interchange of the ϵ-limit and the analytic continuation of the matrix elements of the Green Operator G_0 would give rise to solutions with the wrong boundary condition. The procedure expressed by equations (4-6) has the remarkable advantage that it does not make use of any arbitrary long-range cut-off of the potential so that the occurrence of spurious resonances is avoided[30].

A complete and orthogonal set of basis states can be chosen to consist of discrete bound states $|\phi_\nu^B\rangle$, quasistationary Gamow states $|\phi^{(+)}\rangle$ and a continuum of complex scattering states. With the assumption that the continuum is dominated by Gamow states the solution of the time-dependent Schrödinger equation (2) can be expanded as

$$|\psi(t)\rangle = \sum_\nu a_\nu(t)|\phi_\nu^B(\lambda(t))\rangle + \sum_\mu b_\mu(t)|\phi_\mu^{(+)}(\lambda(t))\rangle . \tag{7}$$

From the Schrödinger equation a set of coupled first-order differential equations for the coefficients $a_\nu(t)$, $b_\mu(t)$ can be derived which has to be solved with the initial conditions $a_\nu = \delta_{\nu\nu_0}$, $b_\mu = 0$ for all μ. During the course of the time particle emission occurs due to the population of Gamow states by inelastic excitations from bound states or by pushing the position of the occupied adiabatic states upwards into the continuum region. If one defines a time-dependent occupation probability $n_\mu(t)$ of a Gamow state $|\phi_\mu^{(+)}(t)\rangle$ as $n_\mu(t) = |b_\mu(t)|^2$, the exponential decay of each Gamow state with complex energy $E_\mu(t) - i\Gamma_\mu(t)$ gives rise to a Lorentzian shaped contribution to the emission spectrum. Integrating over time and performing an incoherent sum over all participating Gamow states one obtains a phasically transparent expression for the instantaneous angle integrated emission spectrum at time t [11],

$$\frac{dW(t)}{d\epsilon} = \frac{2}{\pi} \sum_\mu \int_{-\infty}^{t} dt' \frac{n_\mu(t')\Gamma_\mu^2(t')}{(E_\mu(t') - \epsilon)^2 + \Gamma_\mu^2(t')} . \tag{8}$$

A similar procedure using occupied Gamow states in a diabatic basis has been applied both for the angle integrated spectrum[16] and for the double differential emission probability[29].

2.2 Exact Emission Spectrum

An exact solution of (2) can be found in terms of the time evolution operator $\Gamma(t, t_0)$

$$|\psi(t)\rangle = \Gamma(t, t_0)|\psi(t_0)\rangle \tag{9}$$

which obeys the equations

$$i\frac{\partial}{\partial t}\Gamma(t, t') = H(t)\Gamma(t, t') , \tag{10}$$

$$i\frac{\partial}{\partial t'}\Gamma(t, t') = -\Gamma(t, t')H(t') .$$

After defining the time-dependent scattering states $|\psi_{\vec{k}}^{(-)}(t)\rangle$ with the help of the Möller operator $\Omega^{(-)}(t)$ (for details see [19])

$$|\psi_{\vec{k}}^{(-)}(t)\rangle = \Omega^{(-)}(t)|\vec{k}(t)\rangle =$$

$$= \lim_{r \to \infty} [\Gamma(t,r)\, \Gamma_0(r,t)]\, e^{-\frac{k^2}{2m}t}\, |\vec{k}\rangle \qquad (11)$$

the differential probability for the emission of a particle with momentum $\vec{k}$ is obtained by projecting $|\psi(t)\rangle$ into $|\psi_{\vec{k}}^{(-)}(t)\rangle$,

$$\frac{d^3W}{d\vec{k}} = |\langle \psi_{\vec{k}}^{(-)}(t)|\psi(t)\rangle|^2 . \qquad (12)$$

In equation (11) $\Gamma_0(t,t')$ denotes the free time-evoluation operator, $\Gamma_0(t,t') = \exp(-iH_0(t-t'))$. Due to the specific time-dependence of the potential as expressed in (1), for $t > T$ ($V = \text{const}$) one can simply insert

$$|\psi_{\vec{k}}^{(-)}(t)\rangle = |\Phi_{\vec{k}}^{(-)}\rangle\, e^{-i\frac{k^2}{2m}t} , \qquad (13)$$

where $|\Phi_{\vec{k}}^{(-)}\rangle$ is a stationary scattering state referring to the potential $V(\lambda(t > T))$. An angle integration in (12) yields the emission probability

$$\frac{dW}{d\epsilon} = (2m)^{3/2}\sqrt{\epsilon} \int d\vec{k}\, \frac{d^3W}{d\vec{k}} , \quad k = \sqrt{2m\epsilon} \qquad (14)$$

which has to be compared to the approximate expression (8) for $t > T$.

3. MODEL WITH TIME-DEPENDENT SEPARABLE POTENTIALS

For the present computations the attractive potential V appearing in (1) has been represented by a sum of separable terms ($\lambda_{n\ell}(t) > 0$)

$$V(t) = -\sum_{n\ell m} |n\ell m\rangle\, \lambda_{n\ell}(t)\, \langle n\ell m| . \qquad (15)$$

The single particle states $|n\ell m\rangle$ are specified by the nodal quantum number n (which has been restricted·to be $n = 0$ in the following so that it may be suppressed in the notation) and the angular momentum quantum numbers ℓ, m. The time-dependence of the potential results from the time-dependence of the parameter λ which mainly determines the binding energy of the ℓ-state in the potential.

Such a separable expansion of a potential invented by Revai[21] and Gyarmati et al.[22] has been successfully applied later on in various contexts as for the calculation of bound[23] and Gamov states [24] in a deformed potential (including the Coulomb case [25]) for scattering states[26], for the treatment of the two-center problem[20,11] or for the representation of realistic nucleon-nucleon interactions[27].

3.1. Adiabatic Basis

Due to the orthogonality of the partial waves ℓ, m, the condition for the existence of non-trivial solutions of the secular equation,

$$1 + \lambda_\ell\, \langle \ell m|\, G_0(E)|\, \ell m\rangle = 0 \qquad (16)$$

holds for each partial wave separately. Because G_0 contains only the momentum operator it is convenient to work in momentum representation. If one chooses

$$\langle \vec{k}\,|\, \ell m\rangle \sim g_{0\ell}(k)\, Y_{\ell m}(\hat{\vec{k}}) \qquad (17)$$

with $g_{0\ell}(k)$ representing a radial oscillator function in momentum representation, the matrix elements appearing in (16) can be calculated analytically for bound states (G_0 from (5)) as well as for Gamow states (G_0 from (6)) (for details see appendix A of [11]). One should emphasize that due to the presence of G_0 in (4) the bound state possesses the correct asymptotic behaviour, although oscillator states have been utilized to expand the potential. Moreover, it turns out that it is not necessary to introduce complex separable terms (e.g., complex parameter λ) in order to describe resonances as has been done by Baldo et al. [27,28].

In figure 1 the energy eigenvalues computed from (16) are presented in the complex k-plane ($k = K_r - iK_i$, $K_r \geq 0$) (pole trajectories[31]) for the angular momentum $\ell = 0, 1, 2, 3$ in dependence on the potential parameter (expressed in units of 20.7 MeV),

$$E = (K_r^2 - K_i^2)/2m , \quad \Gamma = K_r \cdot K_i/m. \qquad (18)$$

The oscillator parameter used to calculate the functions $g_{0\ell}$ in (17) is $b = \sqrt{\hbar/m\omega} = 1$ fm. For s-waves ($\ell = 0$) only virtual states ($K_r = 0$, $K_i < 0$), which have no physical meaning,
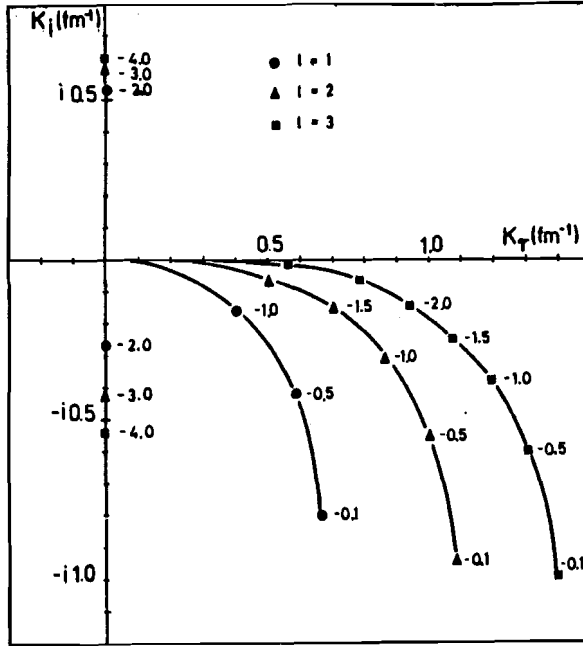
Fig. 1. Pole trajectories of bound, virtual, and Gamow states in the complex K-plane for separable potentials with angular momentum ℓ = 1, 2, 3. The dimensionless potential-strength parameter λ is shown by the the numbers at the curves.
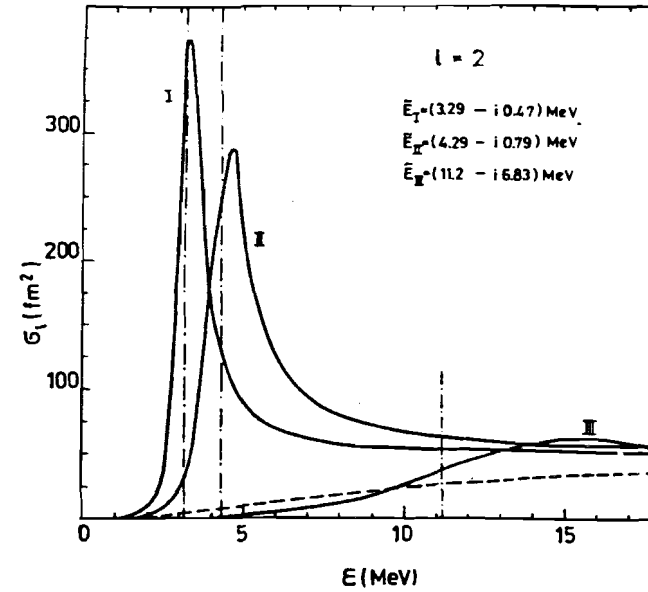

Fig. 2. Cross sections for the elastic scattering process from a separable potential with ℓ =2 as a function of the incident energy ϵ. The dashed curve corresponds to the scattering from a bound system with a potential strength λ =-2.7. The three full curves are the results for the scattering from potentials possessing Gamow states with complex energy E corresponding to λ = -2.2 (I), -2.1 (II), -1.3 (III). For comparison the positions of these Gamow states are indicated by dash-dotted lines.

and bound states (K_r = 0, K_i > 0) have been found in the region of the k-plane covered by the figure. For ℓ > 0 and sufficiently large values of λ_ℓ all states become bound. But by lowering at a certain value $\lambda_\ell^{(c)}$ a transition from a bound state to a Gamow state occurs. For example, if λ_ℓ = -2.0 the p-state is still bound (E_B = -5.6 MeV) while the d- and the f-state appear in the continuum with a complex energy E = 5.07 MeV, Γ = 1.45 MeV and E = 17.5 MeV and Γ = 5.8 MeV, respectively. Above this value $\lambda_\ell^{(c)}$, $\lambda_\ell \gtrsim \lambda_\ell^{(c)}$, the poles tend to stay near to the real axis ($\Gamma/E \ll 1$) so that the Gamow states give rise to sharp resonances in the elastic scattering cross section of a particle with kinetic energy ϵ = $k^2/2m$. The scattering state in (13) is given by

$$|\Phi_{\vec{k}}^{(-)}> = |\vec{k}> + A_\ell(k)G_0^{(-)}(k^2)|\ell m>, \tag{19}$$

$$A_\ell(k) = \frac{\lambda_\ell <\vec{k}|\ell m>}{1 + \lambda_\ell <\ell m|G_0^{(-)}(k^2)|\ell m>}, \tag{20}$$

and the cross section for elastic scattering reads

$$\sigma_\ell(k) = \frac{m^2}{4\pi^2}(2\ell + 1)|t_\ell(k)|^2, \tag{21}$$

$$t_\ell(k) = 2\pi^2 A_\ell(k). \tag{22}$$

For demonstration, the resonance behaviour of the d-wave elastic scattering cross section is shown in figure 2 for various position of the pole along the trajectory given in figure 1, λ_2 = -2.8 (E_B = -4.0 MeV), λ_2 = -2.2 (Γ/E = 0.14), λ_2 = -2.1 (Γ/E = 0.18), λ_2 = 1.3 (Γ/E = 0.60). Because for λ_2 = -2.8 the pole represents a bound state, only potential scattering without forming a quasistationary state occurs, which gives a smooth background increasing with bombarding energy. If the energy eigenvalue becomes complex, sharp resonances are observed, the position and width of which are directly related to the parameters of the Gamow state as far as $\Gamma/E \ll 1$. Already for Γ/E = 0.6 the position and width of the resonances in the cross section disagree appreciably with the parameters of the quasistationary state.

3.2. Exact Solution

The exact solution of the time-dependent Schrödinger equation (2) with the ansatz (15) for the potential can be found applying the techniques described in [19].

From the equation (10) a Volterra integral equation follows for $\Gamma(t, t')$,

$$\Gamma(t, t') = \Gamma_0(t, t') - i \int_{t'}^{t} dt'' \Gamma_0(t, t'') V(\lambda(t'')) \Gamma(t'', t') \qquad (23)$$

so that $|\psi(t)\rangle$ can be expressed as

$$|\psi(t)\rangle = \Gamma_0(t, t_0)|\psi(t_0)\rangle - i \int_{t_0}^{t} dt' \Gamma_0(t, t') V(\lambda(t'))|\psi(t')\rangle. \qquad (24)$$

Inserting the separable expansion (15) of the potential, one obtains

$$|\psi(t)\rangle = \Gamma_0(t, t_0)|\psi(t_0)\rangle - i \sum_{\ell m} \int_{t_0}^{t} dt' \Gamma_0(t, t') \lambda_\ell(t')|\ell m\rangle X_{\ell m}(t') \qquad (25)$$

with

$$X_{\ell m}(t) = \langle \ell m | \Gamma_0(t, t_0)|\psi(t_0)\rangle -$$

$$\qquad (26)$$

$$-i \sum_{\ell' m'} \int_{t_0}^{t} dt' \langle \ell m | \Gamma_0(t, t')|\ell' m'\rangle \lambda_{\ell'}(t') X_{\ell' m'}(t').$$

Again, due to the orthogonality in the angular part of $|\ell m\rangle$ the set of equations for the quantities $X_{\ell m}(t)$ reduces to decoupled Volterra integral equations for each $X_{\ell m}(t)$ which can be solved numerically after calculating the inhomogeneous terms and the integral kernels according to the appendix.

4. NUMERICAL CALCULATIONS

The numerical calculations have been performed for a single particle in a one-term separable potential with quantum numbers $n = 0$, $\ell = 2$, $m = 0$. Initially, the particle occupies a bound state at $E_B = -4$ MeV, which correponds to the parameter value $\lambda(t = 0) = \lambda_0 = -56$ MeV. By decreasing λ in time, a smooth transition of the adiabatic state from bound state to a Gamow state can be achieved for $\lambda^{(c)} = -50$ MeV at $t = t_c$. Such a time-dependence of λ may be parametrized according to

$$\lambda(t) = \lambda_0 + \Delta\lambda F(t), \quad \Delta\lambda > 0 \qquad (27)$$

so that different regimes of the potential change can be distinguished by the choice of $F(t)$ as displayed in figure 3. If one uses the ansatz

$$F(t) = \begin{cases} \frac{1}{2}(1 - \cos(2\pi t/\tau)) & t < \tau \\ & \text{for} \\ 0 & t \geq \tau, \end{cases} \qquad (28)$$

the Gamow state appears as a transient state only (case A). So, the system "irradiates" during a certain time period while it finally at $t = T$ comes back to the initial bound state. For

$$F(t) = \begin{cases} \frac{1}{2}(1 - \cos(\pi t/\tau)) & t < \tau \\ & \text{for} \\ 1 & t \geq \tau \end{cases} \qquad (29)$$

the system remains unbound also for $t > T$, with the position and width of the Gamow state being determined by the parameter $\lambda = \lambda_0 + \Delta\lambda$ (case B). For example, $\Delta\lambda = 16$ MeV (26 MeV) gives a final position of the Gamow state at $E_m = 5.1$. MeV (9.6 MeV) with a width of $\Gamma_m = 1.3$ MeV (4.7 MeV). In both cases the time-dependent occupation probability $n(t)$ of the Gamow state has been computed as

$$n(t) = \exp(-2 \int_{t_c}^{t} dt' \Gamma(t')), \quad t > t_c. \qquad (30)$$

Within this picture the behaviour of the system is governed by three time scales. The initial binding energy E_B of the particle determines the period T_B of the unperturbed internal motion of the particle, $T_B = \hbar/E_B$. Then, one has a "reaction time" T given by the time interval in which
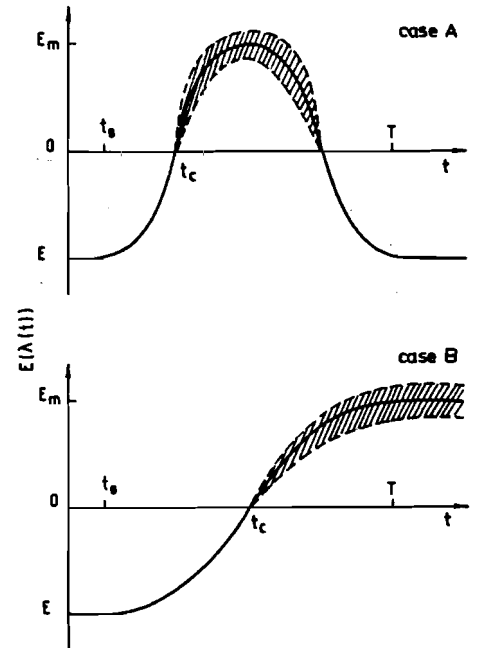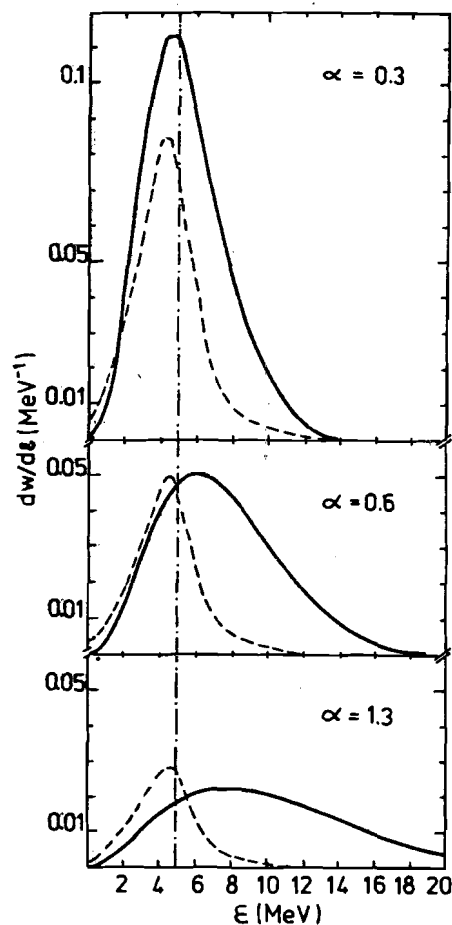

Fig. 3. Illustration of the adiabatic basis for the types of dynamics used in the numerical calculations. Case A: The Gamow state is reached as a transient state for a limited time interval. Case B: The Gamow state appears as the final state. The width of the complex eigenstates involved are indicated by the shadowed area.

the change of the potential between the stationary situations is performed. The ratio $a = T_B/T$ can be used to express the degree of non-adiabaticity of the process. The time scale of emission is fixed by the mean life time of the Gamow states involved, $\tau = \hbar/\Gamma$, so that the ratio T/τ finally determines the shape of the peak structure of the emission spectrum if it is calculated according to the expression (8). For both regimes A, B the numerical calculations have been performed for $E_B = -4$ MeV corresponding to $T_B = 1.6 \cdot 10^{-22}$ s. The reaction time T has been chosen to select the parameter values $a = 0.3$, 0.6, and 1.3 for the non-adiabaticity parameter. The parameter λ has been varied according to equations (27)-(29) in order to lift the Gamow state in the continuum until a maximum position characterized by E_m and Γ_m is reached.

The emission spectra calculated according to the expressions (12, 14) based on the exact solution of the time-dependent Schrödinger equation and the approximate formula (8) making use of the concept of Gamow states are compared in figures 4 and 5.

For case F (figure 4) a growing agreement between the predictions of both approaches can be observed if the reaction time T of the process increases, e.g., the parameter a decreases. For a slow process (upper part of figure 4) one can observe a coincidence

Fig. 4. Case A. Differential emission probability varsus escape energy for various values of the non-adiabaticity parameter a defined in the text. Full curves: results from exact calculations including fully the continuum. Dashed curves: spectra deduced from an emission from Gamov states. The dash-dotted line marks the maximum real part E_m of the complex energy of the intermediate Gamow state (compare to figure 3).

of the peak position and a qualitative agreement, within an error of 30%, of the width and the total amount of emission probability. The coincidence of the peak positions both in the exact and in the approximate Gamow state treatment is remarkable because the peak does not correspond to the position of E_m (compare to figure 3) for which the width has its maximum value Γ_m. This means that for slow processes the emission prior or post to reaching the highest-lying Gamow state is not negligible. For a fast process (lower part of figure 4) the high non-adiabaticity creates a maximum in the emission spectrum which appears at higher energies than has been reached by the Gamow state. In addition, a high-energy component develops in the exact spectrum, which cannot be covered by the decay of the Gamow state because the width Γ involved is too much small. Therefore, one can conclude that in such a case for several reasons the picture cannot be maintained that the Gamow state is occupied and then decays independently from the mode of its preparation. First, the mean life time of the Gamow state becomes less than the reaction time T. For example, the average values of the mean life time of the Gamow states involved turns out to be of about $6 \cdot 10^{-22}$ s ($\Gamma = 1$ MeV) which can be compared to the reaction time $T = 5 \cdot 10^{-22}$ s for $a = 0.4$, but is significantly larger than the reaction time $T = 1 \cdot 10^{-22}$ s corresponding to $a = 1.3$. Second, due to the high non-adiabaticity of the process, in an expansion of $|\psi(t)\rangle$ in terms of an adiabatic basis the strong coupling to the background parts of the
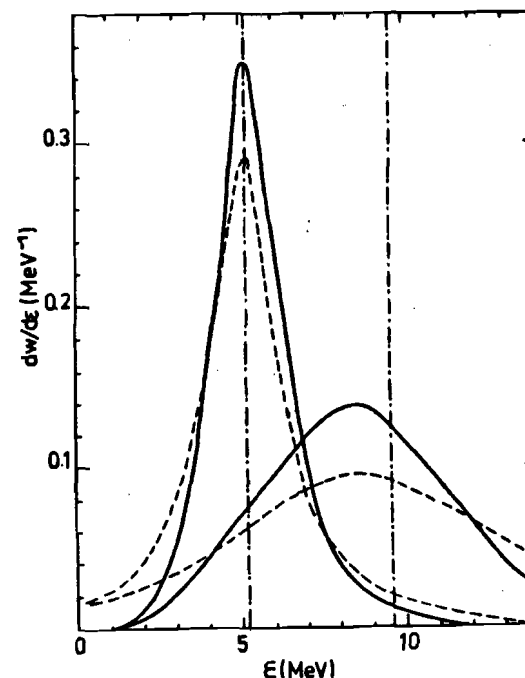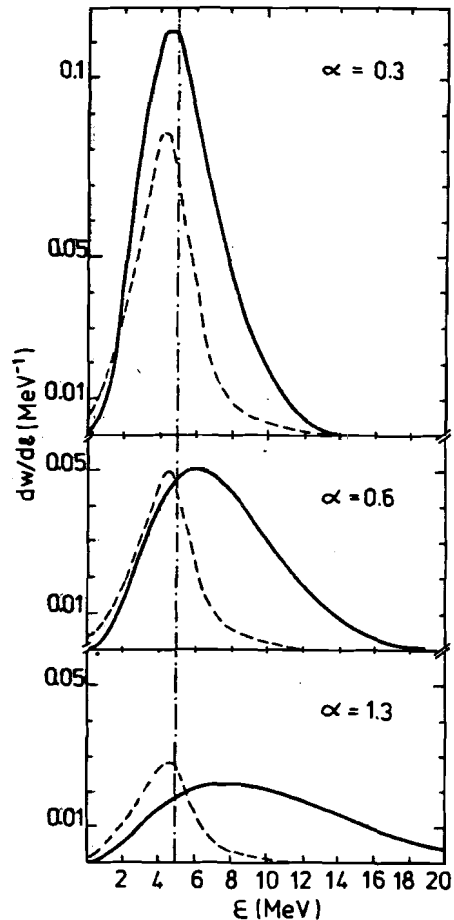

Fig. 5. Case B: Differential probability versus escape energy for two different positions of the final Gamow state marked by the dash-dotted line. In both cases the non-adiabaticity parameter is $a = 0.4$.

the change of the potential between the stationary situations is performed. The ratio $a = T_B/T$ can be used to express the degree of non-adiabaticity of the process. The time scale of emission is fixed by the mean life time of the Gamow states involved, $\tau = \hbar/\Gamma$, so that the ratio T/τ finally determines the shape of the peak structure of the emission spectrum if it is calculated according to the expression (8). For both regimes A, B the numerical calculations have been performed for $E_B = -4$ MeV corresponding to $T_B = 1.6 \cdot 10^{-22}$ s. The reaction time T has been chosen to select the parameter values $a = 0.3$, 0.6, and 1.3 for the non-adiabaticity parameter. The parameter λ has been varied according to equations (27)-(29) in order to lift the Gamow state in the continuum until a maximum position characterized by E_m and Γ_m is reached.

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of the peak position and a qualitative agreement, within an error of 30%, of the width and the total amount of emission probability. The coincidence of the peak positions both in the exact and in the approximate Gamow state treatment is remarkable because the peak does not correspond to the position of E_m (compare to figure 3) for which the width has its maximum value Γ_m. This means that for slow processes the emission prior or post to reaching the highest-lying Gamow state is not negligible. For a fast process (lower part of figure 4) the high non-adiabaticity creates a maximum in the emission spectrum which appears at higher energies than has been reached by the Gamow state. In addition, a high-energy component develops in the exact spectrum, which cannot be covered by the decay of the Gamow state because the width Γ involved is too much small. Therefore, one can conclude that in such a case for several reasons the picture cannot be maintained that the Gamow state is occupied and then decays independently from the mode of its preparation. First, the mean life time of the Gamow state becomes less than the reaction time T. For example, the average values of the mean life time of the Gamow states involved turns out to be of about $6 \cdot 10^{-22}$ s ($\Gamma = 1$ MeV) which can be compared to the reaction time $T = 5 \cdot 10^{-22}$ s for $a = 0.4$, but is significantly larger than the reaction time $T = 1 \cdot 10^{-22}$ s corresponding to $a = 1.3$. Second, due to the high non-adiabaticity of the process, in an expansion of $|\psi(t)>$ in terms of an adiabatic basis the strong coupling to the background parts of the
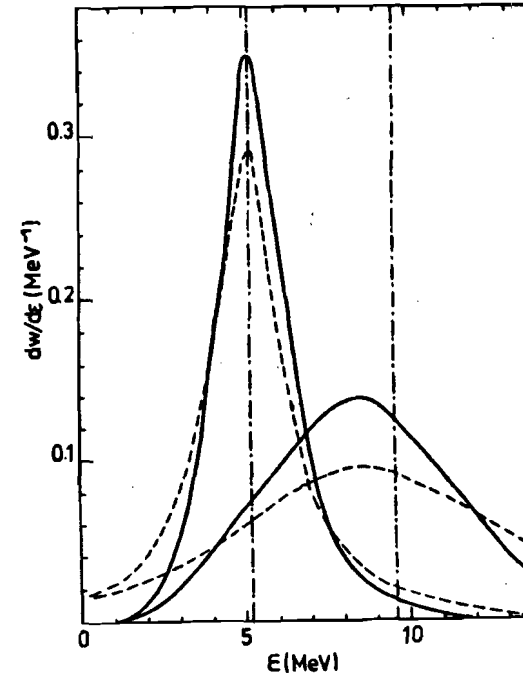

Fig. 5. Case B: Differential probability versus escape energy for two different positions of the final Gamow state marked by the dash-dotted line. In both cases the non-adiabaticity parameter is $a = 0.4$.

continuum (complex scattering states) also at higher energies obviously cannot be neglected.

For case B the general shapes of the emission spectrum computed exactly and according to the ansatz (8) are in good agreement with each other. The peak in the spectrum appears near the final position of the Gamow state. In figure 5 some numerical results for case B are shown for an almost adiabatic process ($\alpha = 0.4$) taking into account two final positions of the Gamow states E_m, Γ_m. For the higher-lying final quasi-stationary state a large fraction of the ejected particles are appearing before the final position in the continuum is reached. Consequentely, the peak in the emission spectrum is shifted to a position below the final position of the Gamow state. For a fast process (not shown in figure 5) the exactly calculated emission spectrum exhibits in addition to the dominating peak a high-energy tail which in magnitude is comparable to case A (about 10% of the total emission probability).

In all calculations one has to state a principal overestimation of the low-energy part in the Gamow approximation. This fact reflects (i) the neglect of the background contribution far from the peak region in the transient high-lying Gamow states and (ii) the fact that a Gaussian shape for the instantaneous differential emission probability is probably better suited for the large values of Γ/E under consideration rather than the Lorentzian distribution used in the calculations.

5. SUMMARY

For the first time the relevance of Gamow states for decay processes of explicitly time-dependent system has been investigated numerically by employing the exact solutions of the time-dependent Schrödinger equation. The concept of Gamow states has been used earlier in order to describe semiclassically the break-up of a bound system in nucleus-nucleus collisions. The treatment leads to a closed expression for the emission probability which is based on a physically transparent mechanism of a radioactive decay with time-dependent decay parameters. Such a procedure is easy feasible compared to an exact incorporation of the continuum but suffers from a lack of rigorous foundation.

In order to study this problem more quantitatively, complex quasistationary states have been investigated in angular-dependent real separable potentials with fixed angular momen-

tum. The use of separable potentials allows to a large extent analytical work. Although not very relativistic, the separable potential gives a behaviour of the pole trajectories of Gamow states which is known from three-dimensional local potential with a barrier in the effective radial potential shape. It turns out that for $\Gamma/E \leq 0.1$ the position and width of the resonance in the scattering cross section coincides with the parameters of the Gamow state. For $\Gamma/E > 0.5$ this correlation is completely lost which means that the lifetime of an intermediate state in the scattering process becomes too small in order to create a pronounced resonance pattern expressing an exponential decay law as well as a "forgetting" of the incoming plane wave.

After constructing the adiabatic basis with Gamow states an approximate instantaneous emission spectrum has been computed for two dynamical situations. In both regimes the particle initially occupies a bound state. Then, the potential has been varried in time in such a way that in the adiabatic basis a Gamow state occurs as a transient state (case A) or as a final state (case B). The exact solution of the time-dependent single-particle Schrödinger equation has been calculated for these processes in the framework of a time-evolution operator technique applied earlier in more complicated dynamical systems. A comparision of the approximate and exact results support the following statements:

(i) For a small degree of non-adiabaticity and for a time interval during which the initially bound state stays in the continuum which is large compared to the characteristic lifetimes of the Gamow states the approximate spectrum reflects quantitatively the features of the exact solutions (peak position, peak width, amount of emission).

(ii) For the considered break-up of a bound system under the influence of external forces the physical relevance of a quasistationary state is maintained in contrast to a scattering process, even for a ratio $\Gamma/E \geq 0.5$. This result reflects the fact that its asymptotic boundary condition of purely outgoing waves is much more suited for the decay problem under consideration than for the scattering process.

(iii) The non-adiabatic coupling to the background of high-lying scattering states, which basically cannot be reproduced by the emission from Gamow states, results in a high-energetic component in the emission spectrum.

Finally, we can conclude that under certain conditions the use of Gamow states in explicitly time-dependent processes allows at least qualitatively the prediction of typical fea-

tures of the emission spectrum. Such a mechanism could be important in parameter-dependent systems known from molecular, nuclear, and solid-state physics. But a general mathematical investigation of this question is still missing.

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Appendix MATRIX ELEMENTS

We consider the inhomogeneity f and the integral kernel K of the Volterra equation (26), respectively.

$$f = <\ell m | \Gamma_0(t,t_0) | \psi(t_0)>, \tag{A.1}$$

$$K = <\ell m | \Gamma_0(t,t_0) | \ell m>. \tag{A.2}$$

With the use of the normalized bound state $|\phi^B>$ with energy E_B

$$|\phi^B> = N_\ell G_0(E_B) | \ell m>, \tag{A.3}$$

$$N_\ell = (<\ell m | G_0^2(E_B) | \ell m>)^{-1/2}, \tag{A.4}$$

as the initial state and using the explicit form of $\Gamma_0(t,t_0)$ and $G_0(E_B)$ follows for f

$$f = N_\ell <\ell m | e^{-i\frac{p^2}{2m}(t-t_0)} \epsilon [E_B - \frac{p^2}{2m}]^{-1} | \ell m>. \tag{A.5}$$

Introducing the oscillator wave functions (17) in momentum representation

$$g_{0\ell}(x) = b^{3/2} \sqrt{\frac{\Gamma(\ell+3/2)}{2}} x^\ell e^{-x^2/2} \tag{A.6}$$

from (A.5) one obtains integrals of the type

$$I_s = \frac{1}{a(t)^{\ell+1/2}} \int_0^\infty dx \frac{e^{-a(t)x^2}(x^2)^s}{a(t)\gamma^2 + x^2}, \quad s = \ell+1, \tag{A.7}$$

$$a(t) = 1 + i(t-t_0)/(2mb^2), \quad \gamma^2 = -2mE_B b^2,$$

which obeys the recurrence formula

$$I_s = a(t)\gamma^2 I_{s-1} + \frac{1}{2} \Gamma(s + \frac{1}{2}) \tag{A.8}$$

with the basic integral

$$I_0 = \text{erfc}(\sqrt{a(t)}\gamma) \pi [2\sqrt{a(t)}\gamma]^{-1} e^{-a(t)\gamma^2}, \tag{A.9}$$

The integral kernels are found in a similar way.

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