89-432

объединенный
институт
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исследований
дубна

P97

E4-89-432

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PROTON-PROTON REACTION THEORY WITH PROTON POLARIZABILITY

Submitted to journal "Few-Body Systems"

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1989

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Теория протон-протонной реакции с эффектом поляризуемости протона

Эффект поляризуемости протона в pp-рассеянии и в pp-реакции рассматривается посредством включения в pp-взаимодействие поляризационного потенциала. В рамках метода фазовых функций получены удобные низкоэнергетические представления функции pp-рассеяния. Эти представления используются для детального аналитического и численного анализа матричного элемента pp-реакции, записанного в стандартном импульсном приближении. Доказано, что для астрофизически низких энергий квадрат этого матричного элемента и вклад поляризационного потенциала в фактор S_{11} могут быть аппроксимированы линейными функциями энергии E, а часть этого вклада, связанная с областью расстояний квазиклассически допустимых для pp-рассеяния, имеет $E^{8/3}$-пороговое ($E \to 0$) поведение.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

Pupyshev V.V., Solovtsova O.P. E4-89-432
Proton-Proton Reaction Theory with Proton Polarizability

The effect of proton polarizability in pp-scattering and in pp-reaction is considered with including a polarization potential into pp-interaction. Convenient low-energy representations of the pp-scattering function are derived within the variable phase approach and are used for a detailed analytical and numerical analysis of the pp-reaction matrix element considered in the standard impulse approximation. It is proved that for low astrophysical energies this squared matrix element and the contribution from the polarization potential to the factor S_{11} may be approximated by linear functions of energy E, while the part of this contribution associated with the region of distances quasiclassically avoided for the pp-scattering has the $E^{8/3}$ threshold ($E \to 0$) behaviour.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989

1. Introduction

As is known[1], a significant discrepancy between the predicted (7.9 ± 2.6 SNU) and measured (2.0 ± 0.3 SNU) capture rate of the solar neutrinos in the ^{37}Cl -detector exists. The predicted capture rate is more sensitive ($\sim S^{-5/2}$[2]) to the cross section factor S_{11} of the initial reaction $pp \to de\nu$ of the solar pp-chain [3]. Therefore the investigations of any corrections to the above factor are important. One of these corrections is due to the electric polarizability[4] of a proton.

After the work[5], where an unsuccessful attempt to take into account of the deuteron polarizability effect on the pd-radiative capture was made, the question about the role of nuclear polarizability on the solar nucleosynthesis reactions has been place in focus of attention. In a series of papers [6-11], stimulated by works[5,12,13], it was shown that contrary to the claims of these works the nuclear polarizability has a small effect on the cross sections of nucleosynthesis reactions. However, the polarizability effects on these inelastic process were studied in the framework of various low-energy approximations, namely: the WKB[6,7], simple optical model [8] and the Born [9-11] ones. Moreover, in all the works [6-11] a series of intermediate approximations were used without a detailed inspection of their applicability range. Numerical results of the recent work[14], where it has been shown that the contribution from the proton polarizability to the factor S_{11} is smaller than $2 \cdot 10^{-6}$, have confirmed the results obtained previously[6-11], however, they do not contribute anything new to the issue. Also, it is necessary to stress that the authors of works [6-11] concentrated their attention on estimations of upper bounds of nuclear polarizability effects on total cross sections to nucleosynthesis reactions. The question about this effect on the low-energy behaviour of the S-factors is still open.

In view of all the above reasons it is necessary to analyse the low-energy expansions of the S-factors, with taking into account of the nuclear polarizability effect and using as few assumption and approximations as possible. In the present work we realise this program for the factor S_{11} as follows. In Sec. 2 we describe the model used for pp-reaction and, in Sec. 3 we formulate the problems under consideration. In Sec. 4 we derive and analyse the low-energy representations of the pp-scattering function, pp-reaction matrix element and of the contribution from the proton polarizability effect to the factor S_{11}. In Sec. 5 we construct a perturbation theory for this effect. In Sec. 6 we report same results of numerical investigation of the factor S_{11} and in Sec. 7 we summarize our main results.

2. The model for pp-reaction

We use the standard model /15,16/ in which the factor S_{11} is proportional to the square of a dimensionless radial matrix element usually denoted by Λ. In the impulse approximation the definition of Λ is /16/

$$\Lambda = (\gamma^3/8\pi\kappa C_0^2(\eta))^{1/2} \int_0^\infty u(\kappa,r)v(r)dr \quad . \qquad (1)$$

Here γ =0.2316 fm^{-1} /17/ is the inverse deuteron radius, κ is the two-proton relative momentum corresponding to the c.m.s. energy $E=\kappa^2$, C_0^2 /17/ is the Coulomb barrier factor and $\eta = 1/2\kappa R$, where R =28.81 fm /17/ is the Bohr radius for one proton. In the asymptotic region ($r \to \infty$) the $^3S_1$ deuteron radial function v behaves like $\exp(-\gamma r)$ and the $^1S_0$ pp-scattering radial function u is defined to have the form:

$$u(\kappa,r) = \cos\delta(\kappa)\, F_0(\kappa,r) + \sin\delta(\kappa)\, G_0(\kappa,r) \quad , \qquad (2)$$

where F_0 and G_0 are the S-wave regular and irregular Coulomb functions /18/ and δ is the phase shift relative to the pure Coulomb phase shift and caused by interaction V.

Physically, V is a sum of a short-range nuclear potential V_s and a polarization potential V_p, which is due to the proton polarizability effect and decays like r^{-4} as $r \to \infty$. For the nuclear potential we use only the fact that it satisfies the conditions /17/:

$$\lim_{r\to\infty} r^n \exp(4\sqrt{r/R})\, V_s(r) = 0 \;, \quad n = 0,1,\ldots \qquad (3)$$

and therefore /19/ may be put equal to identically zero in the region $r > r_s$, where r_s is any finite radius called the action radius /17,20,21/ of this potential. For $r \leq r_s$ the explicit form of the polarization potential is unknown, therefore we are compelled to limit ourselves to the usually used /5-14/ representation

$$V_p(r) = (-\alpha/Rr^4)\,\theta(r-r_p) \quad , \qquad (4)$$

where α is the electric polarizability for one proton /4/, θ is the step function /18/ and r_p is an arbitrary but fixed radius, of course, such that $r_p \geq r_s$.

3. The problems under consideration

We denote matrix element (1) by Λ_p or by Λ when potential (4) is present ($V = V_s + V_p$) or absent ($V = V_s$).

For the temperature 1.5·10^{7}K corresponding to the solar interior the most effective energy of pp-reaction in c.m.s. is ~ 6 keV and for $E \leq$ 20 keV it is found /3/ that

$$\Lambda^2(E) = \Lambda^2(0)\,(1 + A\,E(\mathrm{MeV})) + O(E^2) \qquad (5)$$

and

$$\Lambda^2(E) = (7.08 \pm 0.18)(1 + 2.2\,E(\mathrm{MeV})) + O(E^2) \quad . \qquad (6)$$

The constants $\Lambda^2(0)$ and A are known /22/ with uncertainties /23/ due to: 2.5% - uncertainty from the nuclear potential V_s, 2% - uncertainty from the exchange mesonic currents and some other effects except the proton polarizability effect that generates the additional interaction V_p.

As is known /24/, all the special effects caused by long-range asymptotics (4) of the potential $V = V_s + V_p$ are associated with the region $r \geq r_c$, where $r_c = 1/\kappa^2 R$ is the Coulomb classical turning point /25/. When $\kappa \to 0$, the dominant part δ_t (t is an abbreviation of "tail") of the phase shift δ is due to the tail ($r \geq r_c$) of the potential V_p. In the Born approximation /25/ δ_t is given by the formula /24/

$$\tan\delta_t = -\kappa^{-1} \int_{r_c}^{\infty} V_p(r)\, F_0^2(\kappa,r)\, dr, \qquad (7)$$

which provides the first term of the low-energy expansion of $\tan\delta$ exactly:

$$\tan\delta(\kappa) = \tan\delta_t(\kappa) + O(\kappa C_0^2) = -16\alpha\kappa^5/15R^2 + O(\kappa^5) \quad . \qquad (8)$$

When $V = V_s$ and V_s satisfies eqs. (3), the threshold ($\kappa \to 0$) behaviour of $\tan\delta$ is described by the law /25,26/

$$\tan\delta(\kappa) = -\kappa C_0^2(\eta)\,(a^{-1} + (\kappa^2/2)(2R/3 - r_e) + O(\kappa^4))^{-1} \quad . \qquad (9)$$

Since laws (8) and (9) are quite different, two questions arise immediately: what is the threshold behaviour of Λ_p^2 and what is the threshold behaviour of the part C_t of the contribution

$$C(E) = (\Lambda_p(E)/\Lambda(E))^2 - 1 \qquad (10)$$

from the potential V_p to the factor S_{11} caused by the tail ($r \geq r_c$) of this potential? We attempt to give a mathematically rigorous proof of the facts that as $\kappa \to 0$

$$\Lambda_p^2(E) = \Lambda_p^2(0)(1 + A_p E(\text{MeV})) + O(E^2) \qquad (11)$$

where A_p is independent of the energy, and

$$C_t(E) = (\Lambda_p(0)/3\Lambda(0))^2 (\alpha/2^{2/3} R^3)\Gamma(1/3)(\kappa R)^{16/3}(1 + O(\eta^{1/3})), \qquad (12)$$

where Γ is the Gamma-function[18], and α and R are the parameters of potential (4).

Along with proof of eqs. (11) and (12) we recover some known results within the variable phase approach[20,21] which is more adapted for solution of all our problems. One of them is to find a rough sufficient condition unsuring the applicability of the Born approximation over potential (4) for the evaluation of the functions u, Λ_p and C. This condition being established analytically makes the constructions of the works[9-11] mathematically correct. To explain this critical statement we remind two facts. First, the Born approximation may be incorrect[27] even it provide a small correction to any studied function. Second, all results of works[9-11] were actually established within the Born approximation. As shown numerically[28,29], this approximation is good for the calculation of the phase shift caused by the polarization potential. However, this result has been used in works [9-11] to assume that the Born approximation is quite suitable for the construction of the scattering wave functions. Therefore it is necessary to prove analytically that this assumption was valid. The assumptions of all works[6-11] seem to be correct in the low-energy limit. However upper bounds of the energy region, where the approximations used in these works are suitable, have not been found. Therefore, using the results of refs.[6-11] one can estimate contribution (10) if and only if the energy of pp-collision is sufficiently small. However, according to refs.[6,14], contribution (10) is a growing function of energy. Hence, for completeness it is necessary to give a justifical answer to the question: what is the upper bound of contribution (10)? We estimate this bound by high-accuracy calculations of Λ and Λ_p.

4. The theory

To obtain eqs. (11) and (12) we derive the suitable low-energy representations for the pp-function in the presence of polarization potential (4).

4.1. Low-energy expansion of the pp-function for $r < r_c$. At first we prove that in the region $r < r_c$ the pp-function has a following low-energy expansion:

$$u(\kappa, r) = \kappa C_0(\eta)\Big(\sum_{n=0}^{1} \kappa^{2n} u_n(r) + \Delta u(\kappa, r)\Big), \qquad (13)$$

where the residual term Δu vanishes like κ^4 as $\kappa \to 0$.

To derive eq. (13) and to give a simple method for evaluation of the functions u_0 and u_1, we use one of the methods of the variable phase approach[20,21]. In this method the phase functions s and c are defined as solutions of two coupled equations[21]:

$$\partial_r s(\kappa,r) = -\kappa^{-1} V(r) U(\kappa,r) F_0(\kappa,r),$$
$$\partial_r c(\kappa,r) = \kappa^{-1} V(r) U(\kappa,r) G_0(\kappa,r) \qquad (14)$$

with the bilinear form

$$U(\kappa,r) = c(\kappa,r) F_0(\kappa,r) + s(\kappa,r) G_0(\kappa,r) \qquad (15)$$

and boundary conditions:

$$s(\kappa,0) = 0, \quad c(\kappa,0) = 1. \qquad (16)$$

Using the relations

$$s(\kappa,r) = \kappa C_0^2(\eta)\, sn(\kappa,r), \quad c(\kappa,r) \equiv cs(\kappa,r),$$

where sn and cs are the phase functions of ref.[30], we get that the form U of (15) multiplied by the norm factor

$$N(\kappa) = \cos\delta(\kappa)/c(\kappa,\infty), \qquad (17)$$

where the phase shift δ is defined by

$$\tan\delta(\kappa) = s(\kappa,\infty)/c(\kappa,\infty) \qquad (18)$$

provides the pp-function u satisfying eq. (2).

As it will be clear below, the knowledge of the first three terms of low-energy expansions of the function s for $r < r_c$ and the function c for any r is quite sufficient to obtain eq. (13). To find these terms, we use the method that is more similar to the method of ref.[19], and based on the Bessel-Clifford expansions [18] for the Coulomb functions. The above expansions may be written as

$$F_0(\kappa,r) = \kappa C_0(\eta)\Big(\sum_{n=0}^{2} \kappa^{2n} f_n(r) + \Delta f(\kappa,r)\Big), \qquad (19)$$

$$G_0(\kappa,r) = C_0^{-1}(\eta)\left(\sum_{n=0}^{2}\kappa^{2n}g_n(r) + \Delta g(\kappa,r)\right) \qquad (19)$$

The residual terms Δf and Δg of (19) of the order κ^6 if of course $r<r_c$. The Bessel-Clifford functions f_n and g_n with $n=0,1,2$ may be easily found explicitly (for instance, by using the results of ref.[31]) and read as

$$f_n(r) = R^{-n}(xR/2)^{3n+1}\{I_{n+1}(\delta_{n0}-\delta_{n1}/3)+(2I_4/5x+I_5/18)\delta_{n2}\},$$
$$g_n(r) = (2/R^{n+1})(xR/2)^{3n+1}\{K_{n+1}(\delta_{n0}+\delta_{n1}/3)-(2K_4/5x-K_5/18)\delta_{n2}\}, \qquad (20)$$

where I_n and K_n with $n=1,2,$ are the regular and irregular modified Bessel functions of $x=2\sqrt{r/R}$. Let us look for the solutions of problem (14-16) as

$$s(\kappa,r) = \kappa C_0^2(\eta)\left(\sum_{n=0}^{2}\kappa^{2n}s_n(r) + \Delta s(\kappa,r)\right) , \qquad (21)$$

$$c(\kappa,r) = \sum_{n=0}^{2}\kappa^{2n}c_n(r) + \Delta c(\kappa,r) . \qquad (22)$$

Inserting forms (19), (21) and (22) into (14) and (15) we obtain three sets ($n=0,1,2$) of equations:

$$\partial_r s_n(r) = -V(r)\sum_{\ell+m=n}U_\ell(r)f_m(r) , \qquad (23)$$

$$\partial_r c_n(r) = V(r)\sum_{\ell+m=n}U_\ell(r)g_m(r)$$

with the functions

$$U_n(r) = \sum_{\ell+m=n}(c_\ell(r)f_m(r) + s_\ell(r)g_m(r)) \qquad (24)$$

and boundary conditions

$$s_n(0)=0 , \quad c_n(0)=1 \qquad (25)$$

generated by eqs. (16) and ansatz (21) and (22).

One detail is to be stressed: when we introduced representations (21) and (22) and derived eqs. (23-25), we assumed that the residual terms Δs and Δc are of the order κ^6. The fact that this assumption is correct for $r<r_c$ follows immediately from the equations for Δs and Δc which can be got along with eqs. (23). Since the system of equations for Δs and Δc is more long and is not used below, we do not write it.

Clearly, expansions (19), (21) and (22) generate the expansion for the unnormalized pp-function (15) in the form

$$U(\kappa,r) = \kappa C_0(\eta)\left(\sum_{n=0}^{2}\kappa^{2n}U_n(r) + \Delta U(\kappa,r)\right) , \qquad (26)$$

where the functions U_n with $n=0,1,2$ are defined by eqs. (24).

As is known[18], in the region $r<r_c$ the expansions (19) rapidly converge with $\kappa\to 0$, therefore expansions (21), (22) and (26) are quite suitable for $r<r_c$ and when $r<r_c$ their residual terms are of the order κ^6 by construction. Now we must derive the low-energy expansion of norm factor (17). To make this, we study expansions (21) and (22) for $r\geq r_0$, where r_0 is assumed to be a finite and arbitrary radius satisfying the condition $2\sqrt{r_0/R}\gg 1$. Also we assume that the energy is sufficiently small, namely, such that $r_c>r_s$. Due to both the assumptions, we can use for $r\geq r_0$ the asymptotics forms [18] for functions (20) and asymptotic form (4) for the potential V. Substituting these forms into (23) and (24) and changing the variable r by $x=2\sqrt{r/R}$, we found the solutions of eqs.(23) for $r\geq r_0$ as series of elementary functions. In particular we established that due to the long-range behaviour of polarization potential (4) the functions s_n with $n=0,1,2$ diverge like

$$s_n(r) = c_0(\infty)(8\alpha/\pi R^2)(R/2)^{2n}\, x^{3n-6}\exp(2x) \qquad (27)$$

as $r\to\infty$, while the functions c_n with $n=0,1,2$ are finite at $r=\infty$ and satisfy the relations:

$$c_n(r)/c_n(\infty) - 1 = 32\alpha/5R^3x^5 + O(x^{-10}) , \qquad (28)$$

where $r\geq r_0$. As follows from eqs. (27) and (28) representation (21) loses its meaning when $r\to\infty$, while representation (22) is well-defined for any r and therefore $\Delta c(\kappa,r) = O(\kappa^4)$ also for any r. We proved the last statement actually by the way more similar to the one used in ref.[32] devoted to investigation of the low-energy behaviour of the phase shift caused by long-range potentials in the absence of the Coulomb one.

Now, using eqs. (27) and (28) we get that the function

$$a(r_0) \equiv -s_0(r_0)/c_0(r_0) \sim -(\alpha R/8\pi r_0^3)\exp(4\sqrt{r_0/R}) \qquad (29)$$

diverge as $r_0\to\infty$, while the function

$$r_e(r_0) \equiv 2R/3 + 2\left(c_1(r_0) + s_1(r_0)/a(r_0)\right)/s_0(r_0) \qquad (30)$$

has a finite limit $2R/3$ at $r_0=\infty$.

Result (29) is well-known. After report [33] it has been discussed by many authors[6,28-30,34-36]. To explain these statements, let us put the potential V to be identically zero for $r>r_0$, where r_0 is a finite radius. For this potential, owing to eqs. (14) and (21-23) the functions s, c as well as the functions s_n, c_n with $n=0,1,2$ and $\Delta s, \Delta c$ are equal to their corresponding values at $r=r_0$. Hence, we may replace $s(\kappa,\infty)$ and $c(\kappa,\infty)$ in eq. (18) by their expansions (21) and (22) written at $r=r_0$. Thus, we get the expansion of $\tan\delta$ for the potential $V(r)\theta(r_0-r)$ which satisfies eqs.(3). Hence, we may compare this expansion with eq.(9). As a result, we get that: $a(r_0)$ of (29) is the scattering length (denoted by $\alpha(0,r_0)$ in ref./34/ and plotted in Fig.5 of ref./30/), r_0 of (30) is the effective radius for our truncated potential. Due to eq. (29) $a(r_0)$ diverges as $r_0\to\infty$. Hence, the definition of the scattering length as the low-energy limit of the left-hand side of eq.(9) divided by $-\kappa C_0^2$ loses its meaning for the potential V with asymptotic (4). This fact is known/24/ and we only reproved it within eqs. (14) by the way still more similar to the one used in refs./30,34/. Surprisingly, $r_e(\infty)$ of (30) is finite and therefore the low-energy expansion of $\tan\delta$ contains the term behaving like $\kappa^3 C_0^2$ as $\kappa\to 0$, i.e. the effective radius defined as a limit of eq.(30) as $r_0\to\infty$ has the meaning. This definition is quite different from that one used in work/35/. In this work $\tan\delta$ of the left-hand side of eq. (9) was replaced by the integral of eq.(7) and the effective radius was actually considered as a low-energy limit of the second term of the asymptotic ($\kappa\to 0$) of this integral divided by the factor $\kappa^3 C_0^2$. Since the asymptotic expansion of the integral of eq.(9) does not contain/24/ the terms with $\kappa^n C_0^2$ threshold behaviour, the authors of work /35/ have found that the standard definition of the effective radius loses its meaning for the potentials V with asymptotic (4).

For the first time, the receipt of construction of the finite scattering length and effective radius for the potential with tail (4) in the presence of the Coulomb field was given in ref./24/. Recently, another definition of the scattering length having a physical meaning for the above potential has been introduced in ref./30/.

To prove eq.(13), we do not need redefinition of any function. Really, according to eqs. (7) and (18) the ratio $s(\kappa,r)/c(\kappa,r)$ has a finite limit at $r=\infty$, therefore $\cos\delta(\kappa)$ of (17) is a well-defined factor. Moreover, we have shown that the denominator of fraction (17) has a well-defined expansion

$$c(\kappa,\infty)=\sum_{n=0}^{2}\kappa^{2n}c_n(\infty)+O(\kappa^4) \qquad (31)$$

and due to eqs. (28) written at $r=r_c$, the contributions to the constants $c_n(\infty)$ ($n=0,1,2$) from the tail ($r\geq r_c$) of potential (4) behave like κ^5 as $\kappa\to 0$. By using eq. (31) and the fact that $\cos\delta=1+O(\kappa^{10})$, following from eq. (8), we obtain for norm factor (17) the low-energy expansion:

$$N(\kappa)=c_0^{-1}(\infty)-\kappa^2 c_1(\infty)/c_0^2(\infty)+O(\kappa^4) \qquad (32)$$

Further, multiplying $N(\kappa)$ of (32) by U of (26) we have the required result (13) with

$$u_0(r)=U_0(r)/c_0(\infty), \qquad (33)$$
$$u_1(r)=(U_1(r)-c_1(\infty)u_0(r))/c_0(\infty),$$

where U_0 and U_1 are defined by eqs. (24) and may be easily evaluated after solving problem (23-25), which has no special numerical difficulties. To complete the analysis of expansion (13), we stress that its residual term is of the order κ^4, by construction, and also we represent the asymptotics ($x=2\sqrt{r/R}\gg 1$) form of functions (33):

$$u_n(r)=(R^{2n+1}x^{3n}/(1+23n))\sqrt{x/8\pi}\cdot \exp(x)(1+O(x^{-1})), \quad n=0,1 \qquad (34)$$

obtained with the help of eqs. (26-28).

4.2. Low-energy representation of the pp-function for $r\geq r_c$

As a second step, we derive the low-energy representation of the function u for $r\geq r_c\geq r_s$. To do this we rewrite problem (14-16) in the equivalent integral form:

$$s(\kappa,r)=s^{(0)}_{(\kappa)}-\kappa^{-1}\int_{r_c}^{r}V(t)U(\kappa,t)F_0(\kappa,t)dt, \qquad (35)$$
$$c(\kappa,r)=c^{(0)}_{(\kappa)}+\kappa^{-1}\int_{r_c}^{r}V(t)U(\kappa,t)G_0(\kappa,t)dt,$$

where the values of phase functions s and c at $r=r_c$ are denoted by $s^{(0)}$ and $c^{(0)}$ and may be evaluated by using expansions (21) and (22). To analyse the solutions of eqs. (35), which are the Volterra-type integral equations/37/, we use the usual iteration method/38/. We put $s^{(0)}$ and $c^{(0)}$ to be zero approximations and to obtain the results $s^{(m+1)}$ and $c^{(m+1)}$ of $(m+1)^{th}$-iteration, we shall substitute the results $s^{(m)}$ and $c^{(m)}$ of m^{th}-iteration into the right-hand side of eqs. (35). The series of these iterations converge uniformly to the exact solution of system (35), if all its integral operators are contracting mappings/37,38/. In the usual way/38/, i.e. using the midpoint theorem, one can show that this condition is fulfilled, if

$$\varepsilon(\kappa)\equiv\max_{n=1,2,3}\rho_{r_c}(B_n,0)<1/2, \qquad (36)$$

where

$$B_n(\kappa,r_c,r)\equiv\kappa^{-1}\int_{r_c}^{r}V(t)(F_0(\kappa,t)G_0(\kappa,t)\delta_{n1}-$$

$$F_0^2(\kappa,t)\delta_{n2} + G_0^2(\kappa,t)\delta_{n3})\,dt \tag{37}$$

and the metrix on the $C^1_{[r_c,\infty]}$-class of functions, depending on κ parametrically, is defined as /37/

$$\rho_{r_c}(A,B) \equiv \max_{r\geqslant r_c} |A(\kappa,r)-B(\kappa,r)| \quad . \tag{38}$$

Let us show that ineq. (36) is valid for sufficiently small κ. Using the bound /39/

$$|F_0(\kappa,r)\,(F_0(\kappa,t)+i\,G_0(\kappa,t))| \leqslant \kappa\sqrt{2\pi r t}, \tag{39}$$

which is valid for any κ, r and t we obtain from eqs. (37) with $n=1,2$ that

$$\rho_{r_c}(B_n,0) \leqslant \sqrt{\pi/2}\,(\alpha/R r_c^2) \quad . \tag{40}$$

In the region $r\geqslant r_c$ the Coulomb functions are not greater in order of magnitude than their values at $r=r_c$ /18/, therefore

$$\rho_{r_c}(F_0,0) = O(\eta^{1/6}) \ , \ \rho_{r_c}(G_0,0) = O(\eta^{1/6}) \ . \tag{41}$$

By applying the second eq. (41) to eq. (37) with $n=3$ we find that

$$\rho_{r_c}(B_3,0) = O(\eta^{1/3})\,(\alpha/3\kappa R r_c^3) \quad . \tag{42}$$

Thus, due to bounds (40) and (42) we may replace condition (36) by more rough condition

$$\varepsilon(\kappa) \leqslant \sqrt{\pi/2}\,(\alpha/R r_c^2) < 1/2, \tag{43}$$

which is valid, if κ satisfies the inequality

$$\kappa < (\sqrt{2\pi}\,\alpha R)^{-1/4} \tag{44}$$

that we assume for all the subsequent constructions. The first of them is a result of the first iteration of eqs. (35), which read as

$$s^{(1)}_{(\kappa,r)} = s^{(0)}_{(\kappa)}\,(1 - B_1(\kappa,r_c,r)) + c^{(0)}_{(\kappa)}\,B_2(\kappa,r_c,r) \ , \tag{45}$$

$$c^{(1)}_{(\kappa,r)} = c^{(0)}_{(\kappa)}\,(1 + B_1(\kappa,r_c,r)) + s^{(0)}_{(\kappa)}\,B_3(\kappa,r_c,r) \ . \tag{46}$$

Continuing the iteration procedure we find by induction that

$$\max\{\rho_{r_c}(s,s^{(m)}),\ \rho_{r_c}(c,c^{(m)})\} \leqslant d^{(m)}_{(\kappa)}, \tag{47}$$

where

$$d^{(m)}_{(\kappa)} \equiv (\varepsilon^m/(1-\varepsilon))\max\{\rho_{r_c}(s^{(1)},s^{(0)}),\ \rho_{r_c}(c^{(1)},c^{(0)})\} \ . \tag{48}$$

To estimate $d^{(m)}_{(\kappa)}$, we come back to eqs. (14-16) and introduce a finite matching radius r_0 assumed to be sufficiently large ($r_0 \geqslant r_s$, $2\sqrt{r_0/R} \gg 1$) to use, for $r\geqslant r_0$, forms (27) and (28). Then, due to eqs. (22) and (28) we have

$$c(\kappa,r) = \sum_{n=0}^{1} \kappa^{2n} c_n(\infty)\,(1 + O(r^{-5/2})) + O(\kappa^4) \quad , \tag{49}$$

where, of course $r\geqslant r_0$. Inserting form (49) into the first equation of set (14) we get an equation only for $s(\kappa,r)$. Solving this equation for $r\geqslant r_0$ we find

$$s^{(0)}_{(\kappa)} = s(\kappa,r_0) - \sum_{n=0}^{1}\kappa^{2n}c_n(\infty)\exp(B_1(\kappa,r_0,r_c)) \cdot \int_{r_0}^{r_c}\partial_t B_2(\kappa,r_0,t)\exp(-B_1(\kappa,r_0,t))\,(1+O(t^{-5/2})+O(\kappa^4))\,dt\,, \tag{50}$$

where B_n ($n=1,2$) are defined by eqs. (37). Due to eqs. (4) and (19) integrals $\pm B_1$ of (50) have a finite nonzero limits as $\kappa\to 0$ and therefore to find the threshold behaviour of $s^{(0)}_{(\kappa)}$ we may replace in eq. (50) the exponential functions by the constants. After this substitution we get in the right-hand side of eq. (50) the integral $B_2(\kappa,r_0,r_c)$ which behaves /24/ like κC_0^2 as $\kappa\to 0$. Next, owing to eq. (21) we have $s(\kappa,r_0) = O(\kappa C_0^2)$. Hence, from eqs. (22) and (50) it follows that

$$s^{(0)}_{(\kappa)} = c^{(0)}_{(\kappa)}\,O(\kappa C_0^2(\eta))\,(1+O(\kappa^2)) \quad . \tag{51}$$

Now, by using ineq. (36) and eq. (51) we get from eqs. (45) and (46) that both metrics of (48) are of the order $|c^{(0)}_{(\kappa)}(\varepsilon(\kappa)+O(\kappa C_0^2(\eta)))|$, hence

$$d^{(m)}_{(\kappa)} = |c^{(0)}\varepsilon^m\,(\varepsilon + O(\kappa C_0^2))/(1-\varepsilon)| \quad . \tag{52}$$

Thus, due to eq.(47) the functions $s^{(m)}$ and $c^{(m)}$ reproduce the exact solutions s and c of eqs.(35) within $d^{(m)}$-accuracy and $d^{(m)}_{(\kappa)}$ vanish by law (52) as $\kappa\to 0$. Using these results and eqs. (15) and (17) we obtain the low-energy representation of the pp-function, corresponding to m^{th}-iteration of system (35). This representation reads as

$$u(\kappa,r) = u^{(m)}_{(\kappa,r)} + \Delta u^{(m)}_{(\kappa,r)} = N^{(m)}_{(\kappa)}\,(c^{(m)}_{(\kappa,r)}F_0(\kappa,r) + s^{(m)}_{(\kappa,r)}G_0(\kappa,r)) + \Delta u^{(m)}_{(\kappa,r)} \ , \tag{53}$$

where $r\geqslant r_c \gg r_s$, the momentum κ satisfies ineq. (44), the norm factor $N^{(m)}$ is defined by eqs. (17) and (18) in which s and c are replaced by $s^{(m)}$ and $c^{(m)}$, respectively, and the residual term satisfies the relation

$$\rho_{r_c}(\Delta u^{(m)},0) = O(\eta^{1/6}d^{(m)}_{(\kappa)}) \tag{54}$$

obtained with the help of eqs. (41) and (47).

Before to go further, we make one useful remark. Of course, if $V=V_s$, then in the region $r \geq r_c > r_s$ the pp-function is represented by eq. (2) and therefore in this case formulae (53) and (54) may be not used. However, if $V=V_s+V_p$ and $r \geq r_c > r_s$, one has to use the approximation $u \approx u^{(m)}$ with $u^{(m)}$ of (53) or another equivalent approximation, because in this case the low bound $r_{min}(\kappa)$ of the region $r \geq r_{min}(\kappa)$, where u is close to its asymptotics form (2), is an order of magnitude of several r_c and depends on the energy. A detailed discussion and a numerical proof of this fact are given in refs. [29,30].

4.3. Threshold behaviour of the functions Λ_p^2, C and C_t

Owing to eqs. (2) and (34), the functions u and u_n ($n=0,1$) have quite different asymptotics. Hence, in the region $r \geq r_c$ the function u cannot be approximated by a finite sum contained in the right-hand side of eq.(13). However, representation (13) is quite suitable for evaluation of the integrand $u\vartheta$ of eq.(1), for two apperant reasons [37]. First, the contribution $\varkappa(\kappa)$ from region $r \geq r_c$ to Λ_p of (1) is negligible as $\kappa \to 0$. Really, using for $r \geq r_c > r_s$ the formulae $\vartheta(r) \approx \exp(-\gamma r)$, representation (53) at $m=1$, eqs.(45), (46) and rough bounds (41), we find that

$$\varkappa(\kappa) = O(\eta^{1/6}\kappa^{-1}C_0^{-1}(\eta)\exp(-\gamma r_c)) = O(\kappa^{-2/3}\exp((\pi/2-\gamma/\kappa)/\kappa R)). \quad (55)$$

Second, owing to eqs. (34), the products $u_n(r)\vartheta(r)$ with $n=0,1$ decay exponentially as $r \to \infty$. Hence they are functions integrated on the interval $(0, \infty)$.

For the above reasons we may insert u of (13) into (1). Thus we prove the first required result (11) and find the constants:

$$\Lambda_p(0) = (\gamma^3/8\pi)^{1/2} \int_0^\infty u_0(r)\vartheta(r)dr, \qquad (56)$$
$$A_p = \Lambda_p^{-1}(0)(\gamma^3/2\pi)^{1/2} \cdot \int_0^\infty u_1(r)\vartheta(r)dr.$$

Next, combining eq. (5) with eq.(11) we establish the low-energy asymptotics of contribution (10) in the form

$$C(E) = (\Lambda_p(0)/\Lambda(0))^2(1+(A_p-A)E+O(E^2)) - 1. \qquad (57)$$

Let us study $C(E)$. Let r_0 be an arbitrary radius, such that $r_0 \geq r_p$ and $x_0 = 2\sqrt{r_0/R} \gg 1$. Then, owing to eqs.(28) the contributions from the part ($r \geq r_0$) of potential (4) to the constants $c_0(\infty)$ and $c_1(\infty)$ are of the order x_0^{-5}. Hence, in eq.(33) these constants may be replaced by $c_0(x_0)$ and $c_1(x_0)$ within x_0^{-5} accuracy.

Further, the contributions from the region $r \geq r_0$ to integrals (56) are exponentially small ($\sim \exp(x_0 - \gamma r_0)$). Clearly, for the above reasons contribution (57) from potential (4) to the factor S_{11} is mainly caused by a short-range ($r_p \leq r \leq r_p + R$) part of this potential. This conclusion agrees with one of the main results of refs. [9-11]. Now, let us assume potential (4) to be absent. Then one can step by step repeat all the constructions of subsection 4.1 and derive eq.(13) with u_0 and u_1 which may be evaluated after solving problem (23-25) with $V=V_s$. Thus, one can get eq.(5) with $\Lambda(0)$ and A represented by corresponding integrals (56). Note, that in both the cases ($V=V_s+V_p$ or $V=V_s$) the function u_0, owing to eq.(13), is a limit of $u/\kappa C_0$ as $\kappa \to 0$ and is a solution of the Schrödinger equation ($\hbar = c = m = 1$):

$$Hu(\kappa,r) = (\partial_r^2 + \kappa^2 + 1/rR - V(r))u(\kappa,r) = 0 \qquad (58)$$

at $\kappa = 0$. The polarization potential is attractive, hence $V_p + V_s \geq V_s$ for any r. Using this inequality and applying the well-known theorem (see Sec. 6 of handbook [40]) to eqs. (23) with $n=0$ and to eq. (58) with $\kappa=0$ we find that for any r the function u_0 corresponding to $V=V_s+V_p$ is greater than the function u_0 corresponding to $V=V_s$. Therefore, from eqs. (56) we get $\Lambda_p(0) > \Lambda(0)$ and hence, owing to eq.(57), $C(0)>0$. Thus, in the low-energy limit contribution (10) behaves like the linear function (57) of energy and has a non-zero limit at $E=0$. These results agree with the WKB-prediction represented in Fig. 6 of ref. [6] and disagree with the result (14) of ref. [8] according to which contribution (10) has to vanish like E^5 as $E \to 0$. As is noted [9], the work [8] contains an error leading to the E^5 threshold behaviour of C. When this error is taken into account, one find again that $C(0)>0$. Of course, for the reason that $r_c \to \infty$ as $\kappa \to 0$ the part C_t of C associated with the tail ($r \geq r_c$) of potential (4) must vanish as $\kappa \to 0$.

Let us derive the formula for C_t by subsequently representing every functions $F = \delta, c, N, u, \Lambda_p, C$ as a sum $F = F^{(0)} + F_t$ such that $F = F^{(0)}$ if potential (4) is truncated at $r = r_c$. Along this way we represent phase-shift (18) as

$$\delta = \delta^{(0)} + \delta_t = \operatorname{atan}(s^{(0)}/c^{(0)}) + \delta_t. \qquad (59)$$

Then norm factor (17) becomes

$$N = N^{(0)} + N_t = \cos\delta^{(0)}/c^{(0)} + (\cos\delta/(c^{(0)} + c_t) - N^{(0)}), \qquad (60)$$

where $c_t(\kappa) = c(\kappa,\infty) - c^{(0)}(\kappa)$. Decomposition (60) and the identity $u = NU$ with U of (15) generate the representation

$$u = u^{(0)} + u_t =$$

$$N^{(0)}N^{-1}u\,\theta(r_c - r) + (N_t N^{-1} u\,\theta(r_c - r) + u\,\theta(r - r_c)) \quad , \tag{61}$$

where the third term may be approximated by $u^{(m)}$ of (53). Inserting u of (61) into (1) we have

$$\Lambda_p = \Lambda_p^{(0)} + \Lambda_{pt} = N^{(0)}N^{-1}\Lambda_p + (N_t N^{-1}\Lambda_p + \varkappa) \quad , \tag{62}$$

where $\varkappa$ is given by eq.(55). By substituting Λ_p of (62) into (10) we find the decomposition $C = C^{(0)} + C_t$, where

$$C^{(0)} \equiv (\Lambda_p^{(0)}/\Lambda)^2 - 1 = (1 + N_t/N^{(0)})^{-2}(\Lambda_p/\Lambda)^2 - 1 \tag{63}$$

is the contribution from the part $(r < r_c)$ of potential (4) to the factor S_{11} and

$$C_t = (1 - (1 + N_t/N^{(0)})^{-2})(\Lambda_p/\Lambda)^2 + O(\varkappa) \tag{64}$$

is a part of total contribution (10) associated with the tail $(\ r \geqslant r_c\ )$ of this potential.

Now we describe the threshold behaviour of all the terms of eqs. (59-64). According to eq.(51) $\delta^{(0)}(\kappa)$ of (59) behaves like κC_0^2 . Using eqs. (8), (28) and (31) we find that $N^{(0)}$ of (60) behaves like a linear function of energy and $N^{(0)}(0) \neq 0$. As follows from eqs. (46) and (47) with $m = 1$, the part c_t of c satisfies the condition $|c_t(\kappa)| \leqslant d^{(1)}(\kappa)$ with $d^{(1)}(\kappa)$ of (52), therefore N_t of (60) is of the order of $d^{(1)}$. Applying the results obtained for $N^{(0)}$ and N_t to eqs. (62-64) we find that $\Lambda_p^{(0)}$ and $C^{(0)}$ behave like linear functions of energy and have nonzero limits at $\kappa = 0$, while Λ_{pt} and C_t are of the order of $d^{(1)} + \varkappa$ and hence vanish as $\kappa \to 0$ not slowly than $O(\kappa^4)$ which follows from eqs. (43) and (52).

The leading terms of the asymptotics of the functions δ_t , c_t , N_t , Λ_{pt} and C_t cannot be found without inspection of the second iteration of eqs. (35). After dropping the terms associated with $s^{(0)}$ of (51) and therefore behaving like κC_0^2 this iteration yields:

$$\begin{aligned} s^{(2)}(\kappa,\infty) &= c^{(0)}(\kappa)\,(B_2(\kappa,r_c,\infty) + B_{21}(\kappa,r_c,\infty) - B_{12}(\kappa,r_c,\infty)), \\ c^{(2)}(\kappa,\infty) &= c^{(0)}(\kappa)\,(1 + B_1(\kappa,r_c,\infty) + 2^{-1}B_1^2(\kappa,r_c,\infty) + B_{23}(\kappa,r_c,\infty)), \end{aligned} \tag{65}$$

where we have introduced the integrals

$$B_{nm}(\kappa,r_c,r) = \int_{r_c}^{r} B_n(\kappa,r_c,t)\,\partial_t B_m(\kappa,r_c,t)\,dt \tag{66}$$

with $nm = 12, 21, 23$. Using definitions (37) and bounds (40-42) we immediately find that the last two terms of both eqs. (65) and integrals (66) vanish not slowly than $B_2^2(\kappa,r_c,\infty)$ as $\kappa \to 0$.

Continuing the analysis along this way, we prove the important statement: for any $m \geqslant 1$ the contributions from $(\ m+1\ )$-th interation to $s(\kappa,\infty)$ and $c(\kappa,\infty)$ vanish like or rapidly than the squared contributions from m^{th}-iteration as $\kappa \to 0$. According to this fact and eqs. (45), (46), and (51) we may write eqs. (65) in the form

$$s(\kappa,\infty) = s_t(\kappa) + O(\kappa C_0^2) = c^{(0)}(\kappa)\,(B_2(\kappa,r_c,\infty) + O(B_2^2)) \quad , \tag{67}$$

$$c(\kappa,\infty) = c^{(0)}(\kappa)\,(1 + B_1(\kappa,r_c,\infty) + O(B_2^2)) \quad , \tag{68}$$

where

$$s_t(\kappa) = s(\kappa,\infty) - s^{(0)}(\kappa) \ .$$

Since $c^{(0)}(0) = c(0,\infty) \neq 0$ and by definition (37) integrals B_1 and B_2 of (67-68) vanish at $\kappa = 0$, therefore the leading terms of low-energy asymptotics of the functions $s_t(\kappa)$ and $c_t(\kappa)$ are proportional to the leading terms of asymptotics of the integrals B_2 and B_1, respectively. The asymptotics of all integrals (37) as $\kappa \to 0$ may be found by the method described in detail in ref.[24], as well as by application to these integrals of the standard stationary phase method [41]. For these reasons we give only the final results:

$$B_1(\kappa,r_c,\infty) = ((\alpha/9)/2^{5/3}R^3)\,\Gamma(1/3)(\kappa R)^{16/3}(1 + O(\eta^{1/3})) \quad , \tag{69}$$

$$B_n(\kappa,r_c,\infty) = (-1)^{n+1}(16\alpha\kappa^5/15R^2)(1 + O(\eta^{1/3})) \quad , \tag{70}$$

where, of course, $\kappa \to 0$, Γ is the Gamma-function and $n = 2,3$. According to eqs. (69) and (70) all the integrals of eqs. (65-68) vanish more slowly than $O(\kappa C_0^2)$ as $\kappa \to 0$, namely, for this reason we cancelled all the terms of an order of κC_0^2 when we have derived eqs. (65) and (67)-(68).

Now, inserting froms (67) and (68) into (18) and using eq.(70) with $n = 2$ we recover results (7) and (8) proved in ref.[24]. Next, $c_t \approx c^{(0)}B_1$, owing to eq.(68), therefore when $\kappa \to 0$, N_t of (60) reads as

$$N_t = (-1/c^{(0)})(B_1 + \delta_t^2 + O(B_2^2)) \ . \tag{71}$$

By comparison of eq.(8) with eq.(69) we find that B_1 vanishes more rapidly than δ_t but more slowly than δ_t^2. Hence, the functions N_t of (71) and Λ_{pt} of (62) behave like $(\kappa R)^{16/3}$ as $\kappa \to 0$ and inserting N_t of (71) into (64) we obtain the representation

$$C_t = 2(\Lambda_p/\Lambda)^2(B_1 + O(B_2^2)) \quad . \tag{72}$$

Using eqs.(5), (11) and (69) we get from eq.(72) the leading term (12) of asymptotic of C_t . Thus, the second required result is proved.

For comparison, let us replace the polarization potential by any short-range potential $\Delta V_s \sim \exp(-\mu r)$ satisfying eqs.(3), for instance, by the vacuum polarization potential [42] or by the electron-screening one [16]. Then, repeating step by step all the constructions, described above, we find that the contribution from the tail $(r \geq r_c)$ of ΔV_s to the factor S_{11} falls not slowly than $O(\kappa^4 exp(-\mu r_c))$ as $\kappa \to 0$, i.e. more rapidly than C_t of (12). Hence, eq.(12) describes one special effect caused by the r^{-4} long-range behaviour of potential (4). However, in contrast to the pp-scattering this effect is not dominant in the pp-reaction.

5. Perturbation theory for the polarization potential

In subsection 4.2 we have actually constructed a perturbation theory over the tail $(r \geq r_c > r_p)$ of potential (4). by construction, m^{th}-order of this theory is generated by m^{th}-iteration of eqs.(35). Now we generalize our perturbation theory when total potential (4) is a correction to the Coulomb potential. This generalization on the basis of iterations of eqs.(35) seems to be clear and therefore we describe only the most essential details.

For the reasons to be clear below we formally replace r_c by r_p in all the formulae of subsection 4.2. Then $s^{(0)}$ and $c^{(0)}$ stand for s and c at $r=r_p$ and in the framework of m^{th}-iteration of eqs.(35) the pp-function is represented by eq.(53) for any $r \geq r_p$. Of course, this representation is mathematically correct if and only if momentum κ and the parameters of potential (4) satisfy a possible condition that provides uniform convergence of iterations to eqs.(35) in region $r \geq r_p$. When $r_p \geq r_c$ one can use conditions (43) or (44). For sufficiently small κ, of course, $r_p < r_c$ and condition (43) with r_c replaced by r_p loses the meaning, because when we derive it, we have used relations (41) which after replacement of r_c by r_p become incorrect if $r_c > r_p$. So, for the more general case, $r_p < r_c$, one has to study the convergence problem for iterations of eqs.(35) more thoroughly [37], namely to take into account the fact that the functions s and c belong to quite different classes $\{s\}$ and $\{c\}$. The $\{s\}$-class, owing to eqs.(21) and (51), is formed by the functions with the κC_0^2 dependence for $r < r_c$, while owing to eqs.(22) and (31) the $\{c\}$-class is formed by the functions having a nonzero limit as $\kappa \to 0$ for any r. In the interval $r_p \leq r \leq \infty$ the integral operators of eqs.(35) form a contracting mapping, if any functions $\delta s \in \{s\}_r$ and $\delta c \in \{c\}$ obey the inequality [38]:

$$max\{\rho_{r_p}(\int_{r_p}^{r}(\partial_t B_2(\kappa,r_p,t)\delta c(\kappa,t) - \partial_t B_1(\kappa,r_p,t)\delta s(\kappa,t))dt, 0),$$

$$\rho_{r_p}(\int_{r_p}^{r}(\partial_t B_1(\kappa,r_p,t)\delta c(\kappa,t) + \partial_t B_3(\kappa,r_p,t)\delta s(\kappa,t))dt, 0)\} < \qquad (73)$$
$$max\{\rho_{r_p}(\delta s(\kappa,r),0), \rho_{r_p}(\delta c(\kappa,r),0)\},$$

where the functions $B_n(\kappa,r_p,r)$ with $n=1,2,3$ and the metrics are defined by eqs.(37) and (38) in which r_c is replaced by r_p.

If κ is sufficiently small, then $|\delta c| > |\delta s|$ for any r, because $\delta s(\kappa,r) = O(\kappa C_0^2)$ when $r < r_c$ and $\delta s(\kappa,r) \sim \kappa^5$ when $r \geq r_c$, and when $\kappa \to 0$ due to eqs.(19), the integrals $B_n(\kappa,r_p,r)$ with $n=1,2$ converge, while the integral $B_3(\kappa,r_p,r)$ behaves like $(\kappa C_0^2)^{-1}$. Using these facts and applying the midpoint theorem [38] to integrals of (73) we may replace condition (73) by the set of more rough conditions:

$$\rho_{r_p}(B_n, 0) < 1/2 \quad , \quad n=1,2 \quad , \qquad (74)$$

$$\rho_{r_p}(\int_{r_p}^{r}\partial_t B_3(\kappa,r_p,t)\delta s(\kappa,t)dt, 0) < (1/2)\rho_{r_p}(\delta c(\kappa,r),0) \quad . \qquad (75)$$

When $\kappa \to 0$ and r is fixed, then: δc of (75) tends to a nonzero constant, the integral of (75), owing to eqs.(19), (37) and definition of $\{s\}$-class also tends to a nonzero constant if $r_p < r_c$ and, owing to eqs.(41), (68) and (70) vanishes like or rapidly than $(\kappa R)^{16/3}$ until $r_p \geq r_c$. Hence, if $r_p \geq r_c$ and κ is such that $\rho_{r_c}(B_3(\kappa,r_c,r),0) < 1/2$ then ineq.(75) is valid and, therefore when $r_p \geq r_c$ conditions (74) and (75) are reduced to those early used ineqs.(36) or (44). For the case $r_p < r_c$ under consideration let us assume that ineqs.(74) are valid. Then, according to required ineq.(75), the convergence problem is reduced [38] to proving the fact that contributions from the iteration terms associated with B_3 to $s^{(m+1)}$ and $c^{(m+1)}$ are smaller than the corresponding contributions from similar terms to $s^{(m)}$ and $c^{(m)}$. These contributions to the first iteration results (45) and (46) are obviously bounded by

$$w(\kappa) \equiv |\tan\delta^{(0)}(\kappa)\rho_{r_p}(B_3(\kappa,r_p,r),0)| \quad , \qquad (77)$$

where $\delta^{(0)}$ is defined by eq.(59) and is the phase shift caused by nuclear potential V_s, because now $s^{(0)}$ and $c^{(0)}$ are s and c at $r=r_p$ and by assumption $r_p \geq r_s$. Further, by induction one can show that the above contributions to $s^{(m+1)}$ and $c^{(m+1)}$ decrease with growing m if bound (77) is smaller than one-half and ineqs.(74) are valid. Thus, instead of condition (36), we have established one of the rough conditions

$$\varepsilon(\kappa) = max\{\rho_{r_p}(B_1,0), \rho_{r_p}(B_2,0), w\} < 1/2 \qquad (78)$$

that provides convergence of the iterations of eqs.(35) for any $r \geq r_p$ and any $r_p > 0$. Appling ineqs.(39) and inequality $|G_0| \leq C_0^{-1}$ [39] to integrands of (37) we find that ε of (78) satisfies the inequality

$$\varepsilon(\kappa) \leq \sigma \max\{3\sqrt{\pi/2}, |\tan\delta^{(0)}_{(\kappa)}/\kappa r_p C_0^2(\eta)|\} , \quad (79)$$

where $\sigma = \alpha/3r_p^2 R$ is a dimensionless combination of potential (4) parameters to be used below. Further, when $\tan\delta^{(0)}$ represented by eq.(9), ineq.(79) reads

$$\varepsilon(\kappa) \leq \sigma \max\{3\sqrt{\pi/2}, |\alpha/r_p + O(\kappa^2)|\} < 1/2 . \quad (80)$$

Obviously, the right-hand side of ineq.(80) is bounded to one-half if κ and σ are sufficiently small. Hence we have solved the third problem of Sec. 3.

Now, assuming that ineq.(78) is valid we repeat step by step all the construction of subsection 4.2. So, for $s^{(m)}$ and $c^{(m)}$ we have again ineqs. (47) and (48) with r_c replaced by r_p. Analogously, we find that both metrics in eq.(48) are bounded by the function $|c^{(0)}_{(\kappa)}(\varepsilon(\kappa) + O(\kappa C_0^2))|$ and therefore the function $d^{(m)}_{(\kappa)}$ determined the accuracy of m^{th}-iteration of eqs.(35) satisfies eq.(52) with $\varepsilon(\kappa)$ of (78). Hence $d^{(m)}_{(\kappa)} \to 0$ as $m \to \infty$. However, $\varepsilon(0) \neq 0$ (contrary to ε of (36)), hence $d^{(m)}_{(0)} \neq 0$ for any finite m. This important fact is caused by the following apparent reason: the integrals $B_n(\kappa, r_p, r)$, $n=1,3$ do not vanish as $\kappa \to 0$ and therefore the contributions from any iteration of eqs.(35) to the functions s and c are in general nonzero for any κ and r.

Let by definition $F^{(m)} \equiv F(s^{(m)}, c^{(m)})$ be a functional $F(s,c)$ of the m^{th}-order of the developed perturbation theory. Then, for $r < r_p$ the pp-function in m^{th}-order of this theory is $u^{(m)} = N^{(m)} U$ with U defined by eq.(15), where s and c are solutions of problem (14-16) in the region $r \leq r_p$, where $V \equiv V_s$. For $r \geq r_p$ the function $u^{(m)}$ is represented by the first term of eq.(53).

As a next step we replace u by $u^{(m)}$ in eq.(1) and we insert the obtained integral $\Lambda_p^{(m)}$ into (10) instead of Λ_p. Thus we get $\Lambda_p^{(m)}$ and $C^{(m)}$ i.e. Λ_p and C in the m^{th}-order of perturbation theory. Usually, the first order of any perturbation theory is called the Born approximation. Using eqs.(17), (18), (45),(46),(53) one can easily get the pp-function in the Born approximation, i.e. the function $u^{(1)}$. Comparing $u^{(1)}$ with the pp-function obtained in the Born approximation/9-11/ for the Volterra-type equation (see eqs. (5.27-5.31) of ref./10/) one may verify that our representation of $u^{(1)}$ is equivalent to the one used in these works. Therefore, if κ and σ satisfy ineq.(80), the assumption of refs./9-11/ is correct, i.e. the Born approximation may be used for evaluation of Λ_p(1) and C (10). However, using this approximation one has to keep in mind that it cannot provide exactly the leading term of asymptotics as $\kappa \to 0$ of C.

Due to this fact (as was explained above caused by the structure of integrals $B_n(\kappa, r_p, r)$ with $n=1,3$) the final and complete answer to the last question of Sec. 3 cannot be given within any m^{th}-order of the constructed perturbation theory. For this reason it is necessary to perform high-accuracy numerical investigations of Λ_p and Λ working beyond the scope of any perturbation theory, for instance, by the way described in Subsections 4.1 and 4.3.

6. Numerical results

Due to the facts that the polarization potential V_p of (4) is neglegible and long-range correction to the Coulomb potential numerical investigations of effects caused by V_p are no so simple as it may seem at first sight. Therefore it is useful to reconsider the most essential details of the way that has been used for numerical results reported below.

As input data we have used: the $^3S_1$-deuteron function v corresponding to the RSC-potential/43/, the same $^1S_0$-potential as V_s and potential (4) with $\alpha = 10^{-3}$ fm$^{3/4/}$ and r_p equal to its minimal possible value 4 fm that corresponds to the usually used/17/ value for the interaction radius of V_s. For these α and r_p, according to refs./6,7,9-11,14/ contribution (10) is of an order of 10^{-6}. Hence the numerical investigation of expansions (5) and (11) has to be performed with a high-accuracy. For this reason eqs.(14) and (23) were numerically integrated in the intervals $0 \leq r \leq 30 r_c$ and $0 \leq r \leq 100$ R, respectively. The upper bounds of these intervals were choosen sufficiently large to calculate norm factor (17), phase shift (18) and the coefficients of expansions (5), (11) and (32) with the nine-significant digits accuracy. Following ref./14/ we have used the $S_{5,3}$-spline interpolation/44/ to obtain the pp-function satisfying eq.(58) with the accuracy

$$|u^{-1}(\kappa,r) H u(\kappa,r)| < 10^{-10}$$

for any κ and $r \leq r_\Lambda$. The practical upper limit r_Λ of integrals (1) and (56) was 500 fm, which ensured the calculation of these integrals with a relative accuracy 10^{-9}. For a high-accuracy calculation of the coefficients of eqs. (5) and (11) we have integrated eqs.(23) and (56) with $V = V_s$ and $V = V_s + V_p$, respectively. Further, to find the applicability range of expansions (5), (11) and (57), we have compared their parts linear in energy with the corresponding functions Λ^2, Λ_p^2 and C calculated by integration of eqs.(1) and (14).

By the way described above we got the following results: the coefficients of eq.(5) are $\Lambda^2(0)$ =6.96072905, A =2.42552113 MeV^{-1} and are close to $\Lambda^2(0)$=6.934 and A = 2.5 MeV^{-1} calculated in ref./45/ for the same RSC-potential as well as to the corresponding coefficients of eq.(11): $\Lambda_p^2(0)$ =6.96074375, A_p =2.42552961 MeV^{-1}. Therefore the constants of eq.(57) are $C(0)$ =2.1108 $\cdot 10^{-6}$, $A_p - A$ = =8.48 $\cdot 10^{-6}$ MeV^{-1} and are neglegible as compared with known uncertainties of $\Lambda^2(0)$ and A of eqs.(5) and (6). Further, using the coefficients given above we have found that the functions Λ^2, Λ_p^2 and C are linear in energy within 1.6%-accuracy if $E \leqslant 20$ keV and within 0.5%-accuracy if $E \leqslant 10$ keV.

For the used α and r_p of eq.(4) we have established that: the inequality $r_c \geqslant r_s$ assumed in Subsection 4.2 and ineq.(44) are simultaneously valid if E $\leqslant$ 360 keV, the constant σ of (79) is equal to 7.23 $\cdot 10^7$ and for the used nuclear potential $\varepsilon(\kappa)$ of (79) is smaller than 10^{-6} for E $\leqslant$ 1 MeV. These results testify to the fact that for a low astrophysical energy perturbation theories of subsection 4.2 and Sec. 5 are correct and the Born approximation over total potential (4) is undoubtedly good.

For completeness we have established that contribution (10) is positive for any energy, has a broad maximum ($\max_E C(E) \approx 1.6\, C(0)$), achieves its upper bound ($\approx 3 \cdot 10^{-6}$) at E $\approx$ 400 keV and slowly vanishes as $E \to \infty$.

7. Summary and conclusion

So, we have proved the low-energy representations (11) and (12), we have found rough conditions (78-80) that provide the applicability of the Born approximation for a sufficiently low energy and by high-accuracy calculations we have confirmed the conclusion common for all previous works/6-11,14/ that the correction of the proton polarizability effect to the factor S_{11} is small as compared with other known corrections.

In conclusion, we stress that the method of Sects. 3 and 5 is quite suitable for analytical and numerical investigations of low-energy representations of the scattering functions for any two particles interacting via the sum of repulsive Coulomb potential, short-range potential and a long-range potential vanishing as $r \to \infty$ rapidly than a polarization one. Knowledge of these expansions is necessary for analysis of the threshold behaviour of the S-factors for any inelastic reactions when two complex and charged opposite in sign, particles of the input channel are considered as point-like and the effective two-body interaction is asymptotically represented as a

pure Coulomb interaction plus a leading multipole correction to it. Hence the results of Sects. 3 and 5 may be successfully used for the analysis of more general problems than those solved in the present work.

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Received by Publishing Department
of June 15, 1989.